DOMINATION IN FIBONACCI TREES

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ABSTRACT

The distance-k domination and independent domination numbers for Fibonacci trees are determined, both in terms of Fibonacci numbers and in closed form.

1. INTRODUCTION

Fibonacci trees T_n are defined recursively for all integers $n \in Z^+$ as follows: (1) T_1 and T_2 are the trivial trees, and (2) for $n \geq 3$, T_n is the rooted binary tree where T_{n-1} is the left subtree of the root and T_{n-2} is the right subtree. The root of T_n is designated r_n. The *level* of a vertex v of T_n, denoted $level(v)$, is $dist(v, r_n)$ where $dist$ is the standard distance function. Figure 1 shows T_1 through T_6.

Figure 1: The first six Fibonacci trees

Fibonacci trees have been studied by several researchers, including Knisely, Wallis, and Domke [4] who showed their edges can be colored red and blue so that the subgraph induced by the red edges is isomorphic to the subgraph induced by the blue edges, and Grimaldi [1] who investigated several properties of these trees including the distribution of leaves and internal vertices. In this paper we develop formulas for the distance-k domination and independent domination numbers of T_n. Since these invariants are normally NP-hard to evaluate, it is useful to have classes of graphs for which their value is known. This allows algorithms for general graphs to be tested on these classes in order to evaluate their performance.

Let $G = (V, E)$ be a graph. A subset $D \subseteq V$ is a *distance-k dominating set* of G if every vertex of V is distance at most k from some vertex in D. The size of a smallest such set is denoted $\gamma_{\leq k}(G)$ and a set of that size is a $\gamma_{\leq k}$*-set*. An excellent introduction to distance-k domination, due to Henning, can be found in Chapter 12 of [3]. An *independent dominating set* of graph G is a distance-1 dominating set in which no two vertices are adjacent. The minimum size of such a set is the *independent domination number*. It is denoted $\gamma_i(G)$, and an independent dominating set of that size is a γ_i*-set*.

Some notation and concepts are independent of the two types of domination. These are introduced here, using x to represent either "$\leq k$" or "i". Let D be a γ_x-set of T_n. We define D_i to be $D \cap V(T_{n-i})$ for $i = 1, 2$. A *private neighbor* of vertex $v \in D$ is a vertex which is distant-k dominated (independent dominated) by v and is not distance-k dominated (independent dominated) by any other vertex of D [2]. Note that v can be a private neighbor of itself. The smallest level of a vertex in D is denoted L_D, and we define $L_n = min\{L_D\colon D \text{ a } \gamma_x\text{-set of } T_n\}$. Let $\mathcal{D}_n = \{D : D \text{ a } \gamma_x\text{-set of } T_n \text{ with } L_D = L_n\}$, $X_D = \{v \in D : D \in \mathcal{D}_n \text{ and } level(v) = L_n\}$, and $B_D = max\{dist(v, w) : v \in X_D,\ level(w) \geq L_n, \text{ and } w \text{ is a private neighbor of } v\}$. Define $B_n = min\{B_D : D \in \mathcal{D}_n\}$. The ordered pair (L_n, B_n) is well-defined. Our approach to determining the distance-k and independent domination numbers of T_n will be to see how they change as the pair (L_n, B_n) changes with increasing n.

Section 2 determines the distance-k domination number of Fibonacci trees and Section 3 the independent domination number.

2. THE DISTANCE-k DOMINATION NUMBER

We begin with the following lemma which allows a useful restriction of the $\gamma_{\leq k}$-sets that we need to consider.

Lemma 1: *Let $\hat{D}$ be a $\gamma_{\leq k}$-set of T_n. Then there is an associated $\gamma_{\leq k}$-set D such that every vertex $v \in D$, with the possible exception of r_n, has a private neighbor w such that $level(w) = level(v) + k$.*

Proof: Set D to $\hat{D}$ and iteratively execute the following steps until it is no longer possible. Select $v \in D$ having the largest level and which does not have a private neighbor at $level(v)+k$. If $v \neq r_n$, replace v in D by its parent v'. Observe that v' distance-k dominates all the private neighbors of v. Upon completion, all vertices of D, with the possible exception of r_n, possess the required property. □

A $\gamma_{\leq k}$-set D of T_n satisfying the conditions of Lemma 1 and such that $(L_D, B_D) = (L_n, B_n)$ is said to be *representative*. Note that $L_n > 0$ implies, by Lemma 1, that $B_n = k$.

