ELEMENTARY PROBLEMS AND SOLUTIONS

Edited by
Russ Euler and Jawad Sadek

Please submit all new problem proposals and corresponding solutions to the Problems Editor, DR. RUSS EULER, Department of Mathematics and Statistics, Northwest Missouri State University, 800 University Drive, Maryville, MO 64468. All solutions to others' proposals must be submitted to the Solutions Editor, DR. JAWAD SADEK, Department of Mathematics and Statistics, Northwest Missouri State University, 800 University Drive, Maryville, MO 64468.

If you wish to have receipt of your submission acknowledged, please include a self-addressed, stamped envelope.

Each problem and solution should be typed on separate sheets. Solutions to problems in this issue must be received by January 15, 2005. If a problem is not original, the proposer should inform the Problem Editor of the history of the problem. A problem should not be submitted elsewhere while it is under consideration for publication in this Journal. Solvers are asked to include references rather than quoting "well-known results".

BASIC FORMULAS

The Fibonacci numbers F_n and the Lucas numbers L_n satisfy

$$F_{n+2} = F_{n+1} + F_n,\ F_0 = 0,\ F_1 = 1;$$

$$L_{n+2} = L_{n+1} + L_n,\ L_0 = 2,\ L_1 = 1.$$

Also, $\alpha = (1+\sqrt{5})/2,\ \beta = (1-\sqrt{5})/2,\ F_n = (\alpha^n - \beta^n)/\sqrt{5}$, and $L_n = \alpha^n + \beta^n$.

PROBLEMS PROPOSED IN THIS ISSUE

B-976 Proposed by Muneer Jebreel Karameh, Jerusalem, Israel

Prove that $(L_nL_{n+3})^2 + (2L_{n+1}L_{n+2})^2 = (L_{2n+5} - L_{2n+1})^2$.

B-977 Proposed by Walther Janous, Ursulinengymnasium, Innsbruck, Austria

Determine all integers n such that $\alpha^n = a + b\sqrt{5}$ where a and b are integers.

B-978 Proposed by Carl Libis, University of Rhode Island, Kingston, RI

For $n > 0$, let $A_n = [a_{i,j}]$ denote the symmetric matrix with $a_{i,i} = i+1$ and $a_{i,j} = \min\{i,j\}$ for all integers i and j with $i \neq j$. Find the determinant of A_n.

B-979 Proposed by Ovidiu Furdui, Western Michigan University, Kalamazoo, MI

Prove that

$$\lim_{n\to\infty}\left[n\left(\sqrt[n+1]{F_{n+1}} - \sqrt[n]{F_n}\right)\right] = 0.$$

B-980 Proposed by Steve Edwards, Southern Polytechnic State University, Marietta, GA

For any integers m and n, evaluate

$$\left(L_n\alpha^m + L_{n-1}\alpha^{m-1}\right) / \left(L_m\alpha^n + L_{m-1}\alpha^{n-1}\right).$$

SOLUTIONS

A Constant Sum

B-961 Proposed by Steve Edwards, Southern Polytechnic State University, Marietta, GA (Vol. 41, no. 4, August 2003)

Show that $\frac{L_{n+1}}{\alpha^{n-1}} + \frac{L_n}{\alpha^n}$ is a constant for all nonnegative integers n.

Solution by Kathleen E. Lewis, SUNY Oswego, Oswego, NY

$$\frac{L_{n+1}}{\alpha^{n-1}} + \frac{L_n}{\alpha^n} = \frac{\alpha^{n+1}+\beta^{n+1}}{\alpha^{n-1}} + \frac{\alpha^n+\beta^n}{\alpha^n}$$

$$= \alpha^2 + \alpha\beta\frac{\beta^n}{\alpha^n} + 1 + \frac{\beta^n}{\alpha^n}.$$

But $\alpha\beta = -1$, so $\frac{L_{n+1}}{\alpha^{n+1}} + \frac{L_n}{\alpha^n}$ is equal to the constant $\alpha^2 + 1 = \frac{5+\sqrt{5}}{2}$.

Paul Bruckman generalized the problem to all integers and Walther Janous generalized it to include all sequences of the form $x_n = A\lambda^n + \beta M^n$ such that $x_{n+2} = ax_{n+1} + x_n$ where λ and $M(\lambda \neq M)$ are the solutions of $t^2 - at - 1 = 0$.

Almost all other solutions are similar to the featured one.

Also solved by Luay Q. Abdel-Jaber, Gurdial Arora, Charles Ashbaker, Scott H. Brown, Paul Bruckman, Mario Catalani, Charles Cook, Kenneth Davenport, José Luis Diaz-Barrero and Óscar Ciaurri Ramirez (jointly), Ovidiu Furdui, Pentti Haukkanen, Gerald A. Heuer, Walther Janous, Harris Kwong, Carl Libis, Jaroslav Seibert, H.-J. Seiffert, James Sellers, and the proposer.

