ZIGZAG SEQUENCES AND REPRESENTATIONS OF INTEGERS

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Abstract. In this paper, we examine representations of integers by means of particular zero-one sequences, Fibonacci sequences, and signed Fibonacci sequences, obtaining results similar to the Zeckendorf/Lekkerkerker Theorem.

1. Introduction

The following result is known as Zeckendorf's Theorem [8, 6, 7], although it was apparently first proved by Lekkerkerker [4].

Zeckendorf's Theorem (Lekkerkerker (1952))**.** *Let k be a positive integer and write F_m to denote the mth element of the Fibonacci sequence. There is a unique n and a unique sequence of zeros and ones $\varepsilon = (\varepsilon_{n-1}, \varepsilon_{n-2}, \ldots, \varepsilon_1, \varepsilon_0)$ having no consecutive ones and with $\varepsilon_{n-1} = 1$ such that $\sum_{i=0}^{n-1} \varepsilon_i \cdot F_{i+2} = k$.*

The zero-one sequence ε guaranteed by Zeckendorf's Theorem is called the Zeckendorf representation of k. Here, we prove four similar results for representing integers as sums of Fibonacci numbers. Rather than imposing the condition that the binary representation contains no pair of consecutive ones, we instead will require the binary strings to be so-called zigzag sequences.

We discovered these representations in the course of our work on representations of $C_p \times C_p$ in characteristic p, see [2, 3].

2. Zigzag Sequences

We will consider finite sequences all of whose entries are either 0 or 1. Suppose $\varepsilon = (\varepsilon_{n-1}, \varepsilon_{n-2}, \ldots, \varepsilon_1, \varepsilon_0)$ is such a sequence of zeros and ones. We say the length of this sequence is n.

A finite zero-one sequence of length n is a *down-up* sequence if it is a finite sequence $(\varepsilon_{n-1}, \varepsilon_{n-2}, \ldots, \varepsilon_1, \varepsilon_0)$ of zeros and ones such that

$$\varepsilon_{n-1} \geq \varepsilon_{n-2} \leq \varepsilon_{n-3} \geq \varepsilon_{n-4} \leq \cdots \varepsilon_0 \ .$$

Definition 2.1. *We write $\Omega(n)$ to denote the set of down-up sequences of length n.*

The first few of these sets are

$$\begin{aligned}
&\Omega(0) = \{(0)\} \text{ (by convention)},\\
&\Omega(1) = \{(1), (0)\},\\
&\Omega(2) = \{(11), (10), (00)\},\\
&\Omega(3) = \{(111), (101), (001), (100), (000)\},\\
&\Omega(4) = \{(1111), (1011), (0011), (1110), (1010), (0010), (1000), (0000)\}.
\end{aligned}$$

Up-down zero-one sequences are defined similarly: $\varepsilon = (\varepsilon_{n-1}, \varepsilon_{n-2}, \dots, \varepsilon_1, \varepsilon_0)$ is up-down if $\varepsilon_{n-1} \leq \varepsilon_{n-2} \geq \varepsilon_{n-3} \leq \cdots \varepsilon_0$. We define $\overline{\Omega}(n)$ to be the set of all up-down zero-one sequences of length n.

The first few up-down sequences are

$$\begin{aligned}
&\overline{\Omega}(0) = \{(0)\} \text{ (by convention)},\\
&\overline{\Omega}(1) = \{(1), (0)\},\\
&\overline{\Omega}(2) = \{(11), (01), (00)\},\\
&\overline{\Omega}(3) = \{(111), (110), (011), (010), (000)\},\\
&\overline{\Omega}(4) = \{(1111), (1101), (1100), (0111), (0101), (0100), (0001), (0000)\}.
\end{aligned}$$

A sequence that is either an up-down or a down-up sequence is called a *zigzag* sequence. These sequences have been considered before. They are known to be the vertices of the zigzag order polytopes (which we call *ZZ-topes*) associated with zigzag or alternating permutations. See [5] by Richard Stanley.

Natural operations on bit sequences include reversing the order and complementing all the bits. Zigzag sequences and their lengths are preserved under these two operations (and their composites). Depending upon the parity of the length, some of these may interchange up-down sequences with down-up sequences.

3. Representations of Integers

We prove four propositions for representing integers using zigzag sequences in the spirit of Zeckendorf's Theorem.

