BIJECTIVE PROOFS OF FORMULAS WITH $(-1)^n$

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Abstract. We present simple bijective proofs of formulas involving the expression $(-1)^n$ connected to three different combinatorial problems. Our arguments somewhat resemble the combinatorial proofs of Benjamin-Ornstein and Elizalde of the familiar derangement recurrence.

1. Introduction

Let us recall the well-known recurrence $D_n = nD_{n-1} + (-1)^n$ satisfied for $n > 0$ by the derangement numbers D_n, describing the number of fixed-point-free permutations of an n-element set. It is the best-known example of the phenomenon that solutions to various combinatorial problems sometimes lead to formulas that contain the expression $(-1)^n$. This is precisely what causes a challenge when one tries to present a bijective proof of such a formula. Combinatorial proofs of the derangement recurrence were given by Remmel [6], Wilf [7], Désarménien [2], Benjamin-Ornstein [1], and recently by Elizalde [3]. The bijective proofs of this formula often reduce to creating an "almost-1-to-1" correspondence between some sets A_n and B_n, where the word "almost" indicates that there will be an unmapped element of A_n or an unhit element of B_n, depending on the parity of n (cf. [1]).

The purpose of this note is to present a sample of bijective proofs of some well-known formulas containing the expression $(-1)^n$. The formulas seem to belong to the folklore of Discrete Math exercises. The novelty of our approach lies in presenting in each case a bijective argument based on the construction of an "almost-1-to-1" correspondence between suitably chosen sets. This unified approach is intended to further confirm the usefulness of such "almost bijective" proofs in enumerative combinatorics.

In Section 2, we count the number z_n of those subsets in a $3n$-element set whose number of elements is a multiple of 3 (cf. [4, Problem 1.1.2]). We present a combinatorial proof of the formula $z_n = \frac{8^n+2\cdot(-1)^n}{3}$, $n \geq 1$. Its alternative proof first establishes the recurrence $z_{n+1} = 3 \cdot 8^n - z_n$, with the help of a combinatorial argument (see Remark 2.2).

In Section 3, we deal with the number v_n of vertex-colorings of the cycle graph C_n, $n \geq 3$, with $k \geq 2$ colors. We give a bijective proof of the well-known formula $v_n = (k-1)^n + (k-1) \cdot (-1)^n$. Its standard inductive proof uses the deletion-contraction recurrence for the chromatic polynomial (see [5], where three other proofs are also given, including another bijective one, different from ours).

In Section 4, we look at the number w_n of all the words of length $n \geq 0$ over the alphabet $\{a, b, c, d, e\}$ such that each of the letters c, d, e is always preceded by the letter a. We give a bijective proof of the recurrence $w_n = 3w_{n-1} + (-1)^n$ satisfied for $n > 0$ (with $w_0 = 1$). It is an immediate consequence of the recurrence $w_n = 2 \cdot w_{n-1} + 3 \cdot w_{n-2}$, which can be readily justified by a straightforward combinatorial argument.

2. Subsets of a $3n$-element Set

For any nonempty set X, let

$$\begin{aligned} Z(X) &= \{A \subseteq X : |A| \equiv 0 \pmod 3\}, \\ Z^+(X) &= Z(X) \setminus \{\emptyset\}, \\ O(X) &= \{A \subseteq X : |A| \equiv 1 \pmod 3\}, \\ T(X) &= \{A \subseteq X : |A| \equiv 2 \pmod 3\}. \end{aligned}$$

For $n \geq 1$, let $X_n = \{1, 2, 3, \ldots, 3n-2, 3n-1, 3n\}$.

Our goal in this section is to give a bijective proof of the formula given in the following proposition.

Proposition 2.1.

$$|Z(X_n)| = \frac{8^n + 2 \cdot (-1)^n}{3} \quad \text{for} \quad n \geq 1.$$

Proof. For any $n \geq 1$, we have

(1) $|O(X_n)| = |T(X_n)|$,

as witnessed by the bijection $A \mapsto X_n \setminus A$.

