ON THE DISCRIMINANT OF THE k-GENERALIZED FIBONACCI POLYNOMIAL, II

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Abstract. In this paper, we show that the absolute value of the discriminant of the k-generalized Fibonacci polynomial $X^k - X^{k-1} - \cdots - X - 1$ is a member of the k-generalized Fibonacci sequence $(F_n^{(k)})_{n\ge 0}$ only when $k = 2, 3$.

1. Introduction

Let $k \geq 2$ be an integer. The sequence of k-generalized Fibonacci numbers $\{F_n^{(k)}\}_{n\in\mathbb{Z}}$ has initial terms $F_{2-k}^{(k)} = \cdots = F_0^{(k)} = 0$, $F_1^{(k)} = 1$ and satisfies the recurrence

$$F_{n+k}^{(k)} = F_{n+k-1}^{(k)} + \cdots + F_n^{(k)} \qquad \text{for all} \qquad n \in \mathbb{Z}.$$

Here are a few terms of the k-generalized Fibonacci sequence with positive indices.

k	Name	First nonzero terms with positive indices
2	Fibonacci	1, 1, 2, 3, <u>5</u>, 8, 13, 21, 34, 55, 89, 144, 233, 377, 610, ...
3	Tribonacci	1, 1, 2, 4, 7, 13, 24, <u>44</u>, 81, 149, 274, 504, 927, 1705, 3136, ...
4	Tetranacci	1, 1, 2, 4, 8, 15, 29, 56, 108, 208, 401, 773, 1490, 2872, 5536, ...
5	Pentanacci	1, 1, 2, 4, 8, 16, 31, 61, 120, 236, 464, 912, 1793, 3525, 6930, ...
6	Hexanacci	1, 1, 2, 4, 8, 16, 32, 63, 125, 248, 492, 976, 1936, 3840, 7617, ...
7	Heptanacci	1, 1, 2, 4, 8, 16, 32, 64, 127, 253, 504, 1004, 2000, 3984, 7936, ...
8	Octanacci	1, 1, 2, 4, 8, 16, 32, 64, 128, 255, 509, 1016, 2028, 4048, 8080, ...
9	Nonanacci	1, 1, 2, 4, 8, 16, 32, 64, 128, 256, 511, 1021, 2040, 4076, 8144, ...
10	Decanacci	1, 1, 2, 4, 8, 16, 32, 64, 128, 256, 512, 1023, 2045, 4088, 8172, ...

Let

$$f_k(X) := X^k - X^{k-1} - \cdots - X - 1$$

be the characteristic polynomial of the k-generalized Fibonacci sequence. This is sometimes referred to as the k-generalized Fibonacci polynomial. Let $\mathrm{Disc}(f_k)$ be the discriminant of $f_k(X)$. This number has been computed in many places (see, for example Lemma 2.3 in [6]). Its formula is

$$\mathrm{Disc}(f_k(X)) = (-1)^{\binom{k+1}{2}-1}\left(\frac{2^{k+1}k^k - (k+1)^{k+1}}{(k-1)^2}\right).$$

For $k = 2, 3$, we get that $|\mathrm{Disc}(f_k)| = 5,\ 44$ and a quick look at the above table convinces us that $5 = F_5^{(2)}$ and $44 = F_8^{(3)}$. We ask whether there are other instances when $|\mathrm{Disc}(f_k)|$ is a member of $\{F_n^{(k)}\}_{n\geq 0}$? The answer is no and this is the main theorem of this paper.

Theorem 1. *The only $k \geq 2$ such that $|\mathrm{Disc}(f_k)|$ is a member of $\{F_n^{(k)}\}_{n\geq 0}$ are $k = 2, 3$.*

2. Preliminary Results

We label the roots of $f_k(X)$ as $\alpha_1, \ldots, \alpha_k$. It is known that $f_k(X)$ has only one positive real root, we call it $\alpha := \alpha_1$. This root satisfies

$$2\left(1 - 1/2^k\right) < \alpha < 2 \quad \text{for all} \quad k \geq 2. \tag{1}$$

