CONGRUENCES MODULO THE SQUARE OF A PRIME FOR SUMS CONTAINING FIBONACCI NUMBERS

ALEXANDRU GICA

ABSTRACT. Let $p > 5$ be a prime number and let $U = \sum_{k=2}^{p-1} \frac{(-1)^k F_{k-1}}{k}$, $V = \sum_{k=2}^{p} \frac{(-1)^k F_k}{k-1}$. The aim of this paper is to prove that $U \equiv 0 \pmod{p^2}$, $V \equiv \frac{L_p - 1}{p} \pmod{p^2}$ if $p \equiv 1, 4 \pmod 5$ and $U \equiv \frac{1-L_p}{p} \pmod{p^2}$, $V \equiv 0 \pmod{p^2}$ in the case $p \equiv 2, 3 \pmod 5$. We also find similar results for some general Lucas sequences.

1. Introduction

In this paper, we will always consider a prime number $p > 5$. It is fairly easy to prove that $\sum_{k=1}^{p-1} F_k/k \equiv 0 \pmod p$ (see [1] for a proof of this result). In 2014, Hao Pan and Zhi-Wei Sun proved that

$$\sum_{k=1}^{p-1} \frac{F_k}{k^2} \equiv -\frac{1}{5}\left(\frac{p}{5}\right)\left(\frac{L_p - 1}{p}\right)^2 \pmod p \tag{1.1}$$

(see [3], Remark 3.3, which follows the proof of Theorem 1.2); in the above formula, $\left(\frac{p}{5}\right)$ is the Legendre symbol (which is $+1$ if $p \equiv 1, 4 \pmod 5$ and -1 if $p \equiv 2, 3 \pmod 5$). This result of Pan and Sun will play an important role in our paper. In this article, we will denote by s the number

$$s = \sum_{k=1}^{p-1} \frac{F_k}{k^2}.$$

In their paper, Pan and Sun proved the following lemma.

Lemma 1.1. *(Pan, Sun)* $$\sum_{k=1}^{p-1} \frac{(1-x)^k}{k} \equiv \frac{1 - x^p - (1-x)^p}{p} - p\left(\sum_{k=1}^{p-1} \frac{x^k}{k^2}\right) \pmod{p^2}.$$

Remark: They proved a more general result (see [3], Lemma 4.1), but we only need this particular version. We note that Pan and Sun extended a congruence of Granville (see [2]). In his paper, Granville proved a conjecture of Skula: for every prime $p > 3$,

$$\left(\frac{2^{p-1} - 1}{p}\right)^2 \equiv -\sum_{k=1}^{p-1} \frac{2^k}{k^2} \pmod p.$$

We will use the following notations (of Granville):

$$g(x) = \sum_{k=1}^{p-1} \frac{x^k}{k}, G(x) = \sum_{k=1}^{p-1} \frac{x^k}{k^2}, q(x) = \frac{x^p + (1-x)^p - 1}{p}.$$

Granville proved in [2] the following congruences:

$$G(x) \equiv G(1-x) + x^p G\left(1 - \frac{1}{x}\right) \pmod{p}, \tag{1.2}$$

$$q(x)^2 \equiv -2x^p G(x) - 2(1-x^p)G(1-x) \pmod{p}. \tag{1.3}$$

Replacing the value of $G(x)$ given by (1.2) in (1.3), we have:

$$q(x)^2 \equiv -2G(1-x) - 2x^{2p} G\left(1 - \frac{1}{x}\right) \pmod{p}. \tag{1.4}$$

In Section 5, as a simple consequence of Lemma 1.1, we will prove the following proposition (which will be needed later).

Proposition 1.2. *Let $p > 5$ be a prime number. Then,*

a) $\displaystyle\sum_{k=1}^{p-1} \frac{F_k}{k} \equiv ps \pmod{p^2}$,

b) $\displaystyle\sum_{k=1}^{p-1} \frac{(-1)^k F_k}{k} \equiv \frac{F_p - F_{2p}}{p} - ps \pmod{p^2}$.

Looking to all these results of Granville and Pan and Sun, we realized that we are able to prove the following result.

Theorem 1.3. *Let $p > 5$ be a prime number, $U = \displaystyle\sum_{k=2}^{p-1} \frac{(-1)^k F_{k-1}}{k}$, and $V = \displaystyle\sum_{k=2}^{p} \frac{(-1)^k F_k}{k-1}$. The following congruences hold in the case $p \equiv 1, 4 \pmod{5}$:*

$$U \equiv 0 \pmod{p^2}, \quad V \equiv \frac{L_p - 1}{p} \pmod{p^2}.$$

In the case $p \equiv 2, 3 \pmod{5}$, we have the congruences

$$U \equiv \frac{1 - L_p}{p} \pmod{p^2}, \quad V \equiv 0 \pmod{p^2}.$$

Section 2 is devoted to some technical results. The proof of Theorem 1.3 is given in the Sections 3, 4, and 5; in Section 3, we compute $U \pmod{p^2}$ in the case $p \equiv 1, 4 \pmod{5}$; in Section 4, we compute $V \pmod{p^2}$ in the case $p \equiv 2, 3 \pmod{5}$; and in Section 5, we complete the proof of the theorem (after we compute $U + V \pmod{p^2}$). Section 6 is devoted to finding a general pattern.