A key step of our reasoning is the following observation.

Claim. For any $n \geq 1$, we have

(2) $|Z(X_n)| = |O(X_n)| + (-1)^n$.

To see this, let us first note that equality (2) is obvious for $n = 1$ and a straightforward computation shows that

(3) $|Z^+(X)| = |O(X)|$ for any X with $|X| = 6$,

which, in particular, gives (2) for $n = 2$.

So assume now that $n > 2$, let $m = \lfloor \frac{n}{2} \rfloor$, and for each $i = 1, 2, \ldots, m$, let

$$X_{n,i} = \{6(i-1)+1, 6(i-1)+2, 6(i-1)+3, 6(i-1)+4, 6(i-1)+5, 6i\}.$$

Because $|X_{n,i}| = 6$ for each $i = 1, 2, \ldots, m$, we fix three bijections (cf. (1) and (3)):

$$f_i : Z^+(X_{n,i}) \to O(X_{n,i}),\ g_i : O(X_{n,i}) \to T(X_{n,i}),\ h_i : T(X_{n,i}) \to Z^+(X_{n,i}).$$

We describe a bijection $\varphi_n : Z^*(X_n) \to O^*(X_n)$, where

$$Z^*_n = Z^+(X_n) \text{ and } O^*(X_n) = O(X_n) \text{ when } n \text{ is even,}$$

but

$$Z^*_n = Z(X_n) \text{ and } O^*(X_n) = O(X_n) \setminus \{\{3n\}\} \text{ when } n \text{ is odd.}$$

The existence of such a bijection justifies (2).

First, for an arbitrary $A \in Z^+(X_{2m})$, let $i \in \{1, \ldots, m\}$ be the smallest index with $A \cap X_{n,i} \neq \emptyset$ and then let $Y = X_{n,i}$ and $Z = X_{n,i+1} \cup \ldots \cup X_{n,m} = \{6i+1, \ldots, 3n\}$. Moreover, let $Q = \emptyset$ if n is even and $Q = \{6m+1, 6m+2, 6m+3\}$ if n is odd (in which case $3n = 6m+3$). Let $A_1 = A \cap Y$, $A_2 = A \cap Z$, and $A_3 = A \cap Q$. Clearly, we have $A = A_1 \cup A_2 \cup A_3$.

Finally, for an arbitrary $A \in Z^*(X_n)$, we define

$$\varphi_n(A) = \begin{cases} f_i(A_1) \cup A_2 \cup A_3, & \text{if } A \in Z^+(X_{2m}) \text{ and } A_1 \in Z^+(Y), \\ g_i(A_1) \cup A_2 \cup A_3, & \text{if } A \in Z^+(X_{2m}) \text{ and } A_1 \in O(Y), \\ h_i(A_1) \cup A_2 \cup A_3, & \text{if } A \in Z^+(X_{2m}) \text{ and } A_1 \in T(Y), \\ \{6m+1\}, & \text{if } n \text{ is odd and } A = \{6m+1, 6m+2, 6m+3\}, \\ \{6m+2\}, & \text{if } n \text{ is odd and } A = \emptyset. \end{cases}$$

One readily checks that φ_n bijectively maps $Z^*(X_n)$ onto $O^*(X_n)$, which completes the proof of the claim.

Now, by (1) and (2), we have

$$8^n = |Z(X_n)| + |O(X_n)| + |T(X_n)| = 3 \cdot |Z(X_n)| - 2 \cdot (-1)^n$$

and consequently, $|Z(X_n)| = \frac{8^n + 2\cdot(-1)^n}{3}$, which completes the proof of the proposition. □

Remark 2.2. *The formula* $|Z(X_n)| = \frac{8^n+2\cdot(-1)^n}{3}$ *is a straightforward consequence of the recurrence*

(4) $|Z(X_{n+1})| = 3 \cdot 8^n - |Z(X_n)|$,

which may be justified by the following combinatorial argument.

