ADVANCED PROBLEMS AND SOLUTIONS

Edited by
Florian Luca

Please send all communications concerning ADVANCED PROBLEMS AND SOLUTIONS to FLORIAN LUCA, IMATE, UNAM, AP. POSTAL 61-3 (XANGARI), CP 58 089, MORELIA, MICHOACAN, MEXICO, or by e-mail at fluca@matmor.unam.mx as files of the type tex, dvi, ps, doc, html, pdf, etc. This department especially welcomes problems believed to be new or extending old results. Proposers should submit solutions or other information that will assist the editor. To facilitate their consideration, all solutions sent by regular mail should be submitted on separate signed sheets within two months after publication of the problems.

PROBLEMS PROPOSED IN THIS ISSUE

H-609 Proposed by Mario Catalani, University of Torino, Italy

Let $\boldsymbol{R}_n$ be the right-justified Pascal-triangle matrix, that is the $n \times n$ matrix with the (i, j) element r_{ij}, $1 \le i, j \le n$ given by

$$r_{ij} = \binom{i-1}{n-j}.$$

a. Find $tr(\boldsymbol{R}_n^m)$, where $tr(\cdot)$ is the trace operator and m is a positive integer.

b. Find $|\boldsymbol{R}_n + \boldsymbol{R}_n^{-1}|$, where $|\cdot|$ is the determinant operator.

H-610 Proposed by Jayantibhai M. Patel, Ahmedabad, India

If $\overline{\boldsymbol{x}} = (-2F_n^2, 2F_{2n}, L_n^2)$, $\overline{\boldsymbol{y}} = (2F_{2n}, 2F_n^2 - L_n^2, 2F_{2n})$ and $\overline{\boldsymbol{z}} = (L_n^2, 2F_{2n}, -2F_n^2)$ are three vectors, then prove that

a. $\overline{\boldsymbol{x}}$, $\overline{\boldsymbol{y}}$, $\overline{\boldsymbol{z}}$ are mutually perpendicular vectors.

b. $||\overline{\boldsymbol{x}}|| = ||\overline{\boldsymbol{y}}|| = ||\overline{\boldsymbol{z}}|| = 2F_n^2 + L_n^2$.

c. $\overline{\boldsymbol{x}} \cdot (\overline{\boldsymbol{y}} \times \overline{\boldsymbol{z}}) = ||\overline{\boldsymbol{x}}||^3 = (2F_n^2 + L_n^2)^3$.

H-611 Proposed by Ó. Ciaurri Ramírez, Logroño, Spain and J.L. Díaz-Barrero, Barcelona, Spain

Evaluate

$$\sum_{n=0}^{\infty} \frac{1}{(2\alpha)^n (n+2)} \sum_{k=0}^{n} \frac{F_{k+1} F_{n-k+1}}{k+1},$$

where α denotes the golden section.

SOLUTIONS

Roots of the cubic defining Tribonacci sequences

H-597 Proposed by Mario Catalani, University of Torino, Italy (Vol. 41, no. 2, May 2003)

Let α, β, γ be the roots of the trinomial $x^3 - x^2 - x - 1 = 0$. Express

$$U_n = \sum_{i=1}^{n} \sum_{j=0}^{n-i} \alpha^i \beta^j \gamma^{n-i-j}$$

in terms of the Tribonacci sequence $\{T_n\}$ given by $T_0 = 0,\ T_1 = 1,\ T_2 = 1$ and $T_n = T_{n-1} + T_{n-2} + T_{n-3}$ for $n \geq 3$.