Following suggestions made by the referee, we were able to prove a generalization of Theorem 1.3. In the sequel, $(x_n)_n, (y_n)_n$ are the sequences defined by the conditions: $x_0 = 0$, $x_1 = 1$, $x_{n+2} = x_{n+1} + bx_n$, $y_0 = 2$, $y_1 = 1$, $y_{n+2} = y_{n+1} + by_n$ (b is a fixed integer), for any nonnegative integer n. We will denote by U, V the following numbers: $U = \sum_{k=2}^{p-1} \frac{(-1)^k x_{k-1}}{b^{k-1} k}$, $V = \sum_{k=2}^{p} \frac{(-1)^k x_k}{b^k (k-1)}$. The aforementioned generalization is the following.

Theorem 1.4. *Let $p > 3$ be a prime number that does not divide $b(4b+1)$. The following congruence holds:*

$$U + bV \equiv \frac{x_{2p} - x_p}{pb^p} + \frac{p}{b} \sum_{k=1}^{p-1} \frac{x_k}{k^2} \pmod{p^2}.$$

If the discriminant $\Delta = 1 + 4b$ *is a quadratic residue modulo* p*, then*

$$U \equiv \frac{x_{p-1}(1-y_p)}{p} - 2p\sum_{k=1}^{p-1}\frac{x_k}{k^2} \pmod{p^2}.$$

If the discriminant Δ *is a quadratic nonresidue modulo* p*, then*

$$V \equiv \frac{x_{p+1}(y_p-1)}{b^2p} - \frac{2p}{b}\sum_{k=1}^{p-1}\frac{x_k}{k^2} \pmod{p^2}.$$

We will use the standard notation for the Fibonacci and Lucas numbers: $\alpha = \frac{1+\sqrt{5}}{2}$, $\beta = \frac{1-\sqrt{5}}{2}$, $F_n = \frac{\alpha^n-\beta^n}{\alpha-\beta}$, and $L_n = \alpha^n + \beta^n$. If $p > 5$ is a prime, then $\alpha^p \equiv \alpha \pmod{p}$, $\beta^p \equiv \beta \pmod{p}$ in the case $p \equiv 1, 4 \pmod{5}$ and $\alpha^p \equiv \beta \pmod{p}$, $\beta^p \equiv \alpha \pmod{p}$ in the case $p \equiv 2, 3 \pmod{5}$. With the above notation of Granville, we have $q(\alpha) = q(\beta) = \frac{\alpha^p+\beta^p-1}{p} = \frac{L_p-1}{p}$.

2. Some Preliminary Results

We need some technical results.

Lemma 2.1. *Let* $p > 5$ *be a prime number. Then,* $\frac{1-L_p}{p} \equiv -\frac{5}{2}\frac{F_{p-1}}{p} \pmod{p}$ *in the case* $p \equiv 1, 4 \pmod{5}$*, and* $\frac{1-L_p}{p} \equiv -\frac{5}{2}\frac{F_{p+1}}{p} \pmod{p}$ *in the case* $p \equiv 2, 3 \pmod{5}$*.*

Proof. To prove the result in the case $p \equiv 1, 4 \pmod{5}$, we must show that

$$\frac{1-L_p}{p} \equiv -\frac{5}{2}\cdot\frac{F_{p-1}}{p} \pmod{p}.$$

This is equivalent to

$$2L_p - 2 \equiv 5F_{p-1} \pmod{p^2}$$

and (taking into account that $5F_n = L_{n-1} + L_{n+1}$),

$$2L_p - 2 \equiv L_{p-2} + L_p \pmod{p^2}.$$

This is true because

$$L_p - L_{p-2} = L_{p-1} \equiv 2 \pmod{p^2}.$$

The last congruence holds because

$$L_{p-1} - 2 = \alpha^{p-1} + \beta^{p-1} - 2 = (\alpha^{\frac{p-1}{2}} - \frac{1}{\alpha^{\frac{p-1}{2}}})^2 \equiv 0 \pmod{p^2};$$

for the last congruence we used that $\alpha^{p-1} \equiv 1 \pmod{p}$ in the case $p \equiv 1, 4 \pmod{5}$.