Some additional notation will be useful. Since T_n includes both T_{n-1} and T_{n-2} as induced subgraphs, it will be convenient to refer to T_{n-i} where $i = 1$ or 2 and then let $c(i) = 3 - i$. Thus T_{n-i} and $T_{n-c(i)}$ refer, respectively, to T_{n-1} and T_{n-2} if $i = 1$ and to T_{n-2} and T_{n-1} if $i = 2$. The *open distance-k neighborhood* of a vertex $x \in V$ is $N_k(x) = \{v \in V : 0 < dist(v, x) \leq k\}$, and the *closed distance-k neighborhood* of x is $N_k[x] = N_k(x) \cup \{x\}$. Similarly, the *open distance-k neighborhood* of $S \subseteq V$ is $N_k(S) = \cup_{x \in S} N_k(x)$ and the *closed distance-k neighborhood* of S is $N_k[S] = N_k(S) \cup S$.

Let D be a $\gamma_{\leq k}$-set of T_n. In all of the following, the index i is either 1 or 2. A vertex $x \in D - D_i$ is called a *helper* if $N_k(x) \cap \{V(T_{n-i}) - N_k[D_i]\} \neq \emptyset$. The next lemma shows at most one helper need be considered.

Lemma 2: *If there is at least one helper, then there is a helper x such that, for any other helper $y \in D_i$, $N_k[x] \cap V(T_{n-c(i)}) \supseteq N_k(y) \cap V(T_{n-c(i)})$.*

Proof: Let x be a helper closest to r_n and $y \in D_i$ be any other helper. Since $dist(x, r_n) \leq dist(y, r_n)$, x distance-k dominates all vertices of $T_{n-c(i)}$ that y does. □

The following sequence of lemmas provides insight into the $\gamma_{\leq k}$-sets of T_n.

Lemma 3: $|D_i| \geq \gamma_{\leq k}(T_{n-i}) - 1$.

Proof: Suppose $|D_i| \leq \gamma_{\leq k}(T_{n-i})-2$. Then at least two helpers are required to distance-k dominate T_{n-i}, which violates Lemma 2. □

Lemma 4: $\gamma_{\leq k}(T_n) \geq \gamma_{\leq k}(T_{n-1}) + \gamma_{\leq k}(T_{n-2}) - 1$.

Proof: Let D be a $\gamma_{\leq k}$-set of T_n. Certainly $\gamma_{\leq k} \geq |D_1| + |D_2|$. If $|D_i| \geq \gamma_{\leq k}(T_{n-i})$ for some i, we are done by Lemma 3. Otherwise $|D_i| < \gamma_{\leq k}(T_{n-i})$ for $i = 1, 2$ and a helper is required. By Lemma 2, only one helper x is useful. If $x = r_n$, we are done. If $x \in D_i$ for some i, D_i is a distance-k dominating set of T_{n-i} which is smaller than $\gamma_{\leq k}(T_{n-i})$, a contradiction. □

The next lemma and corollary are helpful when at least one $B_{n-i} = k$.

Lemma 5: *If $B_{n-i} = k$, $|D_i| \geq \gamma_{\leq k}(T_{n-i})$.*

Proof: Suppose $|D_i| = \gamma_{\leq k}(T_{n-i})-1$, so a helper $x \in \{r_n\} \cup D_{c(i)}$ is necessary to distance-k dominate T_{n-i}. Then $N_k[D_i \cup \{x\}] \supseteq V(T_{n-i})$, which implies $N_k[D_i \cup \{r_{n-i}\}] \supseteq V(T_{n-i})$. Thus $D_i \cup \{r_{n-i}\}$ is a $\gamma_{\leq k}$-set of T_{n-i}. By assumption, r_{n-i} must have a private neighbor at level k of T_{n-1} which is at level $k+1$ of T_n and hence can't be distance-k dominated by x, a contradiction. □

Corollary 6: *Suppose r_n is in a $\gamma_{\leq k}$-set of T_n and $B_{n-1} = B_{n-2} = k$. Then $\gamma_{\leq k}(T_n) \geq \gamma_{\leq k}(T_{n-1}) + \gamma_{\leq k}(T_{n-2}) + 1$.*

The following four lemmas provide the basis for determining the distance-k domination number of T_n. They address only a subset of the possible conditions which conceivably could arise. However, this subset is exactly what is needed.

Lemma 7: *Suppose $n \geq 3$, $(L_{n-2}, B_{n-2}) = (0, t)$, and $(L_{n-1}, B_{n-1}) = (0, t+1)$ where $0 \leq t \leq k-2$. Then $\gamma_{\leq k}(T_n) = \gamma_{\leq k}(T_{n-1}) + \gamma_{\leq k}(T_{n-2}) - 1$ and $(L_n, B_n) = (0, t+2)$.*

