INFINITE SUMS INVOLVING JACOBSTHAL POLYNOMIALS

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Abstract. We explore the Jacobsthal versions of four finite sums involving Fibonacci polynomials, and then extract their infinite counterparts and some special cases.

1. Introduction

Extended gibonacci polynomials $z_n(x)$ are defined by the recurrence $z_{n+2}(x) = a(x)z_{n+1}(x) + b(x)z_n(x)$, where x is an arbitrary integer variable; $a(x)$, $b(x)$, $z_0(x)$, and $z_1(x)$ are arbitrary integer polynomials; and $n \geq 0$.

Suppose $a(x) = x$ and $b(x) = 1$. When $z_0(x) = 0$ and $z_1(x) = 1$, $z_n(x) = f_n(x)$, the nth *Fibonacci polynomial*; and when $z_0(x) = 2$ and $z_1(x) = x$, $z_n(x) = l_n(x)$, the nth *Lucas polynomial*. Clearly, $f_n(1) = F_n$, the nth Fibonacci number; and $l_n(1) = L_n$, the nth Lucas number [2, 4].

Suppose $a(x) = 1$ and $b(x) = x$. When $z_0(x) = 0$ and $z_1(x) = 1$, $z_n(x) = J_n(x)$, the nth *Jacobsthal polynomial*; and when $z_0(x) = 2$ and $z_1(x) = 1$, $z_n(x) = j_n(x)$, the nth *Jacobsthal-Lucas polynomial*. Correspondingly, $J_n = J_n(2)$ and $j_n = j_n(2)$ are the nth Jacobsthal and Jacobsthal-Lucas numbers, respectively. Clearly, $J_n(1) = F_n$ and $j_n(1) = L_n$ [4].

Gibonacci and Jacobsthal polynomials are linked by the relationships $J_n(x) = x^{(n-1)/2} f_n(1/\sqrt{x})$ and $j_n(x) = x^{n/2} l_n(1/\sqrt{x})$ [3], and [4, p. 566].

In the interest of brevity, clarity, and convenience, we omit the argument in the functional notation, when there is *no* ambiguity; so z_n will mean $z_n(x)$. In addition, we let $g_n = f_n$ or l_n, $c_n = J_n(x)$ or $j_n(x)$, $\Delta = \sqrt{x^2+4}$, and $D = \sqrt{4x+1}$.

1.1. Jacobsthal Limits.

We have $\deg J_n = \lfloor (n-1)/2 \rfloor$ and $\deg j_n = \lfloor n/2 \rfloor$, where $\deg f$ denotes the degree of the polynomial $f(x)$, $\lfloor x \rfloor$ denotes the *floor* of the real number x, and $n \geq 1$. The leading coefficient of J_n is $n/2$ when n is even, and 1 otherwise; and that of j_n is 2 when n is even, and n otherwise. Both J_n and j_n end in 1.

Let x be a positive integer. Because $\deg J_{2m} = m-1$, $\deg J_{2m}^2 = 2m-2$ and $J_{2m}^2 = mx^{2m-2} + \cdots + 1$. So, $\dfrac{x^{2m-2}}{J_{2m}^2} < 1$ and hence, $\lim_{m\to\infty} \dfrac{x^{2m-2}}{J_{2m}^2} = 0 = \lim_{m\to\infty} \dfrac{x^{2m}}{J_{2m}^2}$. Similarly, $\lim_{m\to\infty} \dfrac{x^{2m+1}}{J_{2m+1}^2} = 0$. Likewise, $\lim_{m\to\infty} \dfrac{x^m}{j_m^2} = 0$, where m is odd or even.