Now, we analyze the case $p \equiv 2, 3 \pmod{5}$. To prove the statement, we must show, in this case, that

$$\frac{1-L_p}{p} \equiv -\frac{5}{2}\cdot\frac{F_{p+1}}{p} \pmod{p}.$$

This is equivalent to

$$2L_p - 2 \equiv 5F_{p+1} \pmod{p^2}$$

and (taking into account that $5F_n = L_{n-1} + L_{n+1}$),

$$2L_p - 2 \equiv L_{p+2} + L_p \pmod{p^2}.$$

This is true because

$$L_{p+2} - L_p = L_{p+1} \equiv -2 \pmod{p^2}.$$

The last congruence is true because

$$L_{p+1}+2=\alpha^{p+1}+\beta^{p+1}+2=(\alpha^{\frac{p+1}{2}}+\frac{1}{\alpha^{\frac{p+1}{2}}})^2\equiv 0 \pmod{p^2};$$

for the last congruence we used that $\alpha^{p+1}\equiv -1 \pmod p$ in the case $p\equiv 2,3 \pmod 5$. □

Lemma 2.2. *Let $p>5$ be a prime number.*

a) $s=\sum_{k=1}^{p-1}\frac{F_k}{k^2}\equiv\sum_{k=1}^{p-1}\frac{F_{2k}}{k^2}\equiv -\frac{1}{5}\Big(\frac{p}{5}\Big)\Big(\frac{L_p-1}{p}\Big)^2 \pmod p$.

b) $\sum_{k=1}^{p-1}\frac{F_{2k-1}}{k^2}\equiv 2s \pmod p$, *when* $p\equiv 1,4 \pmod 5$.

c) $\sum_{k=1}^{p-1}\frac{F_{2k+1}}{k^2}\equiv -2s \pmod p$, *when* $p\equiv 2,3 \pmod 5$.

Proof. a) Let $x=\alpha$ in formula (1.4). Because $1-\alpha=\beta$, $1-\frac{1}{\alpha}=1+\beta=\beta^2$, we obtain

$$q(\alpha)^2\equiv -2G(\beta)-2\alpha^{2p}G(\beta^2) \pmod p. \tag{2.1}$$

We also have (because $\beta^{-2}=\alpha^2$ and $(p-k)^2\equiv k^2 \pmod p$)

$$G(\beta^2)=\sum_{k=1}^{p-1}\frac{\beta^{2k}}{k^2}=\sum_{k=1}^{p-1}\frac{\beta^{2(p-k)}}{(p-k)^2}\equiv\beta^{2p}G(\alpha^2) \pmod p. \tag{2.2}$$

If we combine (2.1) and (2.2), we get

$$q(\alpha)^2\equiv -2G(\beta)-2G(\alpha^2) \pmod p. \tag{2.3}$$

In the same way, we obtain the congruence

$$q(\beta)^2\equiv -2G(\alpha)-2G(\beta^2) \pmod p. \tag{2.4}$$

We take into account that $q(\alpha)=q(\beta)=\frac{L_p-1}{p}$ (because $\alpha+\beta=1$). If we subtract (2.3) from (2.4) and divide by $\alpha-\beta$, we obtain the desired congruence $\sum_{k=1}^{p-1}\frac{F_{2k}}{k^2}\equiv\sum_{k=1}^{p-1}\frac{F_k}{k^2} \pmod p$. The last congruence, $\left(\sum_{k=1}^{p-1}\frac{F_k}{k^2}\equiv -\frac{1}{5}\Big(\frac{p}{5}\Big)\Big(\frac{L_p-1}{p}\Big)^2 \pmod p\right)$, was proved by Pan and Sun in [3] (Theorem 1.2 and Remark 3.3).

b) In the case ($p\equiv 1,4 \pmod 5$), we have $\beta^p\equiv\beta \pmod p$. Using formula (2.2), we obtain

$$G(\beta^2)\equiv\beta^2G(\alpha^2)\equiv\sum_{k=1}^{p-1}\frac{\alpha^{2k-2}}{k^2} \pmod p. \tag{2.5}$$

In the same way, we obtain

$$G(\alpha^2)\equiv\sum_{k=1}^{p-1}\frac{\beta^{2k-2}}{k^2} \pmod p. \tag{2.6}$$

We subtract (2.6) from (2.5) and obtain (dividing by $\alpha - \beta$ and using the first statement of the Lemma)

$$\sum_{k=1}^{p-1} \frac{F_{2k-2}}{k^2} \equiv -\frac{G(\alpha^2) - G(\beta^2)}{\alpha - \beta} = -\sum_{k=1}^{p-1} \frac{F_{2k}}{k^2} \equiv -s \pmod{p}. \tag{2.7}$$

The result follows at once:

$$\sum_{k=1}^{p-1} \frac{F_{2k-1}}{k^2} = \sum_{k=1}^{p-1} \frac{F_{2k} - F_{2k-2}}{k^2} \equiv 2s \pmod{p}.$$

Remark: We obtain, in the same way, $\sum_{k=1}^{p-1} \frac{F_{2k-1}}{k^2} \equiv -3s \pmod{p}$ in the case $p \equiv 2, 3 \pmod{5}$. We will not use this result in our paper.

c) In the case ($p \equiv 2, 3 \pmod{5}$), we have $\beta^p \equiv \alpha \pmod{p}$. Using formula (2.2), we obtain

$$G(\beta^2) \equiv \alpha^2 G(\alpha^2) \equiv \sum_{k=1}^{p-1} \frac{\alpha^{2k+2}}{k^2} \pmod{p}. \tag{2.8}$$