Proof: Let D_{n-1} and D_{n-2} be representative $\gamma_{\leq k}$-sets of T_{n-1} and T_{n-2}, respectively. Observe that $[(D_{n-2} \cup D_{n-1}) - \{r_{n-1}, r_{n-2}\}] \cup \{r_n\}$ is a $\gamma_{\leq k}$-set by Lemma 4, and it also satisfies Lemma 1. Obviously $L_n = 0$ is as small as possible. Suppose $B_n < t+2$. Let D be a representative $\gamma_{\leq k}$-set of T_n. Since $L_n = 0$, $r_n \in D$ which implies $|D_i| = \gamma_{\leq k}(T_{n-i}) - 1$ for $i = 1, 2$ and $D_1 \cup \{r_n\}$ distance-k dominates T_{n-1}. Thus $\hat{D} = D_1 \cup \{r_{n-1}\}$ is a $\gamma_{\leq k}$-set of T_{n-1} satisfying Lemma 1. Furthermore, $B_{n-1} \leq b_{\hat{D}} \leq B_n - 1 < t+1 = B_{n-1}$, a contradiction. □

Lemma 8: *Suppose $n \geq 3$. If $(L_{n-2}, B_{n-2}) = (0, k-1)$ and $(L_{n-1}, B_{n-1}) = (0, k)$, then $(L_n, B_n) = (0, k)$. If $(L_{n-2}, B_{n-2}) = (k, k)$ and $(L_{n-1}, B_{n-1}) = (0, 0)$, then $(L_n, B_n) = (0, 1)$. In either case, $\gamma_{\leq k}(T_n) = \gamma_{\leq k}(T_{n-1}) + \gamma_{\leq k}(T_{n-2})$.*

Proof: Again let D_{n-1} and D_{n-2} be representative $\gamma_{\leq k}$-sets of T_{n-1} and T_{n-2}, respectively. Let $min(B_{n-2}, B_{n-1}) = B_{n-i}$. In this case $[(D_{n-2} \cup D_{n-1}) - \{r_{n-i}\}] \cup \{r_n\}$ is a $\gamma_{\leq k}$-set which satisfies Lemma 1. To see its minimality, observe that $|D_{n-c(i)}| \geq \gamma_{\leq k}(T_{n-c(i)})$

by Lemma 5. Furthermore, in these two cases no vertex of $D_{n-c(i)}$ can distance-k dominate all the private neighbors of r_{n-i} in T_{n-i}. Therefore no helper exists in $D_{n-c(i)}$ and $\gamma_{\leq k}(T_n) = \gamma_{\leq k}(T_{n-1}) + \gamma_{\leq k}(T_{n-2})$. The above $\gamma_{\leq k}$-set shows $L_n = 0$, implying r_n is in any representative $\gamma_{\leq k}$-set. Since $|D_{n-c(i)}| \geq \gamma_{\leq k}(T_{n-c(i)})$, it follows that $|D_{n-i}| = \gamma_{\leq k}(T_{n-i}) - 1$. The value of B_n now follows from an argument similar to that in the proof of Lemma 7. □

Lemma 9: *Suppose* $n \geq 3$, $B_{n-1} = B_{n-2} = k$, *and* $min(L_{n-1}, L_{n-2}) = L_{n-i} < k$. *Then* $\gamma_{\leq k}(T_n) = \gamma_{\leq k}(T_{n-1}) + \gamma_{\leq k}(T_{n-2})$ *and* $(L_n, B_n) = (L_{n-i} + 1, k)$.

Proof: Let D_{n-1} and D_{n-2} be representative $\gamma_{\leq k}$-sets of T_{n-1} and T_{n-2}, respectively. Then $D = D_{n-2} \cup D_{n-1}$ is a $\gamma_{\leq k}$-set by Lemma 5 and it satisfies Lemma 1. Since r_n is in no $\gamma_{\leq k}$-set by Corollary 6, D is a representative $\gamma_{\leq k}$-set and the values for (L_n, B_n) follow immediately. □

Lemma 10: *Suppose* $n \geq 3$ *and* $(B_{n-2}, L_{n-2}) = (B_{n-1}, L_{n-1}) = (k, k)$. *Then* $\gamma_{\leq k}(T_n) = \gamma_{\leq k}(T_{n-1}) + \gamma_{\leq k}(T_{n-2}) + 1$ *and* $(L_n, B_n) = (0, 0)$.

Proof: Let D_{n-1} and D_{n-2} be representative $\gamma_{\leq k}$-sets of T_{n-1} and T_{n-2}, respectively. Then $D = D_{n-2} \cup D_{n-1} \cup \{r_n\}$ is a representative $\gamma_{\leq k}$-set by Corollary 6 and the fact that no $\gamma_{\leq k}$-set of T_{n-i} can distance-k dominate r_n for $i = 1, 2$. Clearly $(L_n, B_n) = (0, 0)$. □

Lemmas 7, 8, 9, and 10 can now be employed to determine the distance-k domination number of T_n. It may be helpful in following the reasoning to be discussed below to refer to Table 1 which gives results for $k = 5$. Each entry of the table is composed of four parts: the tree T_n; (L_n, B_n); $\gamma_{\leq k}(T_n)$, usually in terms of Fibonacci numbers; and the lemma leading to that result.