Solution by V. Mathe, Marseille, France

We replace the subscript $i = 1$ by $i = 0$ in the proposed identity.We then have

$$\sum_{j=0}^{n-i} \beta^j \gamma^{-j} = -\gamma \frac{\beta^{n-i+1}\gamma^{-n+i-1} - 1}{-\beta + \gamma},$$

therefore

$$\sum_{i=0}^{n} \sum_{j=0}^{n-i} \alpha^i \beta^j \gamma^{n-i-j} = -\sum_{i=0}^{n} \alpha^i \frac{\beta^{n-i+1} - \gamma^{n-i+1}}{\gamma - \beta}$$

$$= -\frac{-\beta\alpha^{n+2} + \gamma\alpha^{n+2} + \beta^{n+2}\alpha - \beta^{n+2}\gamma - \gamma^{n+2}\alpha + \gamma^{n+2}\beta}{(-\beta+\gamma)(\alpha-\beta)(-\alpha+\gamma)}$$

$$= \frac{\alpha^{n+2}}{(\alpha-\beta)(\alpha-\gamma)} + \frac{\beta^{n+2}}{(\beta-\alpha)(\beta-\gamma)} + \frac{\gamma^{n+2}}{(\gamma-\alpha)(\gamma-\beta)}. \tag{1}$$

We know (see, for example, [1]), that this last expression equals T_{n+1}, where $(T_n)_{n \geq 0}$ is the Tribonacci sequence.

V. Mathe provides also the following generalization.

Let $x_1, \ldots, x_k$ be distinct complex numbers, roots of the equation

$$f(x) = \prod_{i=1}^{k} (x - x_i) = x^k - a_1 x^{k-1} - \cdots - a_k = 0.$$

Let $(T_n^{(k)})_{n\geq 0}$ be the linearly recurrent sequence given by $T_n^{(k)} = 0$ for $n = 0, \ldots, k-2$, $T_{k-1}^{(k)} = 1$, and

$$T_n^{(k)} = a_1 T_{n-1}^{(k)} + \cdots + a_k T_{n-k}^{(k)}$$

for all $n \geq k$. Let also $(U_n^{(k)})_{n\geq 0}$ be the sequence whose general formula is

$$U_n^{(k)} = \sum_{\substack{i_1+\cdots+i_k=n \\ i_j\geq 0,\ j=1,\ldots,k}} x_1^{i_1}\ldots x_k^{i_k}. \tag{2}$$

Then the identity

$$U_n^{(k)} = T_{n+k-1}^{(k)} \tag{3}$$

holds for all $n \geq 0$.

Note that the sequence $(T_n^{(k)})_{n\geq 0}$ defined above for the choice of roots $(x_1, x_2, x_3) = (\alpha, \beta, \gamma)$ is not exactly the sequence $(T_n)_{n\geq 3}$, but the formula $T_n = T_{n+1}^{(3)}$ holds for all $n \geq 0$. To prove (3), V. Mathe notes that by the method of [1], it follows that

$$T_n^{(k)} = \sum_{j=1}^{k} x_j^n \prod_{i\neq j} \frac{1}{(x_j - x_i)}. \tag{4}$$

We first supply a short proof of (4). Writing $T_n^{(k)} = \sum_{j=1}^k c_j x_j^n$ with some unknown coefficients c_j for $j = 1, \ldots, k$ and treating this as a system of linear equations in the unknowns c_j for $j = 1, \ldots, k$ we note that the determinant of this system is precisely the Vandermonde determinant whose jth column is $[1, x_j, \ldots, x_j^{k-1}]^T$ and whose value is

$$\Delta_k(x_1, \ldots, x_k) = \prod_{1\leq i<j\leq k} (x_i - x_j).$$

By Kramer's rule, we have $c_j = \Delta_{j,k}(x_1, \ldots, x_k)/\Delta_k(x_1, \ldots, x_k)$, where $\Delta_{j,k}$ is the determinant obtained from $\Delta_k(x_1, \ldots, x_k)$ by replacing its jth column by the column of coefficients $[T_0^{(k)}, \ldots, T_{k-1}^{(k)}]^T = [0, \ldots, 0, 1]^T$. Expansing this determinant using the jth column, we get that

$$\Delta_{j,k}(x_1, \ldots, x_k) = (-1)^{j-1}\Delta_{k-1}(x_1, \ldots, x_{j-1}, x_{j+1}, \ldots, x_k)$$

$$= (-1)^{j-1} \prod_{\substack{1\le i\le \ell\\ i,\ell\neq j}} (x_i - x_\ell).$$

Performing the obvious simplifications, we get

$$c_j = \frac{\Delta_{j,k}(x_1,\ldots,x_k)}{\Delta_k(x_1,\ldots,x_k)} = \prod_{i\neq j} \frac{1}{(x_j - x_i)},$$

which proves (4).