In the same way, we obtain

$$G(\alpha^2) \equiv \sum_{k=1}^{p-1} \frac{\beta^{2k+2}}{k^2} \pmod{p}. \tag{2.9}$$

We subtract (2.9) from (2.8) and obtain (dividing by $\alpha - \beta$ and using the first statement of the Lemma)

$$\sum_{k=1}^{p-1} \frac{F_{2k+2}}{k^2} \equiv -s \pmod{p}. \tag{2.10}$$

The result follows at once:

$$\sum_{k=1}^{p-1} \frac{F_{2k+1}}{k^2} = \sum_{k=1}^{p-1} \frac{F_{2k+2} - F_{2k}}{k^2} \equiv -2s \pmod{p}.$$

Remark: We obtain, in the same way, $\sum_{k=1}^{p-1} \frac{F_{2k+1}}{k^2} \equiv 3s \pmod{p}$ in the case $p \equiv 1, 4 \pmod{5}$. We will not use this result in our paper. □

3. $U \equiv 0 \pmod{p^2}$ in the Case $p \equiv 1, 4 \pmod{5}$

In Lemma 1.1, we let $x = \alpha^2$ and obtain (because $(1 - \alpha^2)^p = (-\alpha)^p = -\alpha^p$)

$$\sum_{k=1}^{p-1} \frac{(-\alpha)^k}{k} \equiv \frac{1 - \alpha^{2p} + \alpha^p}{p} - pG(\alpha^2) \pmod{p^2}. \tag{3.1}$$

We multiply the last equality by $-\beta$ and obtain (taking into account that $-\beta \cdot \alpha = 1$)

$$\sum_{k=1}^{p-1} \frac{(-1)^k \alpha^{k-1}}{k} \equiv \frac{-\beta - \alpha^{2p-1} + \alpha^{p-1}}{p} - p\sum_{k=1}^{p-1} \frac{\alpha^{2k-1}}{k^2} \pmod{p^2}. \tag{3.2}$$

In the same way, we obtain the congruence

$$\sum_{k=1}^{p-1} \frac{(-1)^k \beta^{k-1}}{k} \equiv \frac{-\alpha - \beta^{2p-1} + \beta^{p-1}}{p} - p \sum_{k=1}^{p-1} \frac{\beta^{2k-1}}{k^2} \pmod{p^2}. \tag{3.3}$$

If we subtract (3.3) from (3.2) and divide by $\alpha - \beta$, we get

$$U \equiv \frac{1 + F_{p-1} - F_{2p-1}}{p} - p \sum_{k=1}^{p-1} \frac{F_{2k-1}}{k^2} \pmod{p^2}. \tag{3.4}$$

We are now ready to compute U modulo p^2 in the case $p \equiv 1, 4 \pmod 5$ (which is one of the statements from Theorem 1.3).

Proof. The case $p \equiv 1, 4 \pmod 5$. We use Lemma 2.2, part b) $\sum_{k=1}^{p-1} \frac{F_{2k-1}}{k^2} \equiv 2s \pmod p$. To prove

$$U \equiv 0 \pmod{p^2},$$

we must show that

$$ps \equiv \frac{1 + F_{p-1} - F_{2p-1}}{2p} \pmod{p^2}.$$

This is the same as proving

$$2p^2 s \equiv 1 + F_{p-1} - F_{2p-1} \pmod{p^3}. \tag{3.5}$$

In this case, $s \equiv -\frac{1}{5}\left(\frac{L_p - 1}{p}\right)^2 \pmod p$ (according to formula (1.1) of Pan and Sun; here, the Legendre symbol $\left(\frac{p}{5}\right) = +1$). We use Lemma 2.1:

$$\frac{1 - L_p}{p} \equiv -\frac{5}{2} \frac{F_{p-1}}{p} \pmod p.$$

We get $2p^2 s \equiv -\frac{2}{5}(L_p - 1)^2 \equiv -\frac{5}{2} F_{p-1}^2 \pmod{p^3}$. Now, to prove (3.5), we must show

$$5F_{p-1}^2 \equiv 2(F_{2p-1} - F_{p-1} - 1) \pmod{p^3}. \tag{3.6}$$

Because $5F_{p-1}^2 = L_{2p-2} - 2 = F_{2p-1} + F_{2p-3} - 2$, formula (3.6) is equivalent to

$$F_{2p-1} + F_{2p-3} \equiv 2F_{2p-1} - 2F_{p-1} \pmod{p^3}$$

and to

$$F_{2p-2} - 2F_{p-1} \equiv 0 \pmod{p^3}.$$

The last congruence is true because

$$F_{2p-2} - 2F_{p-1} = F_{p-1}(L_{p-1} - 2),$$

$F_{p-1} \equiv 0 \pmod p$, and $L_{p-1} \equiv 2 \pmod{p^2}$ (see the proof of Lemma 2.1). □