The computation of T_n for general k is now described. Observe that the height of T_1 is 0 and the height of T_n is $n - 2$ for $n \geq 2$. It follows at once that $\gamma_{\leq k}(T_n) = 1$ for $1 \leq n \leq k$ and $(L_n, B_n) = (0, n - 2)$ (except $(L_1, B_1) = (0, 0)$).

Beginning with $n = k+1$ and assuming knowledge of $\gamma_{\leq k}(T_{n-1})$ and $\gamma_{\leq k}(T_{n-2})$, the values $\gamma_{\leq k}(T_n)$ and (L_n, B_n) are determined recursively by a reasoning based only on the values of (L_{n-1}, B_{n-1}) and (L_{n-2}, B_{n-2}). It follows that if, for some positive integer t, $(L_{n-2}, B_{n-2}) = (L_{n+t-2}, B_{n+t-2})$ and $(L_{n-1}, B_{n-1}) = (L_{n+t-1}, B_{n+t-1})$, there will be a cyclic repetition of the reasoning and the cycle will have length t. We will see that $t = 3k + 2$. Table 1 illustrates this by presenting each cycle (of length $3k + 2 = 17$) as a column. Notice that along any row the lemma employed stays the same. Of course, there are an infinite number of columns, but exactly $3k + 2 = 17$ rows.

The method for computing T_n, $n \geq k + 1$, is given in the following steps for $k \geq 3$, steps that are repeated for each cycle. When $k = 1$ or $k = 2$, a similar but not identical pattern occurs. Tables 2 and 3 present results for those two cases, respectively. These cases are discussed briefly below. The final Theorem 11 and its corollary are valid for all $k \geq 1$. In the following, the variable i, $i \geq 0$, can be interpreted as the column of the table in which the corresponding result would be displayed. The first column is column 0. We will see that $\gamma_{\leq k}(T_{k+1})$ and $\gamma_{\leq k}(T_{k+2})$ are Fibonacci numbers. It follows from Lemmas 7 to 10 that $\gamma_{\leq k}(T_n)$ is a sum of Fibonacci numbers for all successive n, with small modifications due to the plus or minus 1 that the lemmas sometimes require.

1. Two applications of Lemma 7 so, for $k + 1 + i(3k + 2) \leq n \leq k + 2 + i(3k + 2)$, $(L_{k+1+i(3k+2)}, B_{k+1+i(3k+2)}) = (0, k - 1)$ and $(L_{k+2+i(3k+2)}, B_{k+2+i(3k+2)}) = (0, k)$. Now $\gamma_{\leq k}(T_{n-2})$ and $\gamma_{\leq k}(T_{n-1})$ each include a 1 in their sum. Thus a 1 also will result in the

sum for these cases. When $n = k + 1 + i(3k + 2)$ we interpret the 1 as F_1, and when $n = k + 2 + i(3k + 2)$ we interpret the 1 as F_2.

2. One application of Lemma 8 so $(L_{k+3+i(3k+2)}, B_{k+3+i(3k+2)}) = (0, k)$.
3. $2k$ applications of Lemma 9 so, for $k + 4 + i(3k + 2) \le n \le 3k + 3 + i(3k + 2)$, (L_n, B_n) ranges over the values $(1, k), (1, k), (2, k), (2, k), \ldots, (k, k), (k, k)$.
4. One application of Lemma 10 yields $(L_{3k+4+i(3k+2)}, B_{3k+4+i(3k+2)}) = (0, 0)$. This adds a 1 to the sum of Fibonacci numbers.
5. One application of Lemma 8, so $(L_{3k+5+i(3k+2)}, B_{3k+5+i(3k+2)}) = (0, 1)$. The added 1 remains.
6. $k-3$ applications of Lemma 7, so, for $3k+6+i(3k+2) \le n \le 4k+2+i(3k+2)$, (L_n, B_n) ranges from $(0, 2)$ to $(0, k-2)$. The added 1 stays on all of the sums.

It can be checked that exactly $3k+2$ values are determined by one pass through the above steps. When $k = 1$, $\gamma_{\le k}(T_2) = \gamma_{\le k}(T_3) = 1$. Furthermore, step 4 results in $(L_n, B_n) = (0, 0)$ which is a repeat of the value for T_2, so this represents the beginning of a new cycle. Thus, any cycle beginning with the second employs only step 4 for one value, step 2 for two values, and step 3 for two values. The cycle then has length $5 = 3k + 2$. When $k = 2$, $\gamma_{\le k}(T_3) = 1$. In this case, step 5 produces the value which starts a new cycle, so, beginning with the second cycle, we employ step 5 for one value, step 1 but only for one value, step 2 for one value, step 3 for four values, and step 4 for one value. Hence each cycle has length $8 = 3k + 2$.

