FURTHER PROPERTIES OF GENERALIZED BINOMIAL COEFFICIENT k-EXTENSIONS

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1. INTRODUCTION

Ollerton and Shannon [3] have developed "k-extensions" of the generalized binomial coefficients, written $\binom{n}{m}_q^k$ with $n, m, q \geq 0$ and $0 \leq k = abc_2 \leq 7$, by considering the ways in which m objects can be placed in n cells, each of which holds a maximum of q objects, according to certain rules indicated by the binary variables a, b, c (where $1 \equiv$ Yes, $0 \equiv$ No). These correspond to:

a. whether empty cells give new arrangements (if not then empty cells may be placed at the end of the arrangement),
b. whether the order of objects across the cells is free (if not then each cell must contain objects all of which are less than those in the next non-blank cell to the right), and
c. whether the order of objects within the cells gives new arrangements (if not then they may be listed in increasing order).

k-extensions thus have very concrete combinatorial interpretations and are a natural development of the usual generalized binomial coefficients described by Bondarenko [1]. These latter have $k = 4$ (whence $q = 1$ gives the ordinary binomial coefficients). A unifying recurrence relation is given by

$$\binom{n}{m}_q^k = \sum_{i=l-a}^{q} C_{ib}^{m} i!^c \binom{n-1}{m-i}_q^k \tag{1.1}$$

for $0 \leq k = abc_2 \leq 7$, n & $m > 0$ and $q \geq 0$, with boundary conditions

$$\binom{n}{m}_q^k = \begin{cases} 0 & \begin{cases} n < 0 \text{ or } m < 0 \\ n = 0 \ \& \ m > 0 \end{cases} \\ 1 & n \geq 0 \ \& \ m = 0. \end{cases}$$

For each k and q, arrays can be formed with rows given by n and columns given by m. Table 1 shows the 0-extension for $q = 2$. A number of properties of the k-extensions were developed in [3], including row and diagonal sum recurrence relationships as well as the generating function:

$$\sum_{m=0}^{qn} m!^{-b} \binom{n}{m}_q^k x^m = \begin{cases} \frac{T_q^k(x)^{n+1}-1}{T_q^k(x)-1} & (= n+1 \text{ for } T_q^k(x) = 1) \quad k = 0 \text{ to } 3 \\ T_q^k(x)^n & k = 4 \text{ to } 7 \end{cases} \tag{1.2}$$

where $T_q^k(x) = \sum_{i=l-a}^{q} i!^{c-b} x^i$.

In the following, we develop further properties of these extensions by use of the generating function. Diagonal sums of the arrays produced by the k-extensions may be considered to be extensions of the corresponding generalized Fibonacci sequences. We also determine the diagonal sum generating functions and describe some of their properties.

2. FURTHER k-EXTENSION PROPERTIES

Let $g(k,n,q;x) = \sum_{m=0}^{qn} m!^{-b} \binom{n}{m}_q^k x^m$.

i. Differentiating (1.2) with respect to x leads to

$$g'(k,n,q;x) = \sum_{m=1}^{qn} m!^{-b} \binom{n}{m}_q^k m x^{m-1}$$

$$= \begin{cases} \frac{nT_q^k(x)^{n+1} - (n+1)T_q^k(x)^n + 1}{(T_q^k(x)-1)^2} T_q'^k(x) & k = 0 \text{ to } 3 \\ \left(= \frac{n(n+1)}{2} T_q'^k(x) \text{ for } T_q^k(x) = 1\right) & \\ nT_q^k(x)^{n-1} T_q'^k(x) & k = 4 \text{ to } 7, \end{cases}$$

noting that $T_q'^k(x) = \sum_{i=1}^{q} i!^{c-b} i x^{i-1}$.

For example, setting $x = 1$ and $b = c$ ($k = 0, 3, 4, 7$) gives

$$T_q^k(1) = q + a, \ T_q'^k(1) = q(q+1)/2 \text{ and}$$

$$\sum_{m=1}^{qn} m!^{-b} \binom{n}{m}_q^k m = \begin{cases} \frac{nq^{n+1} - (n+1)q^n + 1}{(q-1)^2} q(q+1)/2 \quad \left(= \frac{n(n+1)}{2} \text{ for } q = 1\right) & k = 0, 3 \\ nq(q+1)^n/2 & k = 4, 7 \end{cases}$$

which is well-known for $k = 4$. For $k = 0$ and $q = 2$ in particular,

$$\sum_{m=1}^{2n} \binom{n}{m}_2^0 m = 3(1 + (n-1)2^n)$$

which may be verified using Table 1.