1.2. Fibonacci Polynomial Sums.

In [5], we studied the following finite sums involving Fibonacci polynomials:

$$\sum_{n=1}^{m} \frac{x f_{2(2n+1)}}{\left(f_{2n+1}^2 - 1\right)^2} = \frac{1}{x^2} - \frac{1}{f_{2m+2}^2}; \tag{1.1}$$

$$\sum_{n=1}^{m} \frac{(x^3+2x) f_{2(2n+2)}}{\left(f_{2n+2}^2 - x^2\right)^2} = \frac{1}{x^2} + \frac{1}{(x^3+2x)^2} - \frac{1}{f_{2m+2}^2} - \frac{1}{f_{2m+4}^2}; \tag{1.2}$$

$$\sum_{n=1}^{m} \frac{(x^3+2x)f_{2(2n+1)}}{\left(f_{2n+1}^2+x^2\right)^2} = 1+\frac{1}{(x^2+1)^2}-\frac{1}{f_{2m+1}^2}-\frac{1}{f_{2m+3}^2}; \tag{1.3}$$

$$\sum_{n=1}^{m} \frac{xf_{2(2n+2)}}{\left(f_{2n+2}^2+1\right)^2} = \frac{1}{(x^2+1)^2}-\frac{1}{f_{2m+3}^2}. \tag{1.4}$$

We will now find their Jacobsthal counterparts and then extract their infinite versions.

2. Jacobsthal Sums

We begin our discourse with sum (1.1) involving odd-numbered Fibonacci polynomials.

2.1. **Jacobsthal Version of Sum** (1.1)**.**

Proof. Let $A = \dfrac{xf_{2(2n+1)}}{\left(f_{2n+1}^2-1\right)^2}$. Replacing x with $1/\sqrt{x}$, and then multiplying the numerator and denominator with $x^{(4n+1)/2}$, we get

$$\begin{aligned} A &= \frac{x^{(4n-1)/2}\left[x^{(4n+1)/2}f_{2(2n+1)}\right]}{\sqrt{x}\left[\left(x^{2n/2}f_{2n+1}\right)^2-x^{2n}\right]^2} \\ &= \frac{x^{2n-1}J_{2(2n+1)}}{\left(J_{2n+1}^2-x^{2n}\right)^2}; \\ \text{LHS} &= \sum_{n=1}^{m} \frac{x^{2n-1}J_{2(2n+1)}}{\left(J_{2n+1}^2-x^{2n}\right)^2}, \end{aligned}$$

where $f_n = f_n(1/\sqrt{x})$ and $J_n = J_n(x)$.

Now, let $B = \dfrac{1}{x^2}-\dfrac{1}{f_{2m+2}^2}$. Replace x with $1/\sqrt{x}$, and then multiply the numerator and denominator with x^{2m+1}. This yields

$$\begin{aligned} B &= x-\frac{1}{f_{2m+2}^2} \\ &= x-\frac{x^{2m+1}}{\left[x^{(2m+1)/2}f_{2m+2}\right]^2}; \\ \text{RHS} &= x-\frac{x^{2m+1}}{J_{2m+2}^2}, \end{aligned}$$

where $f_n = f_n(1/\sqrt{x})$ and $J_n = J_n(x)$.

Equating the two sides, we get

$$\sum_{n=1}^{m} \frac{x^{2n}J_{2(2n+1)}}{\left(J_{2n+1}^2-x^{2n}\right)^2} = x^2-\frac{x^{2m+2}}{J_{2m+2}^2}. \tag{2.1}$$

□

The next sum involves even-numbered Fibonacci polynomials.

2.2. **Jacobsthal Version of Sum** (1.2).

Proof. Let $A = \dfrac{(x^3+2x)f_{2(2n+2)}}{\left(f_{2n+2}^2 - x^2\right)^2}$. Replacing x with $1/\sqrt{x}$, and then multiplying the numerator and denominator with x^{4n} yields

$$\begin{aligned} A &= \frac{\sqrt{x}(2x+1)f_{2(2n+2)}}{\left(xf_{2n+2}^2 - 1\right)^2}; \\ &= \frac{(2x+1)x^{2n}\left[x^{(4n+3)/2}f_{2(2n+2)}\right]}{x\left[\left(x^{(2n+1)/2}f_{2n+2}\right)^2 - x^{2n}\right]^2}; \\ \text{LHS} &= \sum_{n=1}^{m} \frac{(2x+1)x^{2n}J_{2(2n+2)}}{x\left(J_{2n+2}^2 - x^{2n}\right)^2}, \end{aligned}$$

where $f_n = f_n(1/\sqrt{x})$ and $J_n = J_n(x)$.