4. $V \equiv 0 \pmod{p^2}$ in the Case $p \equiv 2, 3 \pmod 5$

We multiply formula (3.1) by $-\alpha$ and obtain

$$\sum_{k=1}^{p-1} \frac{(-\alpha)^{k+1}}{k} \equiv \frac{-\alpha + \alpha^{2p+1} - \alpha^{p+1}}{p} + p \sum_{k=1}^{p-1} \frac{\alpha^{2k+1}}{k^2} \pmod{p^2}. \tag{4.1}$$

In the same way, we obtain the congruence

$$\sum_{k=1}^{p-1} \frac{(-\beta)^{k+1}}{k} \equiv \frac{-\beta + \beta^{2p+1} - \beta^{p+1}}{p} + p \sum_{k=1}^{p-1} \frac{\beta^{2k+1}}{k^2} \pmod{p^2}. \tag{4.2}$$

If we subtract (4.2) from (4.1) and divide by $\alpha - \beta$, we get

$$V \equiv \frac{-1 - F_{p+1} + F_{2p+1}}{p} + p \sum_{k=1}^{p-1} \frac{F_{2k+1}}{k^2} \pmod{p^2}. \tag{4.3}$$

We are now ready to compute V modulo p^2 in the case $p \equiv 2, 3 \pmod 5$, which is another statement of Theorem 1.3.

Proof. The case $p \equiv 2, 3 \pmod 5$. We use Lemma 2.2, part c) $\sum_{k=1}^{p-1} \frac{F_{2k+1}}{k^2} \equiv -2s \pmod p$. To prove

$$V \equiv 0 \pmod{p^2},$$

we must show that

$$ps \equiv \frac{-1 - F_{p+1} + F_{2p+1}}{2p} \pmod{p^2}.$$

This is the same as proving

$$2p^2 s \equiv -1 - F_{p+1} + F_{2p+1} \pmod{p^3}. \tag{4.4}$$

According to formula (1.1) of Pan and Sun (here, the Legendre symbol $\left(\frac{p}{5}\right) = -1$), in this case $s \equiv \frac{1}{5}\left(\frac{L_p - 1}{p}\right)^2 \pmod p$. We use Lemma 2.1:

$$\frac{1 - L_p}{p} \equiv -\frac{5}{2} \frac{F_{p+1}}{p} \pmod p.$$

We get $2p^2 s \equiv \frac{2}{5}(L_p - 1)^2 \equiv \frac{5}{2} F_{p+1}^2 \pmod{p^3}$. Now, it is obvious that to prove (4.4), we must show

$$5F_{p+1}^2 \equiv 2(F_{2p+1} - F_{p+1} - 1) \pmod{p^3}. \tag{4.5}$$

Because $5F_{p+1}^2 = L_{2p+2} - 2 = F_{2p+1} + F_{2p+3} - 2$, formula (4.5) is equivalent to

$$F_{2p+1} + F_{2p+3} \equiv 2F_{2p+1} - 2F_{p+1} \pmod{p^3}$$

and to

$$F_{2p+2} + 2F_{p+1} \equiv 0 \pmod{p^3}.$$

The last congruence is true because

$$F_{2p+2} + 2F_{p+1} = F_{p+1}(L_{p+1} + 2),$$

$F_{p+1} \equiv 0 \pmod p$, and $L_{p+1} \equiv -2 \pmod{p^2}$ (see the proof of Lemma 2.1). □

5. The Sum $U+V \pmod{p^2}$

In Sections 3 and 4, we saw that $U = \sum_{k=2}^{p-1} \frac{(-1)^k F_{k-1}}{k} \equiv 0 \pmod{p^2}$ if $p \equiv 1,4 \pmod 5$ and $V = \sum_{k=2}^{p} \frac{(-1)^k F_k}{k-1} \equiv 0 \pmod{p^2}$ when $p \equiv 2,3 \pmod 5$.

How do we prove that

$$U \equiv \frac{1-L_p}{p} \pmod{p^2}$$

when $p \equiv 2,3 \pmod 5$? And how do we prove that

$$V \equiv \frac{L_p-1}{p} \pmod{p^2}$$

in the case $p \equiv 1,4 \pmod 5$? The idea is simple. We combine the facts that we already know ($U \equiv 0 \pmod{p^2}$ when $p \equiv 1,4 \pmod 5$ and $V \equiv 0 \pmod{p^2}$ when $p \equiv 2,3 \pmod 5$) with the result about the sum $U+V$ modulo p^2. It is easy to check that

$$U+V = -\sum_{k=1}^{p-1} \frac{(-1)^k F_k}{k}.$$

We need to find $\sum_{k=1}^{p-1} \frac{(-1)^k F_k}{k} \pmod{p^2}$; therefore, we have to prove Proposition 1.2.