ii. Recurrence relations in terms of q may also be developed from (1.2). For $k = 4$ to 7 and $q \geq 1$,

$$g(k,n,q;x) = T_q^k(x)^n = (T_{q-1}^k(x) + q!^{c-b} x^q)^n$$

$$= \sum_{j=0}^{n} C_j^n T_{q-1}^k(x)^j (q!^{c-b} x^q)^{n-j}$$

$$= \sum_{j=0}^{n} C_j^n (q!^{c-b} x^q)^{n-j} g(k, j, q-1; x)$$

with $g(k,n,0;x) = T_0^k(x)^n = \left(\sum_{i=0}^{0} i!^{c-b}x^i\right)^n = 1$. Setting $x = 1$ gives recurrence relations for (weighted) row sums in terms of row sums of arrays with smaller values of q. Alternating-sign row sum recurrence relations are found similarly by setting $x = -1$. Individual terms may be explored further by equating powers of x.

iii. The cases $k = 0$ to 3 may be related to $k = 4$ to 7 respectively in the following manner. For $k = 0$ to 3, $T_q^k(x) = T_q^{k+4}(x) - 1$ and

$$\begin{aligned} g(k,n,q;x) &= \frac{T_q^k(x)^{n+1} - 1}{T_q^k(x) - 1} = \sum_{j=0}^{n} T_q^k(x)^j \\ &= \sum_{j=0}^{n} (T_q^{k+4}(x) - 1)^j \\ &= \sum_{j=0}^{n} \sum_{h=0}^{j} C_h^j T_q^{k+4}(x)^h (-1)^{j-h} \\ &= \sum_{h=0}^{n} g(k+4,h,q;x) \sum_{j=h}^{n} C_h^j (-1)^{j-h}. \qquad (2.1) \end{aligned}$$

Once again, setting $x = \pm 1$ provides relationships between the (weighted) row sums of the respective values of k. Individual coefficients may be related by equating powers of x. For instance, it may be verified from Table 1 that $\binom{n}{n}_2^0 = F_{n+1}$. (Indeed, Table 1 indicates that there are F_{m+1} ways of placing m objects in $n \geq m$ cells, where each cell holds at most 2 objects, according to the rules indicated by $(a,b,c) = (0,0,0)$.) For $k = 0$ and $q = 2$, differentiating (2.1) n times with respect to x and then setting $x = 0$ gives

$$g^{(n)}(0,n,2;0) = n!F_{n+1} \text{ and } g^{(n)}(4,h,2;0) = \begin{cases} 0 & h < n/2 \\ n!\binom{h}{n}_2^4 & h \geq n/2 \end{cases}$$

so that

$$F_{n+1} = \sum_{h=\lfloor \frac{n+1}{2} \rfloor}^{n} \binom{h}{n}_2^4 \sum_{j=h}^{n} C_h^j (-1)^{j-h}.$$

This relates the Fibonacci sequence to the generalized binomial coefficients of order $s = q+1 = 3$ as described by Bondarenko. For example, when $n = 4$, $F_5 = 5$ and

$$\sum_{h=2}^{4} \binom{h}{4}_2^4 \sum_{j=h}^{4} C_h^j (-1)^{j-h} = 1(C_2^2 - C_2^3 + C_2^4) + 6(C_3^3 - C_3^4) + 19C_4^4 = 5.$$

Relationships for other values of k and q may be explored in a similar manner.

3. EXTENDED FIBONACCI SEQUENCE GENERATING FUNCTIONS

Let

$$d(k,n,q;x) = \sum_{m=0}^{n} m!^{-b} \binom{n-m}{m}_q^k x^m = \sum_{m=0}^{\lfloor qn/(q+1) \rfloor} m!^{-b} \binom{n-m}{m}_q^k x^m \text{ for } n \geq 0 \tag{3.1}$$

and $d(k,n,q;x) = 0$ otherwise. This extends the concept of the usual generalized Fibonacci sequence (as defined for $n \geq 0$). Substituting (1.1) for $n, m > 0$ gives

$$\begin{aligned}
d(k,n,q;x) &= \binom{n}{0}_q^k + \sum_{m=1}^{n} m!^{-b} \sum_{i=1-a}^{q} C_{ib}^m i!^c \binom{n-m-1}{m-i}_q^k x^m \\
&= 1-a+\sum_{i=1-a}^{q} i!^c \sum_{m=0}^{n} C_{ib}^m m!^{-b} \binom{n-m-1}{m-i}_q^k x^m \\
&= 1-a+\sum_{i=1-a}^{q} i!^{c-b} x^i \sum_{m=0}^{n} (m-i)!^{-b} \binom{n-m-1}{m-i}_q^k x^{m-i} \\
&= 1-a+\sum_{i=1-a}^{q} i!^{c-b} x^i \sum_{m=-i}^{n-i} m!^{-b} \binom{n-i-1-m}{m}_q^k x^m \quad \text{(changing the summation index to } j=m-i \text{ then reverting to } m) \\
&= 1-a+\sum_{i=1-a}^{q} i!^{c-b} x^i \sum_{m=0}^{n-i-1} m!^{-b} \binom{n-i-1-m}{m}_q^k x^m \quad \text{(using the boundary conditions)} \\
&= 1-a+\sum_{i=1-a}^{q} i!^{c-b} x^i d(k,n-i-1,q;x)
\end{aligned} \tag{3.2}$$

with $d(k,0,q;x) = \sum_{m=0}^{0} m!^{-b} \binom{0-m}{m}_q^k x^m = \binom{0}{0}_1^k = 1$.