Now, let $B = \dfrac{1}{x^2} + \dfrac{1}{(x^3+2x)^2} - \dfrac{1}{f_{2m+2}^2} - \dfrac{1}{f_{2m+4}^2}$. Replace x with $1/\sqrt{x}$, and then multiply the numerator and denominator with x^{2m+3}. Then,

$$\begin{aligned} B &= x + \frac{x^3}{(2x+1)^2} - \frac{1}{f_{2m+2}^2} - \frac{1}{f_{2m+4}^2} \\ &= x + \frac{x^3}{(2x+1)^2} - \frac{x^{2m+3}}{x^2\left[x^{(2m+1)/2}f_{2m+2}\right]^2} - \frac{x^{2m+3}}{\left[x^{(2m+3)/2}f_{2m+4}\right]^2}; \\ \text{RHS} &= x + \frac{x^3}{(2x+1)^2} - \frac{x^{2m+1}}{J_{2m+2}^2} - \frac{x^{2m+3}}{J_{2m+4}^2}, \end{aligned}$$

where $f_n = f_n(1/\sqrt{x})$ and $J_n = J_n(x)$.

Equating the two sides yields

$$\sum_{n=1}^{m} \frac{(2x+1)x^{2n}J_{2(2n+2)}}{\left(J_{2n+2}^2 - x^{2n}\right)^2} = x^2 + \frac{x^4}{(2x+1)^2} - \frac{x^{2m+2}}{J_{2m+2}^2} - \frac{x^{2m+4}}{J_{2m+4}^2}. \tag{2.2}$$

□

The next sum contains odd- and even-numbered Fibonacci polynomials.

2.3. **Jacobsthal Version of Sum** (1.3).

Proof. Let $A = \dfrac{(x^3+2x)f_{2(2n+1)}}{\left(f_{2n+1}^2 + x^2\right)^2}$. Replace x with $1/\sqrt{x}$, and then multiply the numerator and denominator with $x^{2(2n-1)}$. This yields

$$\begin{aligned} A &= \frac{x(2x+1)f_{2(2n+1)}}{\sqrt{x}\left(xf_{2n+1}^2+1\right)^2} \\ &= \frac{(2x+1)x^{2n-2}\left[x^{(4n+1)/2}f_{2(2n+1)}\right]}{\left[\left(x^{2n/2}f_{2n+1}\right)^2+x^{2n-1}\right]^2} \\ &= \frac{(2x+1)x^{2n-2}J_{2(2n+1)}}{\left(J_{2n+1}^2+x^{2n-1}\right)^2}; \\ \text{LHS} &= \sum_{n=1}^{m} \frac{(2x+1)x^{2n-2}J_{2(2n+1)}}{\left(J_{2n+1}^2+x^{2n-1}\right)^2}, \end{aligned}$$

where $f_n = f_n(1/\sqrt{x})$ and $J_n = J_n(x)$.

Now, let $B = 1 + \frac{1}{(x^2+1)^2} - \frac{1}{f_{2m+1}^2} - \frac{1}{f_{2m+3}^2}$. Replacing x with $1/\sqrt{x}$,and then multiplying the numerator and denominator with x^{2m+2}, we get

$$\begin{aligned} B &= 1 + \frac{x^2}{(x+1)^2} - \frac{1}{f_{2m+1}^2} - \frac{1}{f_{2m+3}^2} \\ &= 1 + \frac{x^2}{(x+1)^2} - \frac{x^{2m}}{\left(x^{2m/2}f_{2m+1}\right)^2} - \frac{x^{2m+2}}{\left[x^{(2m+2)/2}f_{2m+3}\right]^2}; \\ \text{RHS} &= 1 + \frac{x^2}{(x+1)^2} - \frac{x^{2m}}{J_{2m+1}^2} - \frac{x^{2m+2}}{J_{2m+3}^2}, \end{aligned}$$

where $f_n = f_n(1/\sqrt{x})$ and $J_n = J_n(x)$.