T_6	(0,4)	F_1	(7)	T_{23}	(0,4)	$F_{18}+F_1$	(7)	T_{40}	(0,4)	$F_{35}+F_{18}+F_1$	(7)
T_7	(0,5)	F_2	(7)	T_{24}	(0,5)	$F_{19}+F_2$	(7)	T_{41}	(0,5)	$F_{36}+F_{19}+F_2$	(7)
T_8	(0,5)	F_3	(8)	T_{25}	(0,5)	$F_{20}+F_3$	(8)	T_{42}	(0,5)	$F_{37}+F_{20}+F_3$	(8)
T_9	(1,5)	F_4	(9)	T_{26}	(1,5)	$F_{21}+F_4$	(9)	T_{43}	(1,5)	$F_{38}+F_{21}+F_4$	(9)
T_{10}	(1,5)	F_5	(9)	T_{27}	(1,5)	$F_{22}+F_5$	(9)	T_{44}	(1,5)	$F_{39}+F_{22}+F_5$	(9)
T_{11}	(2,5)	F_6	(9)	T_{28}	(2,5)	$F_{23}+F_6$	(9)	T_{45}	(2,5)	$F_{40}+F_{23}+F_6$	(9)
T_{12}	(2,5)	F_7	(9)	T_{29}	(2,5)	$F_{24}+F_7$	(9)	T_{46}	(2,5)	$F_{41}+F_{24}+F_7$	(9)
T_{13}	(3,5)	F_8	(9)	T_{30}	(3,5)	$F_{25}+F_8$	(9)	T_{47}	(3,5)	$F_{42}+F_{25}+F_8$	(9)
T_{14}	(3,5)	F_9	(9)	T_{31}	(3,5)	$F_{26}+F_9$	(9)	T_{48}	(3,5)	$F_{43}+F_{26}+F_9$	(9)
T_{15}	(4,5)	F_{10}	(9)	T_{32}	(4,5)	$F_{27}+F_{10}$	(9)	T_{49}	(4,5)	$F_{44}+F_{27}+F_{10}$	(9)
T_{16}	(4,5)	F_{11}	(9)	T_{33}	(4,5)	$F_{28}+F_{11}$	(9)	T_{50}	(4,5)	$F_{45}+F_{28}+F_{11}$	(9)
T_{17}	(5,5)	F_{12}	(9)	T_{34}	(5,5)	$F_{29}+F_{12}$	(9)	T_{51}	(5,5)	$F_{46}+F_{29}+F_{12}$	(9)
T_{18}	(5,5)	F_{13}	(9)	T_{35}	(5,5)	$F_{30}+F_{13}$	(9)	T_{52}	(5,5)	$F_{47}+F_{30}+F_{13}$	(9)
T_{19}	(0,0)	$F_{14}+1$	(10)	T_{36}	(0,0)	$F_{31}+F_{14}+1$	(10)	T_{53}	(0,0)	$F_{48}+F_{31}+F_{14}+1$	(10)
T_{20}	(0,1)	$F_{15}+1$	(8)	T_{37}	(0,1)	$F_{32}+F_{15}+1$	(8)	T_{54}	(0,1)	$F_{49}+F_{32}+F_{15}+1$	(8)
T_{21}	(0,2)	$F_{16}+1$	(7)	T_{38}	(0,2)	$F_{33}+F_{16}+1$	(7)	T_{55}	(0,2)	$F_{50}+F_{33}+F_{16}+1$	(7)
T_{22}	(0,3)	$F_{17}+1$	(7)	T_{39}	(0,3)	$F_{34}+F_{17}+1$	(7)	T_{56}	(0,3)	$F_{51}+F_{34}+F_{17}+1$	(7)

Table 1: $\gamma_{\le k}(T_n)$ for $6 \le n \le 56$ and $k = 5$

T_2	(0,0)	F_1		T_7	(0,0)	F_6+F_1	(10)	T_{12}	(0,0)	$F_{11}+F_6+F_1$	(10)
T_3	(0,1)	F_2		T_8	(0,1)	F_7+F_2	(8)	T_{13}	(0,1)	$F_{12}+F_7+F_2$	(8)
T_4	(0,1)	F_3	(8)	T_9	(0,1)	F_8+F_3	(8)	T_{14}	(0,1)	$F_{13}+F_8+F_3$	(8)
T_5	(1,1)	F_4	(9)	T_{10}	(1,1)	F_9+F_4	(9)	T_{15}	(1,1)	$F_{14}+F_9+F_4$	(9)
T_6	(1,1)	F_5	(9)	T_{11}	(1,1)	$F_{10}+F_5$	(9)	T_{16}	(1,1)	$F_{15}+F_{10}+F_5$	(9)