i. Diagonal sum formulae, obtained here by setting $x = 1$, were given in [3].
ii. Differentiating (3.2) with respect to x leads to a diagonal sum (further) weighted by column number m,

$$\begin{aligned}
d'(k,n,q;x) &= \sum_{m=1}^{n} m!^{-b} \binom{n-m}{m}_q^k m x^{m-1} \\
&= \sum_{i=1-a}^{q} i!^{c-b} \left(i x^{i-1} d(k,n-i-1,q;x) + x^i d'(k,n-i-1,q;x) \right).
\end{aligned} \tag{3.3}$$

Substituting $x = 1$ and $k = 4$ in (3.1) leads to the generalized Fibonacci recurrence relation of arbitrary order with $d(4, n-1, q; 1) = U_n$ defined by $U_n = \sum_{i=0}^{q} U_{n-i-1}$ for $n > 1$ and $U_{-n} = 0$ for $n = 0, \cdots, q-1,\ U_1 = 1$. (3.3) then gives

$$\sum_{m=1}^{n} \binom{n-m}{m}_q^4 m = \sum_{i=0}^{q} \left(iU_{n-i} + \sum_{j=1}^{n-i-1} \binom{n-i-1-j}{j}_q^4 j \right).$$

For example, $q = 1$ implies $U_n = F_n$ and $\binom{n}{m}_1^4 = C_m^n$ which leads to

$$\sum_{m=1}^{n} C_m^{n-m} m = F_{n-1} + \sum_{j=1}^{n-1} C_j^{n-1-j} j + \sum_{j=1}^{n-2} C_j^{n-2-j} j = F_{n-1} + \sum_{j=1}^{n-2} \left(C_j^{n-1-j} + C_j^{n-2-j} \right) j.$$

Upper summation limits may be reduced as in (3.1). When $n = 5$ this gives

$$C_1^4 + 2C_2^3 = F_4 + C_1^3 + C_1^2 + 2C_2^2 = 10.$$

Formulae for the other extensions may be explored in a similar manner. Substituting $x = 1$ and $k = 0$ in (3.1) leads to the "dying rabbit" sequence recurrence relations [2] with $d(0, n-1, q; 1) = V_n$ defined by $V_n = 1 + \sum_{i=1}^{q} V_{n-i-1} = V_{n-1} + V_{n-2} - V_{n-2-q}$ for $n > 1$ and $V_{-n} = 0$ for $n = 0, \cdots, q-1$, $V_1 = 1$. (3.3) then gives

$$\sum_{m=1}^{n} \binom{n-m}{m}_q^0 m = \sum_{i=1}^{q} \left(iV_{n-i} + \sum_{j=1}^{n-i-1} \binom{n-i-1-j}{j}_q^0 j \right).$$

For example, $q = 2$ gives $V_n = \{0, 1, 1, 2, 3, 4, 6, 8, 11, 15, \dots\}$ for $n \geq 0$. Values of $\binom{n}{m}_2^0$ are given in Table 1. This leads to

$$\sum_{m=1}^{n} \binom{n-m}{m}_q^0 m = V_{n-1} + 2V_{n-2} + \sum_{j=1}^{n-2} \binom{n-2-j}{j}_q^0 j + \sum_{j=1}^{n-3} \binom{n-3-j}{j}_q^0 j.$$

Upper summation limits may also be reduced as in (3.1). When $n = 5$, this gives

$$1 \times 1 + 2 \times 2 + 2 \times 3 = 3 + 2 \times 2 + (1 \times 1 + 1 \times 2) + 1 \times 1 = 11.$$

iii. Alternating-sign weighted diagonal sums can be investigated by setting $x = -1$.

4. CONCLUSION

The k-extensions of the generalized binomial coefficients provide a wealth of interesting relationships. Importantly, these relationships may also be interpreted in terms of the underlying combinatorial meanings of the k-extensions. Further research as outlined by Ollerton and Shannon [3] should prove fruitful.

$$\begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 1 & 2 & 2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 1 & 2 & 3 & 4 & 3 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 1 & 2 & 3 & 5 & 7 & 7 & 4 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 1 & 2 & 3 & 5 & 8 & 12 & 14 & 11 & 5 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 1 & 2 & 3 & 5 & 8 & 13 & 20 & 26 & 25 & 16 & 6 & 1 & 0 & 0 & 0 & 0 \\
1 & 1 & 2 & 3 & 5 & 8 & 13 & 21 & 33 & 46 & 51 & 41 & 22 & 7 & 1 & 0 & 0 \\
1 & 1 & 2 & 3 & 5 & 8 & 13 & 21 & 34 & 54 & 79 & 97 & 92 & 63 & 29 & 8 & 1
\end{pmatrix}$$

Table 1. The $\binom{n}{m}_2^0$ array for $n = 0$ to 8 ($n \geq 0$ gives rows, $m \geq 0$ gives columns).

REFERENCES

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