Consider the fibers of the mapping $\varphi : A \mapsto A \cap X_n$ *defined for* $A \in Z(X_{n+1})$. *Observe that if* $B \subseteq X_n$, *then* $|\varphi^{-1}(B)|$ *equals 2 if* $B \in Z(X_n)$ *or 3 if* $B \notin Z(X_n)$. *Consequently,*

$$|Z(X_{n+1})| = 2 \cdot |Z(X_n)| + 3 \cdot (2^{3n} - |Z(X_n)|),$$

completing the proof of (4).

3. Vertex Colorings of C_n

Let us assume that the set of vertices of the cyclic graph C_n ($n \geq 3$) is $\{1, 2, \ldots, n\}$. The vertex coloring of C_n with k colors ($k \geq 2$) is any sequence $(a_1, a_2, \ldots, a_n)$ of length n with values in $\{1, \ldots, k\}$ such that $a_i \neq a_{i+1}$ for any $i < n$ and $a_n \neq a_1$.

Let us fix $k \geq 2$ and let v_n be the number of all vertex colorings of C_n with k colors. The goal in this section is to provide a combinatorial proof of the formula given in the following proposition.

Proposition 3.1.

$$v_n = (k-1)^n + (k-1) \cdot (-1)^n \quad \text{for} \quad n \geq 3.$$

Proof. Let X_n be the set of all sequences $(a_1, a_2, \ldots, a_n)$ of length n with values in $\{1, \ldots, k\}$ such that $a_1 = 1$ and $a_i \neq a_{i+1}$ for any $i < n$; clearly, $|X_n| = (k-1)^{n-1}$. For each $m \in \{1, \ldots, k\}$, let

$$X_n^{(m)} = \big\{(a_1, a_2, \ldots, a_n) \in X_n : \ a_n = m\big\}.$$

Let us notice that the set $X_n^{(2)} \cup \ldots \cup X_n^{(k)}$ consists of all the vertex colorings $(a_1, a_2, \ldots, a_n)$ of C_n with $a_1 = 1$. It follows that

(1) $v_n = k \cdot |X_n^{(2)} \cup \ldots \cup X_n^{(k)}|$.

Moreover,

(2) $|X_n^{(2)}| = |X_n^{(l)}|$ for any $l \in \{2, \ldots, k\}$.

Indeed, given l, we can fix a permutation π of $\{1,\dots,k\}$, which cyclically permutes the colors $\{2,\dots,k\}$ so that $\pi(2)=l$. A bijection between $X_n^{(2)}$ and $X_n^{(l)}$ is now provided by composing each coloring from $X_n^{(2)}$ with π.

Consequently, (1) and (2) imply

(3) $v_n = k\cdot(k-1)\cdot|X_n^{(2)}|$.

On the other hand, because $X_n^{(1)} = X_n \setminus (X_n^{(2)} \cup \cdots \cup X_n^{(k)})$, we have (cf. (2))

(4) $|X_n^{(1)}| = (k-1)^{n-1} - (k-1)\cdot|X_n^{(2)}|$.

In view of (3) and (4), a key point of our argument is the following observation, which establishes another relation between $|X_n^{(1)}|$ and $|X_n^{(2)}|$.

Claim. For any $n \geq 1$, we have

(5) $|X_n^{(2)}| = |X_n^{(1)}| + (-1)^n$.

To show this, it suffices to define a bijection

$$\varphi_n : X_n^{(2)} \setminus \{(1,2,\dots,1,2)\} \to X_n^{(1)} \setminus \{(1,2,\dots,1,2,1)\},$$

where the sequence $(1,2,\dots,1,2)$ consists of the pair $(1,2)$ repeated $\lfloor \frac{n}{2} \rfloor$ times (so it has length $2\cdot\lfloor \frac{n}{2} \rfloor$) and the sequence $(1,2,\dots,1,2,1)$ consists of the pair $(1,2)$ repeated $\lfloor \frac{n}{2} \rfloor$ times, followed at the end by the number 1 (so it has length $2\cdot\lfloor \frac{n}{2} \rfloor + 1$).