Table 2: $\gamma_{\le k}(T_n)$ for $2 \le n \le 16$ and $k = 1$

T_3	(0,1)	F_1		T_{11}	(0,1)	F_9+F_1	(8)	T_{19}	(0,1)	$F_{17}+F_9+F_1$	(8)
T_4	(0,2)	F_2	(7)	T_{12}	(0,2)	$F_{10}+F_2$	(7)	T_{20}	(0,2)	$F_{18}+F_{10}+F_2$	(7)
T_5	(0,2)	F_3	(8)	T_{13}	(0,2)	$F_{11}+F_3$	(8)	T_{21}	(0,2)	$F_{19}+F_{11}+F_3$	(8)
T_6	(1,2)	F_4	(9)	T_{14}	(1,2)	$F_{12}+F_4$	(9)	T_{22}	(1,2)	$F_{20}+F_{12}+F_4$	(9)
T_7	(1,2)	F_5	(9)	T_{15}	(1,2)	$F_{13}+F_5$	(9)	T_{23}	(1,2)	$F_{21}+F_{13}+F_5$	(9)
T_8	(2,2)	F_6	(9)	T_{16}	(2,2)	$F_{14}+F_6$	(9)	T_{24}	(2,2)	$F_{22}+F_{14}+F_6$	(9)
T_9	(2,2)	F_7	(9)	T_{17}	(2,2)	$F_{15}+F_7$	(9)	T_{25}	(2,2)	$F_{23}+F_{15}+F_7$	(9)
T_{10}	(0,0)	F_8+1	(10)	T_{18}	(0,0)	$F_{16}+F_8+1$	(10)	T_{26}	(0,0)	$F_{24}+F_{16}+F_8+1$	(10)

Table 3: $\gamma_{\le k}(T_n)$ for $3 \le n \le 26$ and $k = 2$

We now can calculate the value of $\gamma_{\le k}(T_n)$.

Theorem 11: *For $1 \le n \le k$, $\gamma_{\le k}(T_n) = 1$. For $n \ge k+1$,*

$$\gamma_{\le k}(T_n) = \sum_{i=0}^{\lfloor \frac{n-k-1}{3k+2} \rfloor} F_{n-k-i(3k+2)} + \epsilon$$

where $\epsilon = 1$ if $2 + (3k+2)i \le n \le k + (3k+2)i$ for any nonnegative integer $i \ge 1$ and $\epsilon = 0$ otherwise.

Proof: The value of ϵ results from the fact that in each complete cycle (in each column) the last $k-1$ domination numbers include an added 1. Notice that $\gamma_{\le k}(T_n)$ includes the sum of F_{n-k} and successively smaller Fibonacci numbers whose indices differ by $3k+2$. The number of Fibonacci numbers involved is easily seen to be $\left\lfloor \frac{n-k-1}{3k+2} \right\rfloor + 1$ which accounts for the limits on the sum. $\square$

A closed form can be found. Let $\alpha = \frac{1+\sqrt{5}}{2}$ and $\beta = \frac{1-\sqrt{5}}{2}$.

Corollary 12: *For $1 \le n \le k$, $\gamma_{\le k}(T_n) = 1$. For $n \ge k+1$,*

$$\gamma_{\le k}(T_n) = \frac{1}{\sqrt{5}} \left\{ \frac{\alpha^{n+2k+2} - \alpha^{n-k-\lfloor \frac{n-k-1}{3k+2} \rfloor (3k+2)}}{\alpha^{3k+2} - 1} \right\}$$

$$-\frac{1}{\sqrt{5}} \left\{ \frac{\beta^{n+2k+2} - \beta^{n-k-\lfloor \frac{n-k-1}{3k+2} \rfloor (3k+2)}}{\beta^{3k+2} - 1} \right\} + \epsilon.$$

Proof: It is known that $F_n = \frac{\alpha^n - \beta^n}{\sqrt{5}}$. Using this to replace the Fibonacci numbers in the sum of Theorem 11 and recognizing this produces two finite geometric sums, one involving α and one β, yields

$$\gamma_{\le k}(T_n) = \frac{1}{\sqrt{5}} \left\{ \alpha^{n-k-\lfloor \frac{n-k-1}{3k+2} \rfloor (3k+2)} \left[\frac{\alpha^{(3k+2)\left(\lfloor \frac{n-k-1}{3k+2} \rfloor + 1\right)} - 1}{\alpha^{3k+2} - 1} \right] \right\}$$

$$-\frac{1}{\sqrt{5}}\left\{\beta^{n-k-\lfloor\frac{n-k-1}{3k+2}\rfloor(3k+2)}\left[\frac{\beta^{(3k+2)\left(\lfloor\frac{n-k-1}{3k+2}\rfloor+1\right)}-1}{\beta^{3k+2}-1}\right]\right\}+\epsilon.$$

The result then follows from multiplication by the term before the square brackets. □

We specialize Theorem 11 and Corollary 12 for the case $k = 1$ since this corresponds to the standard domination number γ.