Furthermore, $|\alpha_i| < 1$ for $i = 2, \ldots, k$. It is also known that

$$\alpha^{n-2} \leq F_n^{(k)} \leq \alpha^{n-1} \tag{2}$$

holds for all $n \geq 1$ (see [1]). For sharper estimates of $F_n^{(k)}$ in terms of α, we need some more notation. Putting

$$f_k(z) := \frac{z-1}{2+(k+1)(z-2)} \qquad \text{for} \qquad z \geq 2,$$

then

$$F_n^{(k)} = \sum_{i=1}^{k} f_k(\alpha_i)\alpha_i^{n-1} \qquad \text{holds for all} \qquad n \in \mathbb{Z}. \tag{3}$$

Furthermore,

$$|F_n^{(k)} - f_k(\alpha)\alpha^{n-1}| < \frac{1}{2} \qquad \text{holds for all} \qquad n \geq 1. \tag{4}$$

Both (3) and (4) appear in [3]. An even sharper estimate than (4), but in a more restricted range for n in terms of k, appears in [1]. Namely,

$$\text{If} \quad n < 2^{k/2} \quad \text{and} \quad k > 10, \quad \text{then} \quad |f_k(\alpha)\alpha^{n-1} - 2^{n-2}| < \frac{2^n}{2^{k/2}} \tag{5}$$

(see also (15) in [4]). Finally, we need the following formula of Cooper and Howard [2]:

$$F_n^{(k)} = 2^{n-2} + \sum_{j=1}^{\lfloor \frac{n+k}{k+1} \rfloor - 1} C_{n,j} 2^{n-j(k+1)-2}, \quad \text{where} \quad C_{n,j} = (-1)^j \left(\binom{n-jk}{j} - \binom{n-jk-2}{j-2} \right). \tag{6}$$

In the above formulas, the regular assumptions apply, namely that $\binom{a}{b} = 0$ if either $a < b$ or one of a or b is negative.

3. The Proof

We need to solve

$$F_n^{(k)} = \frac{2^{k+1}k^k - (k+1)^{k+1}}{(k-1)^2} \tag{7}$$

for some $k \geq 4$ and some positive integer n. We start with some rough bounds for n in terms of k. First, by (2), we have

$$\alpha^{n-2} < F_n^k < \frac{2^{k+1}k^k}{(k-1)^2}. \tag{8}$$

Since $\alpha > 1.927$ for $k \geq 4$, the above inequality implies $n < 800$ when $k \leq 100$. We check the range $k \in [4, 100]$ and $n \in [2, 800]$ for equation (7) and we do not find solutions. From now on, we assume that $k \geq 101$. From (8) and (1), we get

$$\begin{aligned} n-2 &< \frac{(k+1)\log 2 + k\log k - 2\log(k-1)}{\log \alpha} < \frac{(k+1)\log 2 + k\log k - 2\log(k-1)}{\log 2 + \log(1-1/2^k)} \\ &< \frac{(k+1)\log 2 + k\log k - 2\log(k-1)}{\log 2}\left(\frac{1}{1-1/(2^{k-1}\log 2)}\right) \\ &< \frac{(k+1)\log 2 + k\log k - 2\log(k-1)}{\log 2}\left(1+\frac{1}{2^{k-2}\log 2}\right) \\ &< k+1+\frac{k\log k - 2\log(k-1)}{\log 2} + \frac{(k+1)\log 2 + k\log k}{2^{k-2}(\log 2)^2} \\ &< k+1.01+\frac{k\log k - 2\log(k-1)}{\log 2}. \end{aligned}$$

In the above, we used $\log(1-x) > -2x$, valid for $x \in (0,1/2)$ (with $x := 1/2^k$), as well as the inequality $1/(1-y) < 1+2y$, valid for $y \in (0,1/2)$ (with $y := 1/(2^{k-1}\log 2)$), as well as the fact that

$$\frac{(k+1)\log 2 + k\log k}{2^{k-2}\log 2} < 0.01 \qquad \text{for} \qquad k \geq 101.$$