Proof. Again, we look to formula (3.1):

$$\sum_{k=1}^{p-1} \frac{(-\alpha)^k}{k} \equiv \frac{1-\alpha^{2p}+\alpha^p}{p} - pG(\alpha^2) \pmod{p^2}.$$

In the same way, we obtain

$$\sum_{k=1}^{p-1} \frac{(-\beta)^k}{k} \equiv \frac{1-\beta^{2p}+\beta^p}{p} - pG(\beta^2) \pmod{p^2}. \tag{5.1}$$

We subtract (5.1) from (3.1) and divide by $\alpha-\beta$. Then, we use Lemma 2.2, part a)

$$\frac{G(\alpha^2)-G(\beta^2)}{\alpha-\beta} = \sum_{k=1}^{p-1} \frac{F_{2k}}{k^2} \equiv s \pmod p$$

and we obtain

$$\sum_{k=1}^{p-1} \frac{(-1)^k F_k}{k} \equiv \frac{F_p - F_{2p}}{p} - ps \pmod{p^2}.$$

At that point, we proved part b) of Proposition 1.2. The proof of part a) is straightforward. We let $x=\alpha$ in Lemma 1.1 and get

$$\sum_{k=1}^{p-1} \frac{\beta^k}{k} \equiv \frac{1-L_p}{p} - p\sum_{k=1}^{p-1} \frac{\alpha^k}{k^2} \pmod{p^2}.$$

Now, we let $x = \beta$ in Lemma 1.1 and get

$$\sum_{k=1}^{p-1} \frac{\alpha^k}{k} \equiv \frac{1-L_p}{p} - p\sum_{k=1}^{p-1} \frac{\beta^k}{k^2} \pmod{p^2}.$$

We subtract the last two congruences and obtain part a) of Proposition 1.2. □

Let us return to the sum $U+V$. If we take into account part b) of the Proposition 1.2, we get

$$U + V \equiv \frac{F_{2p} - F_p}{p} + ps \pmod{p^2}. \tag{5.2}$$

The case $p \equiv 1, 4 \pmod 5$. Using (5.2), $U \equiv 0 \pmod{p^2}$, the well-known identity $F_{2p} = F_pL_p$, and the formula of s obtained by Pan and Sun ($s \equiv -\frac{1}{5}\left(\frac{L_p-1}{p}\right)^2 \pmod p$), we get

$$V \equiv U + V \equiv \frac{L_p - 1}{5p}(5F_p - L_p + 1) \pmod{p^2}. \tag{5.3}$$

To find the formula of $V \pmod{p^2}$ in this case, it is enough to prove that

$$5F_p - L_p + 1 \equiv 5 \pmod{p^2}.$$

This is straightforward because

$$5F_p - L_p - 4 = L_{p-1} + L_{p+1} - L_p - 4 = 2(L_{p-1} - 2) \equiv 0 \pmod{p^2};$$

for the last congruence, we used $L_{p-1} \equiv 2 \pmod{p^2}$, in the case $p \equiv 1, 4 \pmod 5$.

The case $p \equiv 2, 3 \pmod 5$. Using (5.2), $V \equiv 0 \pmod{p^2}$, the identity $F_{2p} = F_pL_p$, and the formula of Pan and Sun ($s \equiv \frac{1}{5}\left(\frac{L_p-1}{p}\right)^2 \pmod p$), we get

$$U \equiv U + V \equiv \frac{L_p - 1}{5p}(5F_p + L_p - 1) \pmod{p^2}. \tag{5.4}$$

To find the formula of $U \pmod{p^2}$ in this case, it is enough to prove that

$$5F_p + L_p - 1 \equiv -5 \pmod{p^2}.$$

This is straightforward because

$$5F_p + L_p + 4 = L_{p-1} + L_{p+1} + L_p + 4 = 2(L_{p+1} + 2) \equiv 0 \pmod{p^2};$$

for the last congruence, we used $L_{p+1} \equiv -2 \pmod{p^2}$ in the case $p \equiv 2, 3 \pmod 5$. □

Remark: Pan and Sun (see Theorem 1.2 in [3]) proved the following result:

$$\sum_{k=1}^{p-1} \frac{L_k}{k^2} \equiv 0 \pmod p.$$

Using this result and the same path as in the proof of Theorem 1.3, we can prove the following statement.

Proposition 5.1. *Let $p > 5$ be a prime number. We let* $U_1 = \sum_{k=1}^{p-1} \frac{(-1)^k L_{k-1}}{k}$ *and* $V_1 = \sum_{k=2}^{p} \frac{(-1)^k L_k}{k-1}$. *The following congruences hold:* $U_1 \equiv \frac{L_{p-1} - L_{2p-1} - 1}{p} \pmod{p^2}$, $V_1 \equiv \frac{L_{2p-1} + L_{p-2}}{p} \pmod{p^2}$ *if* $p \equiv 1, 4 \pmod 5$ *and* $V_1 \equiv \frac{L_{2p+1} - L_{p+1} - 1}{p} \pmod{p^2}$, $U_1 \equiv \frac{L_{p+2} - L_{2p+1}}{p} \pmod{p^2}$ *if* $p \equiv 2, 3 \pmod 5$.