Corollary 13: $\gamma(T_1) = 1$. *For* $n \geq 2$,

$$\gamma(T_n) = \sum_{i=0}^{\lfloor\frac{n-2}{5}\rfloor} F_{n-1-5i} = \frac{1}{\sqrt{5}}\left\{\frac{\alpha^{n+4}-\alpha^{n-1-5\lfloor\frac{n-2}{5}\rfloor}}{\alpha^5-1}\right\} - \frac{1}{\sqrt{5}}\left\{\frac{\beta^{n+4}-\beta^{n-1-5\lfloor\frac{n-2}{5}\rfloor}}{\beta^5-1}\right\}$$

3. THE INDEPENDENT DOMINATION NUMBER

Figure 2 shows γ_i-sets (indicated by the circled vertices) for T_1 through T_6.

Figure 2: γ_i-sets for the first six Fibonacci trees

For independent domination there are three possible ordered pairs (L_n, B_n): (1) $(0,0)$ meaning r_n is in a γ_i-set and has only itself as a private neighbor, (2) $(0,1)$ meaning r_n is in a γ_i-set and has at least one child as a private neighbor, and (3) $(1,1)$ meaning no γ_i-set contains r_n. Notice that T_3 has the pair $(0,1)$, T_4 and T_5 have $(1,1)$, and T_6 has $(0,0)$. We will see that, with the exception of T_3, the pair $(0,1)$ will apply to trees where it is possible for a γ_i-set to have exactly one child of r_n as a private neighbor of r_n. T_7 is the first example of this. The following two lemmas lead to the independent domination number of T_n.

Lemma 14: *Let* D *be a* γ_i*-set of* T_n. *If* $B_{n-i} = 1$ *for* $i = 1$ *or* 2, *then* $|D_i| \geq \gamma_i(T_{n-i})$ *and* $\gamma_i(T_n) \geq \gamma_i(T_{n-1}) + \gamma_i(T_{n-2})$.

Proof: Suppose $|D_i| < \gamma_i(T_{n-i})$. Then $D_i \cup \{r_n\}$ is an independent dominating set of T_{n-i} and r_n is necessary to dominate only r_{n-i}. Thus neither child of r_{n-i} is in D_i, so $D_i \cup \{r_{n-i}\}$ is an independent dominating set of T_{n-i} in which r_{n-i} has only itself as a private neighbor, contradicting the fact that $B_{n-i} = 1$. Since $|D_i| \geq \gamma_i(T_{n-i})$ and no vertex of D_i dominates a vertex of $T_{n-c(i)}$, it takes an additional $\gamma_i(T_{n-c(i)})$ vertices of D to dominate $T_{n-c(i)}$ and the bound on $\gamma_i(T_n)$ follows. □

Let us call the pair $(0,1)$ *proper* if, in a γ_i-set of T_n which corresponds to this pair, the root has exactly one child which is a private neighbor.

Lemma 15: *The following transitions hold:*

1. *If* $(L_{n-1}, B_{n-1}) = (L_{n-2}, B_{n-2}) = (1,1)$, *then* $(L_n, B_n) = (0,0)$ *and* $\gamma_i(T_n) = \gamma_i(T_{n-1}) + \gamma_i(T_{n-2}) + 1$.
2. *If* $(L_{n-1}, B_{n-1}) = (0,0)$ *and* $(L_{n-2}, B_{n-2}) = (1,1)$, *then* $(L_n, B_n) = (0,1)$ *is proper and* $\gamma_i(T_n) = \gamma_i(T_{n-1}) + \gamma_i(T_{n-2})$.
3. *If* $(L_{n-1}, B_{n-1}) = (0,1)$ *is proper and* $(L_{n-2}, B_{n-2}) = (0,0)$, *then* $(L_n, B_n) = (0,1)$ *is proper and* $\gamma_i(T_n) = \gamma_i(T_{n-1}) + \gamma_i(T_{n-2})$.
4. *If* $(L_{n-i}, B_{n-i}) = (0,1)$ *and* $B_{n-c(i)} = 1$, *then* $(L_n, B_n) = (1,1)$ *and* $\gamma_i(T_n) = \gamma_i(T_{n-1})$ $+\gamma_i(T_{n-2})$.

Proof: Suppose D is a γ_i-set of T_n. If $(L_n, B_n) = (0,0)$, then $\gamma_i(T_n) \geq \gamma_i(T_{n-1}) + \gamma_i(T_{n-2}) + 1$ since in this case D_i independently dominates T_{n-i} for $i = 1,2$. Assume D_{n-1} and D_{n-2} are γ_i-sets of T_{n-1} and T_{n-2}, respectively, with the stated ordered pairs. For Parts 2 through 4 we will demonstrate an independent dominating set of size $\gamma_i(T_{n-1}) + \gamma_i(T_{n-2})$ which, by Lemma 14, must be a γ_i-set of T_n.