Hence,

$$n < k+3.01+\frac{k\log k - 2\log(k-1)}{\log 2}. \tag{9}$$

But we can also find a similar lower bound for n. Namely, by (2) and (1), we have

$$\begin{aligned} 2^{n-1} > \alpha^{n-1} &> F_n^{(k)} = \frac{2^{k+1}k^k}{(k-1)^2}\left(1-\frac{(k+1)^{k+1}}{k^k 2^{k+1}(k-1)^2}\right) \\ &> \frac{2^{k+1}k^k}{(k-1)^2}\left(1-\frac{e(k+1)}{(k-1)^2 2^{k+1}}\right) > \frac{2^{k+1}k^k}{(k-1)^2}\left(1-\frac{1}{2^{k-1}}\right), \end{aligned} \tag{10}$$

where we used $(1+1/k)^k < e < 4$, valid for all $k \geq 2$ as well as $k+1 \leq (k-1)^2$, valid for $k \geq 4$. Taking logarithms, we get

$$\begin{aligned} n-1 &> \frac{(k+1)\log 2 + k\log k - 2\log(k-1)}{\log 2} + \frac{\log(1-1/2^{k-1})}{\log 2} \\ &> k+1+\frac{k\log k - 2\log(k-1)}{\log 2} - \frac{1}{2^{k-2}\log 2} \\ &> k+0.99+\frac{k\log k - 2\log(k-1)}{\log 2}. \end{aligned}$$

In the above, we again used $\log(1-x) > -2x$, valid for all $x \in (0,1/2)$ with $x := 1/(2^{k-1}\log 2)$, as well as the fact that $2x < 0.01$ since $k \geq 101$. Thus,

$$n > k+1.99+\frac{k\log k - 2\log(k-1)}{\log 2}. \tag{11}$$

From (9) and (11), we record the following lemma.

Lemma 1. *In equation* (7) *with* $k > 100$*, we have*

$$k+1.99+\frac{k\log k - 2\log(k-1)}{\log 2} < n < k+3.01+\frac{k\log k - 2\log(k-1)}{\log 2}.$$

From (9), together with the fact that $k > 100$, we conclude that

$$n < k + 3.01 + \frac{k \log k - 2\log(k-1)}{\log 2} < 2^{k/2},$$

so we are in the range of (5). Thus, from (4) and (5), we get

$$\left|\frac{2^{k+1}k^k - (k+1)^{k+1}}{(k-1)^2} - 2^{n-2}\right| \le \left|F_n^{(k)} - f_k(\alpha)\alpha^{n-2}\right| + \left|f_k(\alpha)\alpha^{n-2} - 2^{n-2}\right| < \frac{2^n}{2^{k/2}} + 1.$$

By (11), we conclude that $n > k/2$, so the right side above is at most $2^{n+1}/2^{k/2}$. Thus,

$$\left|\frac{2^{k+1}k^k}{(k-1)^2} - 2^{n-2}\right| < \frac{2^{n+1}}{2^{k/2}} + \frac{(k+1)^{k+1}}{(k-1)^2}.$$

Let $M := 2^{k+1}k^k/(k-1)^2$ and $N := 2^{n-2}$. Note that

$$\frac{2^{n+1}}{2^{k/2}\max\{M,N\}} \le \frac{2^{n+1}}{2^{k/2}N} = \frac{8}{2^{k/2}}, \quad \frac{(k+1)^{k+1}}{(k-1)^2\max\{M,N\}} \le \frac{(k+1)^{k+1}}{2^{k+1}k^k} < \frac{e(k+1)}{2^{k+1}} < \frac{1}{2^{k/2}},$$

since $k > 100$. We get

$$|1 - (MN^{-1})^\delta| < \frac{8}{2^{k/2}} + \frac{1}{2^{k/2}} = \frac{9}{2^{k/2}}, \tag{12}$$

where $\delta \in \{\pm 1\}$ (so $\delta = 1$ if $N \ge M$ and $\delta = -1$ otherwise). The left side above is

$$|(2^{(k+3-n)}k^k(k-1)^{-2})^\delta - 1|. \tag{13}$$

This expression is not zero, since $k > 100$, so there is an odd prime p dividing $(k-1)k$, which therefore appears with nonzero exponent in the factorization of $2^{k+3-n}k^k(k-1)^{-2}$. To find a lower bound on the above expression, we use Matveev's theorem (see [7], or the formulation of Theorem 3 in [4]). We take $D = 1$, $t = 3$,

$$\gamma_1 := 2, \qquad \gamma_2 := k-1, \quad \gamma_3 := k;$$
$$b_1 := \delta(k+3-n), \quad b_2 := -2\delta, \quad b_3 := \delta k.$$

We take $A_i := \log \gamma_i$ for $i = 1, 2, 3$ and $B = n \ge \max\{|b_1|, |b_2|, |b_3|\}$. So, if we put

$$\Lambda := \gamma_1^{b_1}\gamma_2^{b_2}\gamma_3^{b_2} - 1,$$

we get

$$|\Lambda| > \exp\left(-1.4 \cdot 30^6 \cdot 3^{4.5}(1 + \log n) \cdot \log 2 \cdot \log k \cdot \log(k-1)\right).$$