6. Is There a General Pattern?

The anonymous referee gave us the idea of considering general Lucas sequences to see if similar results to Theorem 1.3 hold. Let a, b be integers and let $(x_n)_n$ and $(y_n)_n$ be sequences defined by the conditions $x_0 = 0$, $x_1 = a$, $y_0 = 2$, $y_1 = a$, and $x_{n+2} = ax_{n+1} + bx_n$, $y_{n+2} = ay_{n+1} + by_n$, for any nonnegative integer n. Is it possible to find the remainder when we divide the numbers

$$U = \sum_{k=2}^{p-1} \frac{(-1)^k x_{k-1}}{k}, \quad V = \sum_{k=2}^{p} \frac{(-1)^k x_k}{k-1}$$

by the square of the prime number p? The results obtained for the Fibonacci numbers suggest the following remarks. To solve the above mentioned problems, we have to be able to compute $\sum_{k=1}^{p-1} \frac{x_k}{k^2}$ modulo a prime p. In the case of the Fibonacci numbers, we were fortunate that Pan and Sun were able to compute $\sum_{k=1}^{p-1} \frac{F_k}{k^2}$ modulo the prime p. Their proof cannot be adapted for the general case (for random integers a, b).

Let us denote by α and β the roots of the equation $x^2 - ax - b = 0$. Suppose, for the moment, that $\alpha \neq \beta$, $\alpha \cdot \beta \neq 0$. Then,

$$x_n = \frac{\alpha^n - \beta^n}{\alpha - \beta}, \quad y_n = \alpha^n + \beta^n.$$

The tools we used for solving our results (Lemma 1.1 of Pan and Sun and the formulas of Granville) suggest that there is an interplay with two Lucas sequences: $z_0 = 0$, $z_1 = c$, $z_{n+2} = cz_{n+1} + dz_n$, $t_0 = 0$, $t_1 = e$, $t_{n+2} = et_{n+1} + ft_n$, for any nonnegative integer n. The rational numbers c, d, e, f could be found easily considering the second degree equations having as roots $1 - \alpha$, $1 - \beta$, and $1 - \frac{1}{\alpha}$, $1 - \frac{1}{\beta}$:

$$c = (1 - \alpha) + (1 - \beta) = 2 - a, \quad d = -(1 - \alpha)(1 - \beta) = a + b - 1,$$

$$e = (1 - \frac{1}{\alpha}) + (1 - \frac{1}{\beta}) = 2 + \frac{a}{b}, \quad f = -(1 - \frac{1}{\alpha})(1 - \frac{1}{\beta}) = -1 + \frac{1-a}{b}.$$

Another factor that needs to be taken into account are two cases, when the discriminant $\Delta = a^2 + 4b$ of the polynomial $f(X) = X^2 - aX - b$ is a quadratic residue (or nonresidue) modulo the prime p.

We looked carefully at our proof and we saw that, indeed, we could prove a more general result (for $a = 1$), but we are not able to settle the general case (with random integers a, b).

The success in the case $a = 1$ follows from $q(\alpha) = q(\beta) = \frac{y_p - 1}{p}$. But, there is a cost (when working with $b \neq 1$). We have to change the numbers U, V a little bit. In the sequel $(x_n)_n$, $(y_n)_n$ are the sequences defined by the conditions: $x_0 = 0$, $x_1 = 1$, $x_{n+2} = x_{n+1} + bx_n$, $y_0 = 2$, $y_1 = 1$, $y_{n+2} = y_{n+1} + by_n$ (b is a fixed integer), for any nonnegative integer n. We will denote by U, V the following numbers:

$$U = \sum_{k=2}^{p-1} \frac{(-1)^k x_{k-1}}{b^{k-1} k}, \quad V = \sum_{k=2}^{p} \frac{(-1)^k x_k}{b^k (k-1)}.$$

It turns out that Theorem 1.4 is true.

Proof. We will not write all the details of the proof of Theorem 1.4 because it is, basically, the same proof as the one given for formulas (3.4), (4.3), and (5.2). The only difference is when we plug in $x = \frac{\alpha^2}{b}$ instead of α^2 in Lemma 1.1 (as we did in the proof of Theorem 1.3). Doing this, we obtain the congruences:

$$U + bV \equiv \frac{x_{2p} - x_p}{pb^p} + \frac{p}{b}\sum_{k=1}^{p-1} \frac{x_k}{k^2} \pmod{p^2}. \tag{6.1}$$

If the discriminant $\Delta = 1 + 4b$ is a quadratic residue modulo p, then

$$U \equiv \frac{b^{p-1} + x_{p-1} - x_{2p-1}}{b^{p-1}p} - 2p\sum_{k=1}^{p-1} \frac{x_k}{k^2} \pmod{p^2}. \tag{6.2}$$

If the discriminant Δ is a quadratic nonresidue modulo p, then

$$V \equiv \frac{-b^p - x_{p+1} + x_{2p+1}}{b^{p+1}p} - \frac{2p}{b}\sum_{k=1}^{p-1} \frac{x_k}{k^2} \pmod{p^2}. \tag{6.3}$$

We will show that formula (6.2) is the same as the formula given in the statement of Theorem 1.4. We have to prove that

$$\frac{b^{p-1} + x_{p-1} - x_{2p-1}}{b^{p-1}p} \equiv \frac{x_{p-1}(1 - y_p)}{p} \pmod{p^2}. \tag{6.4}$$