Denote the mth Fibonacci number by F_m and recall that the Fibonacci numbers satisfy the recursion relation

$$F_m = F_{m-1} + F_{m-2} \text{ for } m \geq 2\,,$$

with $F_0 = 0$ and $F_1 = 1$. If the recursive definition of the Fibonacci numbers is extended to negative indices, we get $F_{-1} = -1$, $F_{-2} = 1$, $F_{-3} = -2, \dots$. In general, $F_m = (-1)^{m-1} F_{-m}$ if $m < 0$. Then, $F_m = F_{m-1} + F_{m-2}$ for all integers m. These numbers, so defined, are sometimes referred to as the signed Fibonacci numbers.

Let $\varepsilon = (\varepsilon_{n-1}, \varepsilon_{n-2}, \dots, \varepsilon_0)$ be a zero-one sequence of length n and define

$$||\varepsilon||_{\text{Fib}} := \sum_{i=0}^{n-1} \varepsilon_i \cdot F_{i+1} \qquad \text{and}$$

$$||\varepsilon||_{\text{sFib}} := \sum_{i=0}^{n-1} \varepsilon_i \cdot F_{-i-2}\ .$$

Definition 3.1. *Let $\varepsilon = (\varepsilon_{n-1}, \varepsilon_{n-2}, \dots, \varepsilon_0)$. Then, ε is a Fibonacci representation of an integer k if $||\varepsilon||_{\text{Fib}} = k$ and ε is a signed Fibonacci representation of k if $||\varepsilon||_{\text{sFib}} = k$.*

Example 3.2. *We have*

$$\begin{aligned}
||(1000100010)||_{\text{Fib}} &= 64 \quad \textit{because} \quad 64 = F_{10} + F_6 + F_2 = 55 + 8 + 1,\\
||(100001)||_{\text{sFib}} &= 12 \quad \textit{because} \quad 12 = F_{-2} + F_{-7} = -1 + 13,\\
||(100100001)||_{\text{sFib}} &= -43 \ \textit{because} \ -43 = F_{-2} + F_{-7} + F_{-10} = -1 + 13 - 55.
\end{aligned}$$

Suppose that n is odd and that $\varepsilon = (\varepsilon_{n-1}, \varepsilon_{n-2}, \ldots, \varepsilon_1, \varepsilon_0) \in \Omega(n)$. Then, $\varepsilon_1 \leq \varepsilon_0$. Hence if $\varepsilon_0 = 0$, then $\varepsilon_1 = 0$. From this, it follows that for n odd,

$$\begin{aligned}\Omega(n) = &\{(\varepsilon_{n-1}, \varepsilon_{n-2}, \ldots, \varepsilon_2, 0, 0) \mid (\varepsilon_{n-1}, \varepsilon_{n-2}, \ldots, \varepsilon_2) \in \Omega(n-2)\}\\ &\sqcup \{(\varepsilon_{n-1}, \varepsilon_{n-2}, \ldots, \varepsilon_2, \varepsilon_1, 1) \mid (\varepsilon_{n-1}, \varepsilon_{n-2}, \ldots, \varepsilon_2, \varepsilon_1) \in \Omega(n-1)\} .\end{aligned}$$

Similarly for n even,

$$\begin{aligned}\Omega(n) = &\{(\varepsilon_{n-1}, \varepsilon_{n-2}, \ldots, \varepsilon_2, 1, 1) \mid (\varepsilon_{n-1}, \varepsilon_{n-2}, \ldots, \varepsilon_2) \in \Omega(n-2)\}\\ &\sqcup \{(\varepsilon_{n-1}, \varepsilon_{n-2}, \ldots, \varepsilon_2, \varepsilon_1, 0) \mid (\varepsilon_{n-1}, \varepsilon_{n-2}, \ldots, \varepsilon_2, \varepsilon_1) \in \Omega(n-1)\} .\end{aligned}$$

From this, it follows by induction that $\#\Omega(n) = F_{n+2}$ for $n \geq 0$.

For each n, there is a natural inclusion $\iota_n : \Omega(n) \hookrightarrow \Omega(n+2)$ given by $\iota_n(\varepsilon) = (0, 0, \varepsilon)$. Clearly, $||\varepsilon||_{\text{Fib}} = ||\iota_n(\varepsilon)||_{\text{Fib}}$. We identify $\Omega(n)$ with the subset $\iota_n(\Omega(n))$ of $\Omega(n+2)$ and define

$$\Omega(\text{even}) := \bigcup_{i=0}^{\infty} \Omega(2i) \quad \text{and}$$

$$\Omega(\text{odd}) := \bigcup_{i=0}^{\infty} \Omega(2i+1) .$$

Proposition 3.3. *Let k be a nonnegative integer. Then, there is a unique down-up zero-one sequence $\varepsilon \in \Omega(odd)$ such that $||\varepsilon||_{\text{Fib}} = k$. Furthermore, there is a unique down-up zero-one sequence $\varepsilon \in \Omega(even)$ such that $||\varepsilon||_{\text{Fib}} = k$.*