Comparing this with (12), we get

$$(k/2)\log 2 - \log 9 < 1.4 \cdot 30^6 \cdot 3^{4.5} \log 2 \left(1 + \frac{1}{\log n}\right)(\log n)(\log k)^2.$$

Using Lemma 1 and the fact that $k \ge 101$, we get $763 \le n \le 2k \log k$. We get

$$\begin{aligned} k &< 2.8 \cdot 30^6 \cdot 3^{4.5}\left(1 + \frac{1}{\log 763}\right)(\log k)^2 \log(2k\log k) + \frac{2\log 9}{\log 2} \\ &< 3.3 \cdot 10^{11}(\log k)^2 \log(2k \log k). \end{aligned}$$

This gives $k < 2 \cdot 10^{16}$, and now Lemma 1 gives $n < 1.5 \cdot 10^{18}$. We record these conclusions.

Lemma 2. *In equation* (7) *for* $k > 100$*, we have* $k < 2 \cdot 10^{16}$ *and* $n < 1.5 \cdot 10^{18}$.

We need to reduce the above bounds. We use a 2-adic argument. Let $r \in \{0, 1, \dots, k\}$ be the residue of n modulo $n-2$ modulo $k+1$. We have the following lemma.

Lemma 3. *We have* $r = k+1-r_1$, *where*

$$0 \le r_1 \le 3 + \frac{5\log k}{\log 2}. \tag{14}$$

Proof. Assume first that k is even. Then $\mathrm{Disc}(f_k) \equiv 1 \pmod 2$. In particular, $F_n^{(k)} \equiv 1 \pmod 2$. The sequence $(F_n^{(k)})_{n\in\mathbb{Z}}$ is periodic modulo 2 with period $k+1$. This is easily seen as $f_k(x) \mid x^{k+1} - 2x^k + 1$, so that

$$F_{n+k+1}^{(k)} = 2F_{n+k}^{(k)} - F_n^{(k)} \qquad \text{holds for all} \qquad n \in \mathbb{Z},$$

which modulo 2 simplifies to $F_{n+k+1}^{(k)} \equiv F_n^{(k)} \pmod 2$. Further,

$$F_1^{(k)} = F_2^{(k)} = 1 \quad \text{and} \quad F_m^{(k)} = 2^{m-2} \quad \text{for} \quad m = 3,4,\ldots,k+1.$$

This shows that if $F_n^{(k)}$ is odd, then $n \equiv 1,2 \pmod{k+1}$, so that $n-2 \equiv (k+1)-0,\,(k+1)-1 \pmod{k+1}$. Thus, $r_1 \in \{0,1\}$ in this case. Assume next that k is odd. Then

$$\mathrm{Disc}(f_k) = 2^{k+1}\left(\frac{k^k - ((k+1)/2)^{k+1}}{(k-1)^2}\right),$$

which implies that

$$\nu_2(F_n^{(k)}) = \nu_2(\mathrm{Disc}(f_k)) = k+1+\nu_2(k^k - ((k+1)/2)^{k+1}) - \nu_2((k-1)^2) \ge k+1-2\nu_2(k-1).$$

The right-most inequality above is an equality if and only if $4 \mid k+1$. We now go to (6) and deduce that

$$\nu_2(F_n^{(k)}) = \nu_2\left(2^{n-2} + \sum_{j=1}^{\lfloor \frac{n+k}{k+1}\rfloor - 1} 2^{(n-2)-j(k+1)} C_{n,j}\right). \tag{15}$$

