PERIOD LENGTHS OF CONTINUED FRACTIONS INVOLVING FIBONACCI NUMBERS

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1. INTRODUCTION

For given nonsquare positive integers $C \equiv 5 \pmod 8$, we investigate families $\{D_k(X)\}_{k\in\mathbb{N}}$ of integral polynomials of the form $D_k(X) = A_k^2X^2+2B_kX+C$ where $(B_k/2)^2-(A_k/2)^2C = 4$, and show that the period length of the simple continued fraction expansion of $(1+\sqrt{D_k(X)})/2$ is a multiple of k, and independent of X. For each member of the families involved, we show how to easily determine the fundamental unit of the underlying quadratic order $\mathbb{Z}[(1+\sqrt{D_k(X)})/2]$. We also demonstrate how the simple continued fraction expansion of $(1+\sqrt{D_k(X)})/2$ is related to that of $(1+\sqrt{C})/2$. As applications, we present infinite families of continued fractions related to the Fibonacci numbers. This continues work in [11]-[12] and corrects errors in [15] (see Theorem 3.1).

In 1949, Nyberg [14] found the first example of a parametric family in which a fundamental unit can be easily produced even though the period length of the continued fraction gets arbitrarily large. Since this discovery, there have been a number of generalizations: Dan Shanks [16]-[17] (see also Yamamoto [22]) in 1969 - 1971, Hendy [5] in 1974, Bernstein [2]-[3] in 1976, Williams [20] in 1985, Levesque and Rhin [7] in 1986, Azuhata [1] in 1987, Levesque [6] in 1988, Halter-Koch [4] in 1989, Mollin and Williams [13] in 1992, Williams [19] in 1995, and van der Poorten and Williams in [15] in 1999.

In [12], we found infinite families of quadratic Pellian polynomials $D_k(X)$ such that the continued fraction expansions $\sqrt{D_k(X)}$ have unbounded period length for arbitrary k, while for fixed k and arbitrary $X > 0$, have constant period length. In this paper, we continue the investigation for simple continued fraction expansions of $(1+\sqrt{D_k(x)})/2$, where $D_k(X)$ are quadratic polynomials of Eisenstein type (see Section 3 below). We are also able to find explicit fundamental units of the order $\mathbb{Z}[(1+\sqrt{D_k(X)})/2]$. A consequence of the result is an infinite family of continued fraction expansions whose fundamental units and discriminant are related to Fibonacci numbers. Ostensibly the first such finding of these types of continued fractions was given in [21], and the findings herein appear to be the second, albeit distinctly different from those given in [21].

The relatively "small" fundamental units for the underlying quadratic order which we explicitly determine means that we have "large" class numbers $h_{D(X)}$ for $\mathbb{Z}[(1+\sqrt{D(X)})/2]$. The reason behind this fact is Siegel's class number result [18] which tells us that for positive discriminants Δ,

$$\lim_{\Delta\to\infty} \log(h_\Delta R)/\log(\sqrt{\Delta}) = 1$$

where h_Δ is the class number of $\mathbb{Q}(\sqrt{\Delta})$, ε_Δ is its fundamental unit, and $R = \log(\varepsilon_\Delta)$ is the regulator of $\mathbb{Q}(\sqrt{\Delta})$. Hence, a "small" fundamental unit will necessarily mean a large class number.

2. NOTATION AND PRELIMINARIES

The background for the following together with proofs and details may be found in most standard introductory number theory texts (see for example [9], and for a more advanced exposition with detailed background on quadratic orders, see [8]).

Let $\Delta = d^2 D_0 (d \in \mathbb{N}, D_0 > 1$ squarefree) be the discriminant of a real quadratic order $\mathcal{O}_\Delta = \mathbb{Z} + \mathbb{Z}[\sqrt{\Delta}] = [1, \sqrt{\Delta}]$ in $\mathbb{Q}(\sqrt{\Delta}), U_\Delta$ the group of units of $\mathcal{O}_\Delta$, and ε_Δ the fundamental unit of $\mathcal{O}_\Delta$.