Proof. We will show that $||\cdot||_{\text{Fib}}$ gives a bijection for all n between $\Omega(n)$ and the interval of integers $[0, F_{n+2} - 1] = [0, F_{n+2})$. We proceed by induction, treating the even and odd cases in parallel. For $n = 0$, we have $\Omega(1) = \{(0)\}$ and $\{||\varepsilon||_{\text{Fib}} \mid \varepsilon \in \Omega(0)\} = \{0\} = [0, F_2)$. For $n = 1$, we have $\Omega(1) = \{(0), (1)\}$ and $\{||\varepsilon||_{\text{Fib}} \mid \varepsilon \in \Omega(1)\} = \{0, 1\} = [0, F_3)$.

Suppose as an induction hypothesis that there is such a bijection for $n - 2$. Clearly $\Omega(n)$ decomposes as a disjoint union:

$$\Omega(n) = \Omega(n)^{00} \sqcup \Omega(n)^{10} \sqcup \Omega(n)^{11},$$

where $\Omega(n)^{ij} = \{\varepsilon \in \Omega(n) \mid \varepsilon_{n-1} = i, \varepsilon_{n-2} = j\}$. Clearly, $\Omega(n)^{00} = \iota(\Omega(n-2)) = \Omega(n-2)$ under our identification. Also, $\Omega(n)^{10} = \{(1, 0, \varepsilon) \mid \varepsilon \in \Omega(n-2)\}$ and $\Omega(n)^{11} \subsetneq \{(1, 1, \varepsilon) \mid \varepsilon \in \Omega(n-2)\}$.

Thus by the induction hypothesis, $\{||\varepsilon||_{\text{Fib}} \mid \varepsilon \in \Omega(n)^{00}\} = [0, F_n)$ and $\{||\varepsilon||_{\text{Fib}} \mid \varepsilon \in \Omega(n)^{10}\} = F_n + [0, F_n) = [F_n, 2F_n)$. Clearly, $||\cdot||_{\text{Fib}}$ is injective when restricted to $\Omega(n)^{11}$. We consider the image of this restriction. This is $\{||\varepsilon||_{\text{Fib}} \mid \varepsilon \in \Omega(n)^{11}\}$, which is contained in $F_n + F_{n-1} + [0, F_n) = [F_{n+1}, F_{n+2})$. Note that if $\varepsilon \in \Omega(n)^{11}$, then $\varepsilon_{n-3} = 1$ and thus, $||\varepsilon||_{\text{Fib}} \geq F_n + F_{n-1} + F_{n-2} = 2F_n$ for all $\varepsilon \in \Omega(n)^{11}$.

Therefore, there are three injective maps

$$||\cdot||_{\text{Fib}} : \Omega(n)^{00} \to [0, F_n),$$
$$||\cdot||_{\text{Fib}} : \Omega(n)^{10} \to [F_n, 2F_n),$$
$$||\cdot||_{\text{Fib}} : \Omega(n)^{11} \to [2F_n, F_{n+2}),$$

which combine to give the injection

$$||\cdot||_{\text{Fib}} : \Omega(n) \to [0, F_{n+2}).$$

Finally, $\#\Omega(n) = F_{n+2} = \#[0, F_{n+2})$ and so $||\cdot||_{\text{Fib}}$ is a bijection. □

Working with up-down sequences, there is a similar family of inclusions $\overline{\iota}_n : \overline{\Omega}(n) \hookrightarrow \overline{\Omega}(n+2)$ satisfying $||\varepsilon||_{\text{Fib}} = ||\overline{\iota}(\varepsilon)||_{\text{Fib}}$. The inclusion $\overline{\iota}_n$ is more complicated than ι_n. It is defined as follows.

(1) If $\varepsilon_{n-1} = 0$, then $\overline{\iota_n}(\varepsilon) = (0, 0, \varepsilon_{n-1}, \varepsilon_{n-2}, \dots, \varepsilon_0)$.
(2) For $\varepsilon = (1, 1, \dots, 1)$, we put

$$\overline{\iota_n}(\varepsilon) = \begin{cases} (0,1,0,1,\dots,0,1,0), & \text{if } n \text{ is odd;} \\ (0,1,0,1,\dots,0,1,0,0), & \text{if } n \text{ is even.} \end{cases}$$