Let

$$J := \left\lfloor \frac{n+k}{k+1} \right\rfloor - 1.$$

Using (11) and the fact that $k > 100$, we get

$$\frac{n+k}{k+1} \ge \frac{2k+1.99}{k+1} + \frac{k\log k - 2\log(k-1)}{(k+1)\log 2} > 2 - \frac{0.01}{k+1} + \left(1 - \frac{3}{k+1}\right)\frac{\log k}{\log 2} > 8.44,$$

which shows that $J \ge 7$. As an upper bound, we have

$$\begin{aligned} J &\le \frac{n-1}{k+1} \le \frac{k+2.01}{k+1} + \frac{k\log k - 2\log(k-1)}{(k+1)\log 2} \\ &< 1 + \frac{1.01 - 2\log(k-1)/\log 2}{(k+1)} + \frac{k\log k}{(k+1)\log 2} < 1 + \frac{\log k}{\log 2} < 2\log k, \end{aligned} \tag{16}$$

since $k > 100$. Since

$$J+1 = \left\lfloor \frac{n+k}{k+1} \right\rfloor, \qquad \text{we get} \qquad (J+1)(k+1) \le n+k < (J+2)(k+1),$$

which implies

$$J+1 \le n - Jk \le k+J+1 \qquad \text{and} \qquad J-1 \le n-Jk-2 \le k+J-1.$$

So, we see that

$$C_{n,J} = \binom{n-Jk}{J} - \binom{n-Jk-2}{J-2} = \left(\frac{(n-Jk)(n-Jk-1)}{J(J-1)} - 1\right)\binom{n-JK-2}{J-2}.$$

Thus,

$$\begin{aligned} \nu_2(C_{n,J}) &\leq \nu_2\binom{n-Jk-2}{J-2} + \nu_2\left((n-Jk)(n-Jk-2) - J(J-1)\right) \\ &\leq \frac{\log(n-Jk-1)}{\log 2} + \frac{\log((n-Jk)(n-Jk-1))}{\log 2} \\ &< \frac{3\log(n-Jk)}{\log 2} < 3\frac{\log(k+2\log k+1)}{\log 2} < \frac{3\log(2k)}{\log 2}. \end{aligned} \tag{17}$$

In the above, we used Kummer's theorem [5] to the effect that the exponent of 2 in $\binom{n}{m}$ is at most the number of carries when adding m and $n-m$ in base 2 (which is at most $\log(n+1)/\log 2$), inequality (16), as well as the fact that $2\log k + 1 < k$ for $k > 100$. □

In the sum appearing in the right side of (15), all powers of 2 appearing there are congruent to the same number, namely r modulo $k+1$. Furthermore, $n-2-(k+1)j \geq k+1$ if $j=0$ or $j \in \{1, 2, \ldots, J-1\}$. Since

$$k+1 > \frac{3\log(2k)}{\log 2} > \nu_2(2^{n-2-J(k+1)}C_{n,J})$$

holds for $k > 100$, we get that

$$\nu_2(F_n^{(k)}) = \nu_2(2^{n-2-J(k+1)}C_{n,J}) = n-2-J(k+1) + \nu_2(C_{n,J}).$$

We study $n-2-J(k+1)$. Note that since

$$J = \left\lfloor \frac{n-1}{k+1} \right\rfloor, \qquad \text{it follows that} \qquad n-1 = J(k+1) + \lambda, \qquad \text{where} \qquad 0 \leq \lambda \leq k+1.$$

If $\lambda \geq 1$, then $n-2 = J(k+1) + (\lambda - 1)$ and $\lambda - 1 \geq 0$, so that $\lambda - 1 = r$. It could be the case that $\lambda = 0$, in which case $n-2 = (J-1)(k+1)+k$, so $r = k$, but in this case we certainly have $r = k = (k+1)-1$, so that $r_1 = 1$ and the conclusion of the lemma holds. So, we may assume that $\lambda \geq 1$; therefore,

$$\nu_2(F_n^{(k)}) = r + \nu_2(C_{n,j}).$$

Comparing the last formula above with (15) and (17), we get

$$k+1-r_1 = r \geq k+1-\frac{2\log(k-1)}{\log 2} - \nu_2(C_{nJ}) > k+1-\frac{2\log k}{\log 2} - 3\frac{\log(2k)}{\log 2} = k-2-5\frac{\log k}{\log 2}.$$

This gives

$$r_1 \leq 3 + 5\frac{\log k}{\log 2},$$

as desired. Since $k \leq 2 \cdot 10^{16}$, the above upper bound on r_1 is at most 273 in our range. At this point, we find it convenient to increase the range of k to $k \leq 300$. Lemma 1 gives us $n \leq 2755$ and a few minutes of computation with Mathematica reveal no additional solutions in the range $k \in [101, 300]$. From now on, $k \geq 301$. We write

$$n-2 = (k+1)(L+1) - r_1, \qquad \text{where} \qquad L = \left\lfloor \frac{n-2}{k+1} \right\rfloor, \quad \text{and} \quad 0 \leq r_1 \leq 273.$$