Combining the two sides, we get

$$\sum_{n=1}^{\infty} \frac{(2x+1)x^{2n-1}J_{2(2n+1)}}{\left(J_{2n+1}^2+x^{2n-1}\right)^2} = x + \frac{x^3}{(x+1)^2} - \frac{x^{2m+1}}{J_{2m+1}^2} - \frac{x^{2m+3}}{J_{2m+3}^2}. \tag{2.3}$$

□

Finally, we explore the Jacobsthal counterpart of sum (1.4); it also involves only even-numbered Fibonacci polynomials.

2.4. **Jacobsthal Version of Sum** (1.4)**.**

Proof. Let $A = \frac{xf_{2(2n+2)}}{\left(f_{2n+2}^2+1\right)^2}$. Replacing x with $1/\sqrt{x}$, and then multiply the numerator and denominator with $x^{2(2n+1)}$, we then get

$$\begin{aligned} A &= \frac{f_{2(2n+2)}}{\sqrt{x}\left(f_{2n+2}^2+1\right)^2} \\ &= \frac{x^{2n}\left[x^{(4n+3)/2}f_{2(2n+2)}\right]}{\left\{\left[x^{(2n+1)/2}f_{2n+2}\right]^2+x^{2n+1}\right\}^2}; \\ \text{LHS} &= \sum_{n=1}^{m} \frac{x^{2n}J_{2(2n+2)}}{\left(J_{2n+2}^2+x^{2n+1}\right)^2}, \end{aligned}$$

where $f_n = f_n(1/\sqrt{x})$ and $J_n = J_n(x)$.

We now let $B = \dfrac{1}{(x^2+1)^2} - \dfrac{1}{f_{2m+3}^2}$. Replacing x with $1/\sqrt{x}$, and then multiplying the numerator and denominator with x^{2m+2}, we then get

$$\begin{aligned} B &= \frac{x^2}{(x+1)^2} - \frac{1}{f_{2m+3}^2} \\ &= \frac{x^2}{(x+1)^2} - \frac{x^{2m+2}}{\left[x^{(2m+2)/2} f_{2m+3}\right]^2}; \\ \text{RHS} &= \frac{x^2}{(x+1)^2} - \frac{x^{2m+2}}{J_{2m+3}^2}, \end{aligned}$$

where $f_n = f_n(1/\sqrt{x})$ and $J_n = J_n(x)$.

Equating the two sides, we get

$$\sum_{n=1}^{m} \frac{x^{2n+1} J_{2(2n+2)}}{\left(J_{2n+2}^2 + x^{2n+1}\right)^2} = \frac{x^3}{(x+1)^2} - \frac{x^{2m+3}}{J_{2m+3}^2}. \tag{2.4}$$

□

3. Infinite Jacobsthal Sums

Equations (2.1) through (2.4) yield the following infinite sums:

$$\sum_{n=1}^{\infty} \frac{x^{2n} J_{2(2n+1)}}{\left(J_{2n+1}^2 - x^{2n}\right)^2} = x^2; \tag{3.1}$$

$$\sum_{n=1}^{\infty} \frac{(2x+1)x^{2n} J_{2(2n+2)}}{\left(J_{2n+2}^2 - x^{2n}\right)^2} = x^2 + \frac{x^4}{(2x+1)^2}; \tag{3.2}$$

$$\sum_{n=1}^{m} \frac{(2x+1)x^{2n-1} J_{2(2n+1)}}{\left(J_{2n+1}^2 + x^{2n-1}\right)^2} = x + \frac{x^3}{(x+1)^2}; \tag{3.3}$$

$$\sum_{n=1}^{\infty} \frac{x^{2n+1} J_{2(2n+2)}}{\left(J_{2n+2}^2 + x^{2n+1}\right)^2} = \frac{x^3}{(x+1)^2}, \tag{3.4}$$

respectively.

It then follows that [5]

$$\sum_{n=1}^{\infty} \frac{F_{2(2n+1)}}{\left(F_{2n+1}^2 - 1\right)^2} = 1; \qquad \sum_{n=1}^{\infty} \frac{F_{2(2n+2)}}{\left(F_{2n+2}^2 - 1\right)^2} = \frac{10}{27};$$

$$\sum_{n=1}^{\infty} \frac{F_{2(2n+1)}}{\left(F_{2n+1}^2 + 1\right)^2} = \frac{5}{12}; \qquad \sum_{n=1}^{\infty} \frac{F_{2(2n+2)}}{\left(F_{2n+2}^2 + 1\right)^2} = \frac{1}{4},$$

respectively. Consequently, we have

$$\sum_{n=1}^{\infty} \frac{F_{2n}}{(F_n^2 - 1)^2} = \frac{37}{27}; \qquad \sum_{n=1}^{\infty} \frac{F_{2n}}{(F_n^2 + 1)^2} = \frac{5}{3},$$

as in [1, 6].