Now we introduce the notation for continued fractions. Let $\alpha \in \mathcal{O}_\Delta$. We denote the simple continued fraction expansion of α (in terms of its *partial quotients*) by:

$$\alpha = \langle q_0; q_1, \cdots, q_n, \cdots \rangle.$$

If α is *periodic*, we use the notation:

$$\alpha = \langle q_0; q_1 \cdot q_2 \cdots, q_{k-1}, \overline{q_k, q_{k+1}, \ldots, p_{\ell+k-1}} \rangle,$$

to denote the fact that $q_n = q_{n+\ell}$ for all $n \geq k$. The smallest such $\ell = \ell(\alpha) \in \mathbb{N}$ is called the *period length* of α. If $k = 0$ is the least such nonnegative value, then α is *purely periodic*, namely

$$\alpha = \langle \overline{q_0; q_1, \ldots, q_{\ell-1}} \rangle.$$

The *convergents* (for $n \geq 0$) of α are denoted by

$$\frac{x_n}{y_n} = \langle q_0; q_1, \ldots, q_n \rangle = \frac{q_n x_{n-1} + x_{n-2}}{q_n y_{n-1} + y_{n-2}}. \tag{2.1}$$

We will need the following facts.

$$x_j = q_j x_{j-1} + x_{j-2} \quad (\text{for } j \geq 0 \text{ with } x_{-2} = 0, \text{ and } x_{-1} = 1), \tag{2.2}$$

$$y_j = q_j y_{j-1} + y_{j-2} \quad (\text{for } j \geq 0 \text{ with } y_{-2} = 1, \text{ and } y_{-1} = 0), \tag{2.3}$$

and

$$x_j y_{j-1} - x_{j-1} y_j = (-1)^{j-1} \quad (j \in \mathbb{N}). \tag{2.4}$$

In particular, we will be dealing with $\alpha = (1 + \sqrt{D})/2$ where D is a radicand. In this case, the *complete quotients* are given by $(P_j + \sqrt{D})/Q_j$ where the P_j and Q_j are given by the recursive formulae as follows for any $j \geq 0$ (with $P_0 = 1$ and $Q_0 = 2$):

$$q_j = \left\lfloor \frac{P_j + \sqrt{D}}{Q_j} \right\rfloor, \tag{2.5}$$

$$P_{j+1} = q_j Q_j - P_j, \tag{2.6}$$

and

$$D = P_{j+1}^2 + Q_j Q_{j+1}. \tag{2.7}$$

Thus, we may write:

$$(1+\sqrt{D})/2 = \langle q_o; q_1, \ldots, q_n, (P_{n+1}+\sqrt{D})/Q_{n+1}\rangle. \tag{2.8}$$

We will also need the following facts for $\alpha = (1+\sqrt{D})/2$. For any integer $j \geq -1$, let

$$g_{j-1} = 2x_{j-1} - y_{j-1}, \tag{2.9}$$

then for any nonnegative integer j,

$$g_{j-1} = P_j y_{j-1} + Q_j y_{j-2}, \tag{2.10}$$

$$D y_{j-1} = P_j y_{j-1} + Q_j y_{j-2}, \tag{2.11}$$

and if $\ell = \ell((1+\sqrt{D})/2)$, then

$$P_1 = P_{j\ell} = 2q_0 - 1 \quad \text{and} \quad Q_0 = Q_{j\ell} = 2, \tag{2.12}$$

and for any $j \in \mathbb{N}$,

$$g_{j\ell-1}^2 - y_{j\ell-1}^2 D = 4(-1)^{j\ell}. \tag{2.13}$$

Also, for any $1 \leq j \leq \ell - 1$,

$$q_j < 2q_0. \tag{2.14}$$

We close this section with a result on Eisenstein's equation, which will be quite useful in establishing results in the next section. In fact, this gives more detail to the fact exhibited in (2.13) for this case. It is know that the Diophantine Equation:

$$|x^2 - Dy^2| = 4 \text{ with } \gcd(x,y) = 1 \tag{2.15}$$

was studied by Eisenstein in search of a criterion for its solvability, a question already asked by Gauss. Several criteria are known (for instance, see [8, pp. 59-61]). In particular, when $D \equiv 5 \pmod 8$, it is known that (2.15) has a solution if and only if the fundamental unit of $\mathbb{Z}[(1+\sqrt{D})/2]$ is *not* in the order $\mathbb{Z}[\sqrt{D}]$. Moreover, it can be shown that (2.15) is solvable if and only if the ideal $I = [4, 1+\sqrt{D}]$ is principal in $\mathbb{Z}[\sqrt{D}]$. For instance, if $D = 37$, there is *no* solution to (2.15) since $\varepsilon_D = 6 + \sqrt{37} \in \mathbb{Z}[\sqrt{D}]$, whereas for $D = 13$, there *is* a solution since $\varepsilon_{13} = (3+\sqrt{13})/2 \notin \mathbb{Z}[\sqrt{13}]$.

Theorem 2.1: *Let $D \equiv 5 \pmod 8$ be a nonsquare positive integer such that $\varepsilon_D \notin \mathbb{Z}[\sqrt{D}]$. For $j \geq 0$, let x_j and y_j be as above in the simple continued fraction expansion of $(1+\sqrt{D})/2$ and let $\ell = \ell((1+\sqrt{D})/2)$. If ℓ is even, then all solutions of*

$$x^2 - Dy^2 = 4 \text{ with } \gcd(x,y) = 1, \tag{2.16}$$

are given by

$$(x,y) = (g_{j\ell-1}, y_{j\ell-1}) \tag{2.17}$$

for $j \geq 1$, *whereas there are no integer solutions of*

$$x^2 - Dy^2 = -4, \text{ with } \gcd(x, y) = 1. \tag{2.18}$$

If ℓ *is odd, then all positive solutions of* (2.16) *are given by*

$$(x, y) = (g_{2j\ell-1}, y_{2j\ell-1})$$

for $j \geq 1$, *whereas all positive solutions of* (2.18) *are given by*

$$(x, y) = (g_{(2j-1)\ell-1}, y_{(2j-1)\ell-1}).$$

Proof: See [9, Theorem 5.3.4, p. 246]. □

For an algorithm, relying only on continued fraction expansions, which finds all primitive solutions of $x^2 - Dy^2 = a$ for *any* integer a and any nonsquare $D > 0$, see [10].

3. RESULTS

We make the following assumptions throughout. We let $A, B, C, k, X \in \mathbb{N}$ and $C \equiv 5(\text{mod } 8)$ not a perfect square. Suppose that (x, y) is the smallest positive solution of $x^2 - Cy^2 = 4$ with $\gcd(x, y) = 1$. Set $A = 2y$ and $B = 2x$ and define, for each $k \in \mathbb{N}$,

$$B_k + A_k\sqrt{C} = (B + A\sqrt{C})^k/4^{k-1}.$$

Also, set

$$\frac{1 + \sqrt{C}}{2} = \langle c_0; \overline{c_1, \ldots, c_n, 2c_0 - 1}\rangle,$$

and for $m \in \mathbb{N}$ define

$$w_m = c_1, \ldots, c_n, 2c_0 - 1, c_1, \ldots, c_n, 2c_0 - 1, \ldots, c_1, \ldots, c_n,$$

which is m iterations of $c_1, \ldots, c_n, 2c_0 - 1$ followed by one iteration of $c_1, \ldots, c_n$.