Because $-b = \alpha \cdot \beta$, $x_n = \frac{\alpha^n - \beta^n}{\alpha - \beta}$, and $y_n = \alpha^n + \beta^n$, we have

$$b^{p-1} + x_{p-1} - x_{2p-1} = x_{p-1}(1 - y_p).$$

Because $x_{p-1} \equiv 1 - y_p \equiv 0 \pmod{p}$ (when Δ is a quadratic residue modulo p) and $b^{p-1} \equiv 1 \pmod{p}$, formula (6.4) follows at once. The formula in Theorem 1.4 for V in the case when Δ is a quadratic nonresidue modulo p follows from formula (6.3) in the same way. □

Remark: We avoided, in the writing of the statement of Theorem 1.4, the formula for U modulo p^2 when Δ is a quadratic nonresidue modulo p (and the formula for V modulo p^2 when Δ is a quadratic residue modulo p). These formulas follow from the above theorem (using $U + bV$ modulo p^2 and that we know the formula for U modulo p^2 when Δ is a quadratic residue modulo p and the formula for V modulo p^2 when Δ is a quadratic nonresidue modulo p).

We want to discuss the following interesting issue. If we are looking at Theorem 1.4, we see (because $x_{p-1} \equiv 1 - y_p \equiv 0 \pmod{p}$) that $U \equiv 0 \pmod{p}$ for any prime p which is not a divisor of $b\Delta$ and such that Δ is a quadratic residue modulo p. The Fibonacci sequence has the property that $U \equiv 0 \pmod{p^2}$ for any prime $p \equiv 1, 4 \pmod{5}$. Is this true in the general case?

Problem: For which values of b can we assert that

$$U \equiv 0 \pmod{p^2}$$

for any prime $p > 3$ that is not a divisor of $b\Delta$ and such that Δ is a quadratic residue modulo p? We can raise the similar problem for the number V.

We will finish this paper by analyzing two different cases of the above problem.

First example, $b = -1$. In this case, $\Delta = -3$ and we have to consider a prime number p such that -3 is a quadratic residue modulo p. Using quadratic reciprocity, this means that $p = 6k + 1$, where k is a positive integer. It is fairly easy to check that

$$x_{6t+1} = x_{6t+2} = 1, \quad x_{6t} = x_{6t+3} = 0, \quad x_{6t+4} = x_{6t+5} = -1 \tag{6.5}$$

for any nonnegative integer t. We know from Theorem 1.4 that

$$U \equiv \frac{x_{p-1}(1-y_p)}{p} - 2p\sum_{k=1}^{p-1}\frac{x_k}{k^2} \pmod{p^2}. \tag{6.6}$$

Because $p = 6k+1$, we get $x_{p-1} = 0$. Therefore, checking $U \equiv 0 \pmod{p^2}$ is the same as proving that

$$\sum_{k=1}^{p-1}\frac{x_k}{k^2} \equiv 0 \pmod{p}. \tag{6.7}$$

It is easy to check that for $p = 7$, this is not true:

$$\sum_{k=1}^{6}\frac{x_k}{k^2} \equiv 4 \pmod{7}.$$

Remark: The problem of deciding if there exists a prime $p = 6k+1$ such that

$$\sum_{k=1}^{p-1}\frac{x_k}{k^2} \equiv 0 \pmod{p} \tag{6.8}$$

seems to be an interesting one.

Second example, $b = 2$. In this case, $\alpha = 2$, $\beta = -1$, $x_n = \frac{2^n-(-1)^n}{3}$, $y_n = 2^n + (-1)^n$, and $\Delta = 9$. Therefore, we can consider any prime $p > 3$. We will see that, in this case,

$$U \equiv 0 \pmod{p^2}.$$

To prove this, according to Theorem 1.4, we have to prove that

$$\frac{2p}{3}\sum_{k=1}^{p-1}\frac{2^k-(-1)^k}{k^2} \equiv \frac{(2^{p-1}-1)(2-2^p)}{3} \pmod{p^2}. \tag{6.9}$$

Because $\sum_{k=1}^{p-1}\frac{(-1)^k}{k^2} \equiv 0 \pmod{p}$, formula (6.9) is equivalent to

$$p\sum_{k=1}^{p-1}\frac{2^k}{k^2} \equiv -\frac{(2^{p-1}-1)^2}{p} \pmod{p^2} \tag{6.10}$$

and to

$$\sum_{k=1}^{p-1}\frac{2^k}{k^2} \equiv -\frac{(2^{p-1}-1)^2}{p^2} \pmod{p}. \tag{6.11}$$

This is Skula's conjecture proved by Andrew Granville in [2].

Remark: From the above discussion, it follows that in some cases (as $b = 1$ and $b = 2$), $U \equiv 0 \pmod{p^2}$ and in others (as $b = -1$), we can have $U \not\equiv 0 \pmod{p^2}$.

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MSC2020: 11B39

Department of Mathematics, University of Bucharest, Bucharest, RO-010014, Romania
Email address: alexgica@yahoo.com