1. $D_{n-1} \cup D_{n-2} \cup \{r_n\}$ is a γ_i-set of T_n since $|D_{n-i}| \geq \gamma_i(T_{n-i})$ for $i = 1,2$ by Lemma 14 and, because $L_{n-i} = 1$, D_{n-i} cannot dominate r_n. Clearly $(L_n, B_n) = (0,0)$.
2. $(D_{n-1} - \{r_{n-1}\}) \cup D_{n-2} \cup \{r_n\}$ is a γ_i-set of T_n. Root r_n has r_{n-1} as a private neighbor but not r_{n-2}. $(L_n, B_n) \neq (0,0)$ by the remark at the beginning of the proof. Thus $(L_n, B_n) = (0,1)$ and is proper.
3. $(D_{n-1} - \{r_{n-1}\}) \cup (D_{n-2} - \{r_{n-2}\}) \cup \{r_n, x\}$ is a γ_i-set of T_n, where x is the private neighbor child of r_{n-1}. Then r_n has r_{n-2} as a private neighbor but not r_{n-1}. Thus $(L_n, B_n) = (0,1)$ is proper by an argument similar to that in the proof of Part 2.
4. $D_{n-1} \cup D_{n-2}$ is a γ_i-set. Since $B_{n-i} = 1$ for $i = 1,2$, no γ_i-set can contain r_n so $(L_n, B_n) = (1,1)$. □

It is straightforward to show that $\gamma_i(T_1) = 1$, $\gamma_i(T_2) = 1 = F_1$, $\gamma_i(T_3) = 1 = F_2$, $\gamma_i(T_4) = 2 = F_3$, and $\gamma_i(T_5) = 3 = F_4$. Furthermore, $(L_4, B_4) = (L_5, B_5) = (1,1)$. Results for T_n, $n \geq 6$, can be obtained by cycling in order through the parts of Lemma 15. Table 4 shows information for $n \leq 15$. The format is similar to that of Table 1, except now the fourth column for each entry refers to the part of Lemma 15 which is employed. The entry $(0,1)$p indicates it is proper.

T_1	(0,0)	1								
T_2	(0,0)	F_1	T_6	(0,0)	F_5+F_1	(1)	T_{11}	(0,0)	$F_{10}+F_6+F_1$	(1)
T_3	(0,1)	F_2	T_7	(0,1)p	F_6+F_2	(2)	T_{12}	(0,1)p	$F_{11}+F_7+F_2$	(2)
T_4	(1,1)	F_3	T_8	(0,1)p	F_7+F_3	(3)	T_{13}	(0,1)p	$F_{12}+F_8+F_3$	(3)
T_5	(1,1)	F_4	T_9	(1,1)	F_8+F_4	(4)	T_{14}	(1,1)	$F_{13}+F_9+F_4$	(4)
			T_{10}	(1,1)	F_9+F_5	(4)	T_{15}	(1,1)	$F_{14}+F_{10}+F_5$	(4)

Table 4: $\gamma_i(T_n)$ for $1 \leq n \leq 15$

It is now possible to state the value of $\gamma_i(T_n)$. The approach mimics that of Theorem 11 and Corollary 12.

Theorem 16: $\gamma_i(T_1) = 1$. *For* $n \geq 2$,

$$\gamma_i(T_n) = F_{n-1} + \sum_{i=1}^{\lfloor \frac{n-1}{5} \rfloor} F_{n-5i}.$$

Corollary 17: $\gamma_i(T_1) = 1$. *For* $n \geq 2$,

$$\gamma_i(T_n) = \frac{1}{\sqrt{5}}\left(\alpha^{n-1} - \beta^{n-1}\right) + \frac{1}{\sqrt{5}}\left[\frac{\alpha^n - \alpha^{n-5\lfloor \frac{n-1}{5} \rfloor}}{\alpha^5 - 1}\right] - \frac{1}{\sqrt{5}}\left[\frac{\beta^n - \beta^{n-5\lfloor \frac{n-1}{5} \rfloor}}{\beta^5 - 1}\right].$$

4. CONCLUDING REMARKS

Of course, the techniques developed here could be applied to other domination parameters, with total domination being a reasonable one to examine. It is expected that all would result in answers involving the Fibonacci numbers. Perhaps a more interesting investigation, especially if it might have applications or provide insights, is to repeat the study for other recurrence relations of the form $a_n = \sum_{i=1}^{k} c_i a_{n-i}$ where the c_i are nonnegative integers. In this case, some structure would have to be defined for the first k trees.

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