Theorem 3.1: *Let* $D_k(X) = A_k^2X^2 + 2B_kX + C$. *Then the fundamental solutions of* $u^2 - D_k(X)v^2 = 4$ *is* $(u, v) = ((A_k^2X + B_k)/2, A_k/2)$. *In other words,*

$$\varepsilon_{D_k(X)} = \frac{A_k^2X + B_k + A_k\sqrt{D_k(X)}}{4}.$$

If $q_0 = A_kX/2 + c_0$, *then:*

(a) *if* $n \geq 0$ *is even,*

$$\frac{1 + \sqrt{D_k(X)}}{2} = \langle q_0; \overline{w_{2k-1}, 2q_0 - 1}\rangle$$

with

$$\ell\left(\frac{1 + \sqrt{D_k(X)}}{2}\right) = 2k(n + 1);$$

(b) *and if n is odd,*

$$\frac{1+\sqrt{D_k(X)}}{2} = \langle q_0; \overline{w_{k-1}, 2q_0 - 1}\rangle$$

with

$$\ell\left(\frac{1+\sqrt{D_k(X)}}{2}\right) = k(n+1).$$

Proof: We observe that

$$(A_k/2)^2 D_k(X) = ((A_k^2 X + B_k))/2)^2 - 4,$$

which is not a perfect square. Thus, by [8, Theorem 3.2.1, p. 78],

$$\varepsilon_{(A_k/2)^2 D_k(X)} = \frac{A_k^2 X + B_k}{4} + \frac{1}{4}\sqrt{A_k^2 D_k(X)} = \frac{A_k^2 X + B_k}{4} + \frac{A_k}{4}\sqrt{D_k(X)}.$$

Let X_j/Y_j be the j^{th} convergent of $(1+\sqrt{D_k(x)})/2$, so from (2.9), $G_j = 2X_j - Y_j$. Since

$$\left(\frac{A_k^2 X + B_k}{2}\right)^2 - \left(\frac{A_k}{2}\right)^2 D_k(X) = 4,$$

then by Theorem 2.1, there is a $j \in \mathbb{N}$ such that

$$\frac{A_k^2 X + B_k}{A_k} = \frac{G_{j\ell-1}}{Y_{j\ell-1}}.$$

We now show that $j = 1$, namely, via Theorem 2.1, that $\varepsilon_{(A_k/2)^2 D_k(X)} = \varepsilon_{D_k(X)}$, since ℓ will be shown to be even, and by so doing that the continued fraction expansions in (a)-(b) hold. First, we deal with part (a).

If x_j/y_j is the j^{th} convergent of $(1+\sqrt{C})/2$, then by (2.1),

$$\langle A_k X/2 + c_0; w_{2k-1}\rangle = \langle A_k X/2 + c_0, w_{k-1}, 2c_0 - 1 + y_{k(n+1)-2}/y_{k(n+1)-1}\rangle =$$

$$\frac{A_k X}{2} + \frac{(2c_0 - 1 + y_{k(n+1)-2}/y_{k(n+1)-1})x_{k(n+1)-1} + x_{k(n+1)-2}}{(2c_0 - 1 + y_{k(n+1)-2}/y_{k(n+1)-1})y_{k(n+1)-1} + y_{k(n+1)-2}} =$$

$$\frac{A_kX}{2}+\frac{((2c_0-1)y_{k(n+1)-1}+y_{k(n+1)-2})x_{k(n+1)-1}+x_{k(n+1)-2}y_{k(n+1)-1}}{((2c_0-1)y_{k(n+1)-1}+y_{k(n+1)-2})y_{k(n+1)-1}+y_{k(n+1)-2}y_{k(n+1)-1}},$$

and using the facts:

$$g_{k(n+1)-1}=(2c_0-1)y_{k(n+1)-1}+2y_{k(n+1)-2}$$

(see (2.10) and (2.12)) and

$$x_{k(n+1)-1}y_{k(n+1)-2}-x_{k(n+1)-2}y_{k(n+1)-1}=(-1)^{k(n+1)-2}=(-1)^k$$

(see (2.4)), this equals:

$$\frac{A_kX}{2}+\frac{x_{k(n+1)-1}(g_{k(n+1)-1}-y_{k(n+1)-2})+x_{k(n+1)-2}y_{k(n+1)-1}}{(g_{k(n+1)-1}-y_{k(n+1)-2})y_{k(n+1)-1}+x_{k(n+1)-1}y_{k(n+1)-1}}=$$

$$\frac{A_kX}{2}+\frac{x_{k(n+1)-1}g_{k(n+1)-1}+(-1)^{k+1}}{g_{k(n+1)-1}y_{k(n+1)-1}}. \tag{3.19}$$