For an arbitrary sequence $(a_1,a_2,\dots,a_n) \in X_n^{(2)} \setminus \{(1,2,\dots,1,2)\}$, let $i \in \{1,\dots,n\}$ be the largest index for which $a_i \notin \{1,2\}$.

Clearly, i is well-defined and $1 < i < n$.

Then, we let φ_n map $(a_1,a_2,\dots,a_n)$ to $(a_1,a_2,\dots,a_i,a_{i+1}^{'},\dots,a_n^{'})$, where for $j > i$, $a_j^{'} = 1$ if $a_j = 2$ and $a_j^{'} = 2$ if $a_j = 1$. One readily checks that this works, which completes the proof of the claim.

Now, by (3), (4), and (5), a straightforward computation leads to the formula $v_n = (k-1)^n + (k-1)\cdot(-1)^n$, completing the proof of the proposition.

□

4. Counting the Number of Words

Let w_n, $n \geq 1$, be the number of all the words of length n that can be formed from letters a,b,c,d,e in such a way that each of the letters c,d,e is always preceded by the letter a. We are going to give a bijective proof of the recurrence given in the following proposition.

Proposition 4.1.

$$w_n = 3w_{n-1} + (-1)^n \quad \text{for} \quad n \geq 2.$$

Proof. Let A_n be the set of words under consideration, and let B_n be the subset of A_{n+1} consisting of words with the endings ac, ad, or ae.

One immediately observes that $|B_n| = 3\cdot|A_{n-1}| = 3w_{n-1}$, so the proof reduces to the following.

Claim. For any $n \geq 2$,

(1) $|A_n| = |B_n| + (-1)^n$.

To prove this, let $x_k = ae\cdots ae$ be the word of length $2k$ consisting of the group of letters ae repeated k times (we assume that x_0 is the empty word).

Let us note that if $m = \lfloor \frac{n+1}{2} \rfloor$, then $x_m \in A_n$ when $n = 2m$ is even and $x_m \in B_n$ when $n = 2m-1$ is odd.

We will describe now a bijection $\varphi_n : B_n^* \to A_n^*$, where $B_n^* = B_n$ and $A_n^* = A_n \setminus \{x_m\}$ when n is even, but $B_n^* = B_n \setminus \{x_m\}$ and $A_n^* = A_n$ when n is odd. The existence of such a bijection justifies (1).

If s and t are words (of lengths $i = lh(s)$ and $j = lh(t)$, respectively), then by $s^\frown t$ we denote their concatenation (of length $i + j$). In particular, we always have $s^\frown x_0 = s$.

The definition of φ_n splits into the following cases.

- if $lh(s) = n - 1$, then
$$\varphi_n(s^\frown ac) = s^\frown a \text{ and } \varphi_n(s^\frown ad) = s^\frown b,$$
- if $1 \leq k < m$ and $lh(s) = n - 2k$, then
$$\varphi_n(s^\frown a^\frown x_k) = s^\frown ac^\frown x_{k-1} \text{ and } \varphi_n(s^\frown b^\frown x_k) = s^\frown ad^\frown x_{k-1}.$$
- if $1 \leq k < m$ and $lh(s) = n - 2k - 1$, then
$$\varphi_n(s^\frown ac^\frown x_k) = s^\frown a^\frown x_k \text{ and } \varphi_n(s^\frown ad^\frown x_k) = s^\frown b^\frown x_k.$$

It can be readily checked that φ_n bijectively maps B_n^* onto A_n^*, which completes the proof of (1) and the proof of the proposition.

□

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