Now, V. Mathe notes that one can now prove (3) by induction. Indeed, one notes that when $k = 2$ we just have

$$\sum_{\substack{i+j=n\\ i,j\ge 0}} x_1^i x_2^j = \frac{x_1^{n+1} - x_2^{n+1}}{x_1 - x_2} = \frac{x_1^{n+(2-1)}}{x_1 - x_2} + \frac{x_2^{n+(2-1)}}{x_2 - x_1}, \tag{5}$$

and one now recognizes, by (4), that the right hand of (5) is the same as the right hand side of (4) at $k = 2$ which proves (3) when $k = 2$. Assuming by induction that (3) holds for $k - 1$ and the set of complex numbers $(x_2, \ldots, x_k)$, we then get, by separating x_1 in (2), that

$$U_n^{(k)} = U_n^{(k)}(x_1,\ldots,x_k) = \sum_{j=0}^{n} x_1^j U_{n-j}^{(k-1)}(x_2,\ldots,x_k) = \sum_{j=0}^{n} x_1^j \sum_{i=2}^{k} x_i^{n-j+k-2} \prod_{\substack{2\le \ell\le k\\ \ell\neq i}} \frac{1}{(x_i - x_\ell)}$$

$$= \sum_{i=2}^{k} x_i^{k-2}\Big(\prod_{\substack{2\le \ell\le k\\ \ell\neq i}} \frac{1}{(x_i - x_\ell)}\Big)\Big(\sum_{j=0}^{n} x_1^j x_i^{n-j}\Big) = \sum_{i=2}^{k} x_i^{k-2} \prod_{\substack{2\le \ell\le k\\ \ell\neq i}} \frac{1}{(x_i - x_\ell)} \cdot \Big(\frac{x_i^{n+1} - x_1^{n+1}}{x_i - x_1}\Big)$$

$$= \sum_{i=1}^{n} x_i^{n+(k-1)} \prod_{\substack{1\le \ell\le k\\ \ell\neq i}} \frac{1}{(x_i - x_\ell)} - x_1^{n+1}\Big(\sum_{i=1}^{n} x_i^{k-2} \prod_{\substack{1\le \ell\le k\\ \ell\neq i}} \frac{1}{(x_i - x_\ell)}\Big),$$

and it remains to remark that

$$\sum_{i=1}^{n} x_i^{k-2} \prod_{\substack{1\le \ell\le k\\ \ell\neq i}} \frac{1}{(x_i - x_\ell)} = 0,$$

formula which holds because $T_{k-2}^{(k)} = 0$. This completes the proof of the generalization.

Editor's remark. Some of the charm of this problem was lost because in the original statement of the problem the outer sum started at $i = 1$, although the proposers solution indicates that the sum should have started at $i = 0$. Some of the solvers noted this and provided solutions with the outer sum started at $i = 0$ as in the solution presented here. Most solutions arrived in some way at formula (1) and then quoted some results from the literature (like [1]) which relate (1) to the Tribonacci numbers. When the outer sum started at $i = 1$, the resulting answer was a constant multiple of T_{n+1}, the constant multiple depending, as noted by W. Janous, of the particular assignment of the symbols $\alpha,\ \beta,\ \gamma$ to the three roots of the trinomial $x^3 - x^2 - x - 1$.

1. Gwang-Yeon Lee, Jin-Soo Kim and Tae Ho Cho, "Generalized Fibonacci Functions and Sequences of Generalized Fibonacci Functions", The Fibonacci Quarterly **41.2** (2003): 108–121.