However, by Theorem 2.1, since n is even and $(B_k/2)^2-(A_k/2)^2C=4$, then $B_k/2=g_{2k(n+1)-1}$ and $A_k/2=y_{2k(n+1)-1}$. Moreover,

$$g_{2k(n+1)-1}+y_{2k(n+1)-1}\sqrt{C}=(g_n+y_n\sqrt{C})^{2k}/2^{2k-1}=(g_{k(n+1)-1}+y_{k(n+1)-1}\sqrt{C})^2/2=$$

$$(g^2_{k(n+1)-1}+y^2_{k(n+1)-1}C+2g_{k(n+1)-1}y_{k(n+1)-1}\sqrt{C})/2=$$

$$g^2_{k(n+1)-1}+2(-1)^{k+1}+g_{k(n+1)-1}y_{k(n+1)-1}\sqrt{C}$$

where the last equality follows from Equation (2.13) given that $\ell((1+\sqrt{C})/2)=n+1$. Hence, $B_k=2g^2_{k(n+1)-1}+4(-1)^{k+1}$ and $A_k=2g_{k(n+1)-1}y_{k(n+1)-1}$. Thus

$$A_k+B_k=2g_{k(n+1)-1}y_{k(n+1)-1}+2g^2_{k(n+1)-1})+4(-1)^{k+1}=$$

$$2g_{k(n+1)-1}(y_{k(n+1)-1}+g_{k(n+1)-1}+4(-1)^{k+1}=$$

$$4g_{k(n+1)-1}x_{k(n+1)-1}+4(-1)^{k+1}$$

by (2.9) so

$$\frac{A_k+B_k}{4}=g_{k(n+1)-1}x_{k(n+1)-1}+(-1)^{k+1}.$$

Plugging this into (3.19), we get,

$$\langle A_k X/2 + c_0; w_{2k-1}\rangle = \frac{A_k X}{2} + \frac{A_k + B_k}{2A_k} = \frac{(A_k^2 X + A_k + B_k)/4}{A_k/2}.$$

By (2.14), $2q_0 - 1 = A_k X + 2c_0 - 1 \neq c_j$ for any $0 < j < \ell$. Thus, since a convergent $X_{j\ell-1}/Y_{j\ell-1}$ can only occur at the end of the j^{th} multiple of a complete period, then $j = 1$. In other words,

$$\langle A_k X/2 + c_0, w_{2k-1}\rangle = \frac{X_{\ell-1}}{Y_{\ell-1}}.$$

Since we have achieved the $(\ell-1)^{th}$ convergent of $\sqrt{D_k(X)}$, then $\sqrt{D}$ is as given in (a), and $\ell = \ell((1 + \sqrt{D_k(X)})/2) = 2k(n+1)$ for all $k \in \mathbb{N}$ and all even $n \in \mathbb{N}$. Moreover, since $X_{\ell-1} = (A_k^2 X + A_k + B_k)/4$ and $Y_{\ell-1} = A_k/2$, then by (2.9),

$$G_{\ell-1} = 2X_{\ell-1} - Y_{\ell-1} = (A_k^2 X + B_k)/2,$$

which completes the proof of part (a).