Since $k \in [301, 2 \cdot 10^{16}]$, Lemma 1 implies $L \in [9, 55]$. We go back to inequalities (12) and (13). Writing $\Lambda := e^{\Gamma} - 1$ and using that $|\Lambda| < 9/2^{k/2}$ implies $|\Gamma| < 18/2^{k/2}$, we get

$$|(n-(k+3))\log 2 - k\log k + 2\log(k-1)| < \frac{18}{2^{k/2}}.$$

The above can be rewritten as

$$|(k+1)(L\log 2-\log k)-r_1\log 2+\log k+2\log(k-1)|<\frac{18}{2^{k/2}},$$

or

$$\left|\log\left(\frac{2^L}{k}\right)\right|<\frac{r_1\log 2+\log k+2\log(k-1)+18/2^{k/2}}{k+1}.$$

Since $k\in[301,2\cdot 10^{16}]$, the numerator of the fraction from the right side above is <265. Hence, taking the exponential, we get

$$\frac{2^L}{k}=\exp(\zeta),\qquad \text{where}\qquad \zeta\in\left(-\frac{265}{k+1},\frac{265}{k+1}\right).$$

Since $265/(k+1)\le 265/302<1.51/(e-1)$, it follows that

$$\exp(\zeta)\in(1-|\zeta|,1+2.51|\zeta|).$$

Thus,

$$2^L\in\left(k-\frac{265k}{k+1},k+\frac{666k}{k+1}\right).$$

In particular, $k\in[2^L-666,2^L+265]$. We now have everything we want to carry out the calculations. Namely, we fix a number $L\in[9,55]$. We fix $k\in[\max\{301,2^L-666\},2^L+265]$. Note that the above maximum is always 2^L-666 except if $L=9$, in which case it is 301. Note that L is determined in at most 50 ways, then k is determined in at most 1000 ways. Lemma 1 then shows that n is in an interval of length 2.02, so there are at most three possibilities for n. Hence, there are less than $50\cdot 1000\cdot 3=1.5\cdot 10^5$ possibilities. We choose a prime p of size 10^{20} and we check, using formula (6), whether

$$2^{n-2}+\sum_{j=1}^{\lfloor\frac{n+k}{k+1}\rfloor-1}2^{(n-2)-j(k+1)}C_{n,j}\equiv\frac{2^{k+1}k^k-(k+1)^{k+1}}{(k-1)^2}\pmod{p}.$$

Since $k<2\cdot 10^{16}<p$, it follows that $k-1$ is invertible modulo p. We used Mathematica and in particular the command `PowerMod` to calculate $2^{n-2-j(k+1)}$, k^k, and $(k+1)^{k+1}$ modulo p. We chose $p=10^{20}+39$. The computations lasted less than one hour and no solution to the above congruence modulo p was found in our range of the variables $L,\ k,\ r_1,\ n$. The theorem is therefore proved.

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References

[1] J. J. Bravo and F. Luca, *On the largest prime factor of the k-Fibonacci sequences*, Int. J. Number Theory, **9** (2013), 1351–1366.

[2] C. Cooper and F. Howard, *Some identities for r-Fibonacci numbers*, The Fibonacci Quarterly, **49.3** (2011), 231–243.

[3] G. Dresden and Z. Du, *A simplified Binet formula for k-generalized Fibonacci numbers*, J. Integer Sequences, **17** (2014), Article 14.4.7.

[4] C. A. Gómez and F. Luca, *On the largest prime factor of the ratio of two generalized Fibonacci numbers*, J. Number Theory, **152** (2015), 182–203.

[5] E. Kummer, *Über die Ergänzungssätze zu den allgemeinen Reciprocitätsgesetzen*, J. Reine Angew. Math., **44** (1852), 93–146.
[6] P. A. Martin, *The Galois group of* $x^n - x^{n-1} - \cdots - 1$, J. Pure App. Algebra, **190** (2004), 213–223.
[7] E. M. Matveev, *An explicit lower bound for a homogeneous rational linear form in logarithms of algebraic numbers*, Izv. Mat., **64** (2000), 1217–1269.
[8] D. A. Wolfram, *Solving generalized Fibonacci recurrences*, The Fibonacci Quarterly, **36.2** (1988), 129–145.

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