An Infinite Fibonacci Product

B-962 Proposed by Steve Edwards, Southern Polytechnic State University, Marietta, GA (Vol. 41, no. 4, August 2003)

Find

$$\prod_{k=1}^{\infty} \frac{F_{2k}F_{2k+2} + F_{2k-1}F_{2k+2}}{F_{2k}F_{2k+2} + F_{2k}F_{2k+1}}.$$

Solution by Charles K. Cook, University of South Carolina Sumter, Sumter, SC

Let

$$P_n = \prod_{k=1}^{n} \frac{F_{2k}F_{2k+2} + F_{2k-1}F_{2k+2}}{F_{2k}F_{2k+2} + F_{2k}F_{2k+1}} = \prod_{k=1}^{n} \frac{F_{2k+1}F_{2k+2}}{F_{2k}F_{2k+3}}.$$

Then

$$\begin{aligned} P_n &= \frac{F_3F_4}{F_2F_5} \cdot \frac{F_5F_6}{F_6F_7} \cdot \frac{F_7F_8}{F_6F_9} \cdots \frac{F_{2n-1}F_{2n}}{F_{2n-2}F_{2n+1}} \cdot \frac{F_{2n+1}F_{2n+2}}{F_{2n}F_{2n+3}} \\ &= \frac{F_3}{F_2} \cdot \frac{F_{2n+2}}{F_{2n+3}} = \frac{2F_{2n+2}}{F_{2n+3}}. \end{aligned}$$

Thus $\lim_{n\to\infty} P_n = \frac{2}{\alpha} = -2\beta = \sqrt{5} - 1 \approx 1.23608$.

Also solved by Paul Bruckman, José Luis Diaz-Barrero and Óscar Ciaurri Ramirez (jointly), Walther Janous, Harris Kwong, Kathleen Lewis, Jaroslav Seibert, H.-J. Seiffert, James A. Sellers. Four incorrect solutions were received.

A Simple Lower Bound for a Fibonacci Fraction

B-963 Proposed by Ovidiu Furdui, Western Michigan University, Kalamazoo, MI
(Vol. 41, no. 4, August 2003)

Prove that

$$\frac{F_{2n+1} - 1}{F_{2n+4} - 3F_{n+2} - L_{n+2} + 3} \geq \frac{1}{n}$$

for all $n \geq 1$.

Solution by Harris Kwong, SUNY Fredonia, Fredonia, NY

Using induction, it is easy to show that for $n \geq 1$,

$$\sum_{k=1}^{n} F_k = F_{n+2} - 1, \quad \sum_{k=1}^{n} L_k = L_{n+2} - 3, \quad \sum_{k=1}^{n} F_{2k} = F_{2n+1} - 1.$$

In addition, $F_{2k} = F_kL_k$ follows immediately from Binet's formulas. Finally, recall that Chebyshev inequality asserts that

$$n \sum_{k=1} a_kb_k \geq \left(\sum_{k=1}^{n} a_k\right)\left(\sum_{k=1}^{n} b_k\right)$$

for all nondecreasing positive sequences $\{a_k\}$ and $\{b_k\}$. Combining these results, we find

$$\frac{F_{2n+1}-1}{F_{2n+4}-3F_{n+2}-L_{n+2}+3} = \frac{F_{2n+1}-1}{(F_{n+2}-1)(L_{n+2}-3)} = \frac{\sum_{k=1}^{n} F_k L_k}{(\sum_{k=1}^{n} F_k)(\sum_{k=1}^{n} L_k)} \geq \frac{1}{n}$$

for all $n \geq 1$.

Also solved by Paul Bruckman, Walther Janous, Jaroslav Seibert, H.-J. Seiffert, and the proposer.

Fibonacci to Lucas

B-964 Proposed by Stanley Rabinowitz, MathPro, Westford, MA (Vol. 41, no. 4, August 2003)

Find a recurrence relation for $r_n = \frac{F_n}{L_n}$.

Solution by H.-J. Seiffert, Berlin, Germany

¿From (I_8) and (I_9) of [1], we know that $L_n = F_{n-1} + F_{n+1}$ and $5F_n = L_{n-1} + L_{n+1}$. Since $F_n + F_{n-1} = F_{n+1}$ and $L_n + L_{n-1} = L_{n+1}$, we then have $F_n + L_n = 2F_{n+1}$ and $5F_n + L_n = 2L_{n+1}$. Hence,

$$r_{n+1} = \frac{F_{n+1}}{L_{n+1}} = \frac{F_n + L_n}{5F_n + L_n} = \frac{r_n + 1}{5r_n + 1}.$$

Reference:

1. V.E. Hoggatt, Jr. Fibonacci and Lucas Numbers. Santa Clara, CA: The Fibonacci Association, 1979.

It is interesting to note that only two other solvers gave the same recurrence formula. Each one of the other solvers gave a different one.