To prove (b), we assume that n is odd. Then by Theorem 2.1, $B_k/2 = g_{k(n+1)-1}, A_k/2 = y_{k(n+1)-1}$, and

$$\langle A_k X/2 + c_0, w_{k-1}\rangle = A_k X/2 + x_{k(n+1)-1}/y_{k(n+1)-1} =$$

$$A_k X/2 + (g_{k(n+1)-1} + y_{k(n+1)-1})/2y_{k(n+1)-1} =$$

$$\frac{A_k^2 X + A_k + B_k}{2A_k},$$

where the penultimate equality comes from (2.9). As in the proof of (a), $X_{\ell-1} = (A_k^2 X + A_k + B_k)/4$ and $Y_{\ell-1} = A_k/2$, with the last equality as in the proof of (a). Hence, $(1 + \sqrt{D_k(X)})/2$ has the continued fraction expansion as given in (b) and $\ell((1 + \sqrt{D_k(X)})/2) = k(n+1)$. Moreover, by (2.9), $G_{\ell-1} = (A_k^2 X + B_k)/2$. This completes the proof. □

Remark 3.1: *We observe that in the proof of Theorem 3.1,* $A_k^2 D_k(X) = (A_k^2 X + B_k)^2 - 4$, *is an RD-type, and we verified that*

$$\varepsilon_{(A_k/2)^2 D_k(X)} = \frac{A_k^2 X + B_k}{4} + \frac{1}{4}\sqrt{A_k^2 D_k(X)} = \frac{A_k^2 X + B_k}{4} + \frac{A_k}{4}\sqrt{D_k(X)}.$$

This is the underlying kernel of the proof and shows that, even though ERD types are avoided for the $D_k(X)$ *since such types have period lengths no bigger than* 12, *we nevertheless exploit the order* $\mathbb{Z}[(1 + \sqrt{(A_k^2 + B_k)^2 - 4})/2]$ *in* $\mathbb{Z}[(1 + \sqrt{D_k(X)})/2]$ *to achieve our goals.*

Remark 3.2: *We see that for a fixed C (and so a fixed $n \in \mathbb{N}$), we may let $k \to \infty$ in which case $\ell((1+\sqrt{D_k(X)})/2) = \ell((1+\sqrt{D_k(1)})/2) \to \infty$ for all $X \in \mathbb{N}$. We also see that we have infinitely many distinct radicands $D_k(X)$ for a fixed $k \in \mathbb{N}$ with $\ell((1+\sqrt{D_k(X)})/2) = \ell((1+\sqrt{D_k(X+1)})/2)$ for all $X \in \mathbb{N}$.*

Remark 3.3: *It is well known that if a discriminant is of ERD-type, then $\log(\varepsilon_\Delta) \sim e(\log\sqrt{\Delta})$ where $e = 1$ or 2 (for example see [8]). Since $\varepsilon_{(A_k/2)^2 D_k(X)} = \varepsilon_{D_k(X)}$, (see Remark 3.1), then we necessarily have "small" regulator $R = \log(\varepsilon_{D_k(X)})$ compared to $\log(\sqrt{D_k(X)})$, and so "large" class number $h_{D_k(X)}$. For example, $h_{D_1(150)} = h_{9507773061} = 1656$, $R = \log(\varepsilon_{D_1(150)}) = 16.76068\ldots$, and $\log(\sqrt{9507773061}) = 11.48768\ldots$.*

We conclude with an application of Theorem 3.1 which motivated this paper's title. In [21], the authors found an infinite sequence of radicands $D_k = (2F_{6k}+1)^2 + (8F_{6k}+4)$, where F_j is the j^{th} Fibonacci number, such that $\ell((1+\sqrt{D_k})/2) = 6k+1$. By employing Theorem 3.1(a), we get

$$D_k(X) = 4F_{2k}^2X^2 + (20F_k^2 + 8(-1)^k)X + 5 = 4L_k^2F_k^2X^2 + (20F_k^2 + 8(-1)^k)X + 5$$

with $\ell((1+\sqrt{D_k(X)})/2) = 2k$ for any such k, where L_k is the k^{th} Lucas number.

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