(3) If $\varepsilon = (1, 1, \dots, 1, 0)$, then n must be odd and we put

$$\overline{\iota}_n(\varepsilon) = (0, 1, 0, 1, \dots, 0, 1, 0).$$

(4) Finally, suppose that $\varepsilon_{n-1} = \cdots = \varepsilon_{n-t} = 1$ and $\varepsilon_{n-t-1} = 0$ with $t < n - 1$. Then, $\varepsilon = (1, 1, \dots, 1, 0, \varepsilon_{n-t-2}, \dots, \varepsilon_0)$ and

$$\overline{\iota_n}(\varepsilon) = (0, 1, 0, 1, \dots, 0, 1, 0, 0, 0, \varepsilon_{n-t-2}, \dots, \varepsilon_0).$$

As before, we identify $\overline{\Omega}(n-2) \subset \overline{\Omega}(n)$ with the subset $\overline{\iota_{n-2}}(\overline{\Omega}(n-2))$. In particular, if $\varepsilon = (\varepsilon_{n-1}, \varepsilon_{n-2}, \dots, \varepsilon_0) \in \overline{\Omega}(n-2) \subset \overline{\Omega}(n)$, then

$$\overline{\iota_n}(\varepsilon) = (0, 0, \varepsilon_{n-1}, \varepsilon_{n-2}, \dots, \varepsilon_0) \in \overline{\Omega}(n+2)\ .$$

Using these inclusions, we define $\overline{\Omega}(\text{odd}) := \bigcup_{i=0}^{\infty} \overline{\Omega}(2i+1)$ and $\overline{\Omega}(\text{even}) := \bigcup_{i=0}^{\infty} \overline{\Omega}(2i)$ as we did above.

We now give three more results on representing integers using the Fibonacci numbers and zigzag sequences. The proofs of these three propositions are quite similar to the proof of Proposition 3.3 and so we omit them.

Proposition 3.4. *Let k be any integer. Then, there is a unique down-up zero-one sequence $\varepsilon \in \Omega(odd) \cup \Omega(even)$ such that $||\varepsilon||_{\text{sFib}} = k$. Moreover, if k is positive, then ε has even length and if k is negative, then ε has odd length.*

Note that in this proposition, we identify the all zeros strings of even and odd length so that there is a unique signed Fibonacci representation of the integer 0.

Proposition 3.5. *Let k be a nonnegative integer. Then, there is a unique up-down zero-one sequence $\varepsilon \in \overline{\Omega}(odd)$ such that $||\varepsilon||_{\text{Fib}} = k$. Furthermore, there is a unique up-down zero-one sequence $\varepsilon \in \overline{\Omega}(even)$ such that $||\varepsilon||_{\text{Fib}} = k$.*

Proposition 3.6. *Let k be any integer. Then, there is a unique up-down zero-one sequence $\varepsilon \in \overline{\Omega}(odd)$ such that $||\varepsilon||_{\text{sFib}} = k$. Furthermore, there is a unique up-down zero-one sequence $\varepsilon \in \overline{\Omega}(even)$ such that $||\varepsilon||_{\text{sFib}} = k$.*

Example 3.7. *The following tables illustrate Propositions 1–4.*

Down-up Representations

Set	Sequence ε	$\|\|\varepsilon\|\|_{\text{Fib}}$	$\|\|\varepsilon\|\|_{\text{sFib}}$
$\Omega(0)$	(0)	0	0
$\Omega(1)$	(1)	1	−1
	(0)	0	0
$\Omega(2)$	(11)	2	1
	(10)	1	2
	(00)	0	0
$\Omega(3)$	(111)	4	−2
	(101)	3	−4
	(100)	2	−3
	(001)	1	−1
	(000)	0	0
$\Omega(4)$	(1111)	7	3
	(1110)	6	4
	(1011)	5	6
	(1010)	4	7
	(1000)	3	5
	(0011)	2	1
	(0010)	1	2
	(0000)	0	0

Up-down Representations

Set	Sequence ε	$\|\|\varepsilon\|\|_{\text{Fib}}$	$\|\|\varepsilon\|\|_{\text{sFib}}$
$\overline{\Omega}(0)$	(0)	0	0
$\overline{\Omega}(1)$	(1)	1	1
	(0)	0	0
$\overline{\Omega}(2)$	(11)	2	1
	(01)	1	−1
	(00)	0	0
$\overline{\Omega}(3)$	(111)	4	−2
	(010)	1	2
	(011)	2	−1
	(110)	3	1
	(000)	0	0
$\overline{\Omega}(4)$	(1111)	7	3
	(0111)	4	−2
	(1101)	6	1
	(0101)	3	−4
	(0001)	1	−1
	(1100)	5	2
	(0100)	2	−3
	(0000)	0	0

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