It also follows that

$$\sum_{n=1}^{\infty} \frac{4^n J_{2(2n+1)}}{\left(J_{2n+2}^2 - 4^n\right)^2} = 4; \qquad \sum_{n=1}^{\infty} \frac{4^n J_{2(2n+2)}}{\left(J_{2n+2}^2 - 4^n\right)^2} = \frac{116}{125};$$
$$\sum_{n=1}^{\infty} \frac{4^n J_{2(2n+1)}}{\left(J_{2n+1}^2 + 2^{2n-1}\right)^2} = \frac{52}{45}; \qquad \sum_{n=1}^{\infty} \frac{4^n J_{2(2n+2)}}{\left(J_{2n+2}^2 + 2^{2n+1}\right)^2} = \frac{4}{9},$$

respectively.

4. Alternate Forms

Using the identity $j_n^2 - D^2 J_n^2 = 4(-x)^n$ [4], we can rewrite the summations (3.1) through (3.4) as follows, where $D = \sqrt{4x+1}$:

$$\sum_{n=1}^{\infty} \frac{D^4 x^{2n} J_{2(2n+1)}}{\left(j_{2n+1}^2 - x^{2n}\right)^2} = x^2;$$
$$\sum_{n=1}^{\infty} \frac{D^4 (2x+1) x^{2n} J_{2(2n+2)}}{\left[j_{2n+2}^2 - (2x+1)^2 x^{2n}\right]^2} = x^2 + \frac{x^4}{(2x+1)^2};$$
$$\sum_{n=1}^{\infty} \frac{D^4 (2x+1) x^{2n-1} J_{2(2n+1)}}{\left[j_{2n+1}^2 + (2x+1)^2 x^{2n-1}\right]^2} = x + \frac{x^3}{(x+1)^2};$$
$$\sum_{n=1}^{\infty} \frac{D^4 x^{2n+1} J_{2(2n+2)}}{\left(j_{2n+2}^2 + x^{2n+1}\right)^2} = \frac{x^3}{(x+1)^2},$$

respectively.

In particular, we then have [5]

$$\sum_{n=1}^{\infty} \frac{F_{2(2n+1)}}{\left(L_{2n+1}^2 - 1\right)^2} = \frac{1}{25}; \qquad \sum_{n=1}^{\infty} \frac{F_{2(2n+2)}}{\left(L_{2n+2}^2 - 9\right)^2} = \frac{2}{135};$$
$$\sum_{n=1}^{\infty} \frac{F_{2(2n+1)}}{\left(L_{2n+1}^2 + 9\right)^2} = \frac{1}{60}; \qquad \sum_{n=1}^{\infty} \frac{F_{2(2n+2)}}{\left(L_{2n+2}^2 + 1\right)^2} = \frac{1}{100},$$

respectively.

They also yield the following results:

$$\sum_{n=1}^{\infty} \frac{4^n J_{2(2n+1)}}{\left(j_{2n+2}^2 - 4^n\right)^2} = \frac{4}{81}; \qquad \sum_{n=1}^{\infty} \frac{4^n J_{2(2n+2)}}{\left(j_{2n+2}^2 - 25 \cdot 4^n\right)^2} = \frac{116}{10,125};$$
$$\sum_{n=1}^{\infty} \frac{4^n J_{2(2n+1)}}{\left(j_{2n+1}^2 + 25 \cdot 2^{2n-1}\right)^2} = \frac{52}{3,645}; \qquad \sum_{n=1}^{\infty} \frac{4^n J_{2(2n+2)}}{\left(j_{2n+2}^2 + 2^{2n+1}\right)^2} = \frac{4}{729},$$

respectively.

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