Also solved by Luay Abdel-Jaber, Paul Bruckman, Mario Catalani, Charles Cook, Ovidiu Furdui, Gerald D. Heuer, Walther Janous, Harris Kwong, Kathleen E. Lewis, Jaroslav Seibert, and the proposer.

A Fancy Integer

B-965 Proposed by José Luis Díaz-Barrero and Juan José Egozcue, Universitat Politècnica de Catalunya, Barcelona, Spain (Vol. 41, no. 4, August 2003)

Let n be a positive integer. Prove that

$$\frac{F_n!(4F_{n+1})!}{(2F_n)!(F_{n-1}+F_{n+1})!(2F_{n+1})!}$$

is an integer.

Solution I by Harris Kwong, SUNY Fredonia, Fredonia, NY

Observe that $2F_{n+1} - (F_{n-1} + F_{n+1}) = F_{n+1} - F_{n-1} = F_n$. Thus

$$\begin{aligned}
\frac{F_n!(4F_{n+1})!}{(2F_n)!(F_{n-1}+F_{n+1})!(2F_{n+1})!} &= \frac{F_n!}{(2F_n)!} \cdot \frac{(2F_{n+1})!}{(F_{n-1}+F_{n+1})!} \cdot \frac{(4F_{n+1})!}{(2F_{n+1})!(2F_{n+1})!} \\
&= \frac{2F_{n+1}(2F_{n+1}-1)\cdots(F_n+1)}{2F_n(2F_n-1)\cdots(F_n+1)} \binom{4F_{n+1}}{2F_{n+1}} \\
&= 2F_{n+1}(2F_{n+1}-1)\cdots(2F_{n+1}-2F_n+1)\binom{4F_{n+1}}{2F_{n+1}} \\
&= 2F_{n+1}(2F_{n+1}-1)\cdots(2F_{n-1}+1)\binom{4F_{n+1}}{2F_{n+1}},
\end{aligned}$$

which is obviously an integer.

Solution II by Walther Janous, Ursulinengymnasium, Innsbruck, Austria

We show a more general result, namely: Let

$$(x_n, n \geq 1)$$

be a sequence of nonnegative entire numbers satisfying

$$x_n \leq 2x_{n+1}.$$

Then all numbers

$$\frac{x_n!(4x_{n+1})!}{(2x_n)!(2x_{n+1}-x_n)!(2x_{n+1})!}$$

are integers.

Indeed, let p be an arbitrary prime.

As the maximum-exponent of p^e dividing a factorial $N!$ equals

$$e = \sum_{j \geq 1} \left[\frac{N}{p^j}\right]$$

where $[x]$ denotes the floor-function, the claimed result will follow from the inequalities

$$\left[\frac{x_n}{p^j}\right] + \left[\frac{4x_{n+1}}{p^j}\right] \geq \left[\frac{2x_n}{p^j}\right] + \left[\frac{2x_{n+1} - x_n}{p^j}\right] + \left[\frac{2x_{n+1}}{p^j}\right]$$

for $j = 1, 2, 3, \ldots$.

All of these inequalities are of the type

$$[a] + [4b] \geq [2a] + [2b - a] + [2b] \qquad (*)$$

where a and b are nonnegative (real) numbers such that $a \leq 2b$.

We now will show the validity of $(*)$ by distinguishing several cases for a and b.

First of all, by cancelling out the entire parts of a and b it's enough to consider $(*)$ for a, b in $[0, 1)$.

- Let $0 < a < 1/2$. Then $(*)$ becomes

$$[4b] \geq [2b - a] + [2b].$$

This is true because

$$[2b - a] + [2b] \leq [2b - a + 2b] \leq [4b]$$

- $1/2 \leq a < 1$. Then $b = a/2 + t \geq 1/4$, where $0 \leq t < 1 - a/2$. Furthermore $(*)$ becomes

$$\left[4\left(\frac{a}{2} + t\right)\right] \geq 1 + \left[2\left(\frac{a}{2} + t\right) - a\right] + \left[2\left(\frac{a}{2} + t\right)\right]$$

or equivalently,

$$2a + 4t] \geq 1 + [2t] + [a + 2t] \qquad (**)$$

(i) If $[2t] = 0$, then $(**)$ becomes

$$[2(a + 2t)] \geq 1 + [a + 2t].$$

But this inequality is evident for any of the two cases $1/2 \leq a + 2t < 1$ and $1 \leq a + 2t < 2$, respectively.

(ii) If $[2t] = 1$ we get for $(**)$

$$[2(a + 2t)] \geq 2 + [a + 2t].$$

This inequality is evident as now $3/2 \leq a + 2t < 2$ and the proof is complete.

The desired problem is the special case $x_n = F_n$.

Also solved by Paul S. Bruckman and the proposer.