Also solved by Paul Bruckman, Kenneth Davenport, Walther Janous, Gurdial Arora and Sindhu Unnithan (jointly) and the proposer.

Find the Eigenvalue

H-598 Proposed by José Díaz-Barrero & Juan José Egozcue, Barcelona, Spain (Vol. 41, no. 2, May 2003)

Show that all the roots of the equation

$$\begin{vmatrix} F_1F_n & \dots & F_1F_3 & F_1F_2 & F_1^2 - x \\ F_2F_n & \dots & F_2F_3 & F_2^2 - x & F_2F_1 \\ \dots\dots\dots & \dots & \dots & \dots & \dots\dots \\ F_n^2 - x & \dots & F_nF_3 & F_nF_2 & F_nF_1 \end{vmatrix} = 0$$

are integers.

Solution based on the solutions of M. Catalani, Torino, Italy and W. Janous, Innsbruck, Austria

Permuting the columns of the above determinant using the permutation $i \longmapsto n - i$ for $i = 1, \dots, n$, the given equation becomes

$$\begin{vmatrix} F_1^2 - x & F_2F_1 & \dots & F_{n-1}F_1 & F_nF_1 \\ F_1F_2 & F_2^2 - x & \dots & F_{n-1}F_2 & F_nF_2 \\ \dots\dots\dots & \dots & \dots & \dots & \dots\dots \\ F_1F_n & F_2F_n & \dots & F_{n-1}F_n & F_n^2 - x \end{vmatrix} = 0,$$

and we recognize that x is an eigenvalue of the matrix whose ith row is $[F_1F_i, F_2F_i, \dots, F_nF_i]$. More generally, let $a_1, \dots, a_n$ be any complex numbers and let A be the matrix whose ith row is $[a_1a_i, a_2a_i, \dots, a_na_i]$. Since all rows of A are multiples of the vector $[a_1, \dots, a_n]$, it is clear that the rank of A is ≤ 1 and therefore $x = 0$ is a root of the equation $\det(xI_n - A) = 0$ with multiplicity $\geq n - 1$. In particular, $\det(A - xI_n) = x^{n-1}(x - c)$. Note now that $c = \operatorname{tr}(A) =$

$\sum_{i=1}^n a_i^2$. In the particular case when $a_i = F_i$ for $i = 1, \ldots, n$, we get that the roots x of the given equation are $x = 0$ with multiplicity $n-1$ and $x = \sum_{i=1}^n F_i^2$.

Note. V. Mathe provided a basis of eigenvectors for the case in which $a_i = F_i$ for $i = 1, \ldots, n$.

Also solved by Paul Bruckman, Kenneth Davenport, L.A.G. Dresel, V. Mathe, and the proposers.

Please Send in Proposals!

PROBLEM SECTION IN HONOUR OF PROFESSOR

RAYMOND E. WHITNEY

Raymond E. Whitney served as the Editor of the Advanced Problem Section of the *Fibonacci Quarterly* for 36 years from Volume **5** no. 3, 1967 until Volume **41** no. 1, 2003. During his time as the Editor, the Advanced Problem Section has published more than 500 problems.

This Department would like to have the Advanced Problem Section of one of the issues for 2005 dedicated to Professor Whitney for his many years of service. We are encouraging the proposers to submit problems to be dedicated to Professor Raymond E. Whitney before December 31, 2004.

Errata: Advanced Problems and Solutions, Vol. 41, no. 5, November 2003, Page 473.

1. Displayed formula (d) should have been

$$5^{n-1}L_{2n+1} = \sum_{\substack{k=0 \\ 5 | 2n-k+2}}^{2n-1} (-1)^{\lfloor (8n+k+2)/5 \rfloor} \binom{4n-1}{k}.$$

2. The last three lines of displayed formula (2) should have been "if $k \equiv 2 \pmod 5$","if $k \equiv 3 \pmod 5$" and "if $k \equiv 4 \pmod 5$", respectively.

The Editor apologizes for these oversights.