# ON CONGRUENCES OF EULER NUMBERS MODULO AN ODD SQUARE

**Guodong Liu**
Department of Mathematics, Huizhou University, Huizhou, Guangdong, 516015, P. R. China
*(Submitted August 2002-Final Revision March 2003)*

## 1. INTRODUCTION AND RESULTS

Let $x$ be a complex number with $|x| < \frac{\pi}{2}$ and let the Euler numbers $E_{2n}$ $(n = 0, 1, 2, \cdots)$ be defined by the coefficients in the expansion of (see [3])

$$\sec x = \sum_{n=0}^{\infty} E_{2n} \frac{x^{2n}}{(2n)!}. \tag{1}$$

That is, $E_0 = 1, E_2 = 1, E_4 = 5, E_6 = 61, E_8 = 1385, E_{10} = 50521, \cdots$.

In [3], W. Zhang obtained an interesting congruence for Euler numbers,

$$E_{p-1} \equiv \begin{cases} 0 \pmod{p}, & p \equiv 1 \pmod{4} \\ -2 \pmod{p}, & p \equiv 3 \pmod{4} \end{cases}. \tag{2}$$

where $p$ is any odd prime.

The main purpose of this paper is to prove some new congruences including a generalization of (2). More specifically, we shall prove the following results in the next section.

**Theorem 1**: Let $n \geq 1, k \geq 1$ be any integers, then

$$E_{2n} \equiv (-1)^{n+k} 2^{2n+1} \sum_{i=1}^{k} (-1)^i i^{2n} \pmod{(2k+1)^2}. \tag{3}$$

**Remark 1**: Set $k = 1, 2, 3, 4$ in Theorem 1, when $n \geq 1$, we have following the congruences for Euler numbers:

$$E_{2n} \equiv (-1)^n 2^{2n+1} \pmod{9},$$
$$E_{2n} \equiv (-1)^n 2^{2n+1}(2^{2n} - 1) \pmod{25},$$
$$E_{2n} \equiv (-1)^n 2^{2n+1}(3^{2n} - 2^{2n} + 1) \pmod{49},$$
$$E_{2n} \equiv (-1)^n 2^{2n+1}(4^{2n} - 3^{2n} + 2^{2n} - 1) \pmod{81}.$$

**Corollary 1.1**: Let $p$ be any odd prime, then

$$E_{p-1} \equiv 2^p \sum_{i=1}^{(p-1)/2} (-1)^i i^{p-1} \pmod{p^2}. \tag{4}$$

**Remark 2**: By Corollary 1.1 and Fermat's little theorem (see [2]), we immediately obtain (2) (see Zhang [3]).

**Corollary 1.2**: Let $n \geq 1, k \geq 0$ be any integers, $p$ be any odd prime, then

$$E_{2n+k(p-1)} \equiv (-1)^{\frac{k(p-1)}{2}} E_{2n} \pmod{p}. \tag{5}$$

**Theorem 2**: For any odd prime $p$ and any nonnegative integer $n$, we have

$$\sum_{j=1}^{(p-1)/2} (-1)^{n+j} E_{2n+2j} \equiv -1 \pmod{p}. \tag{6}$$

**Remark 3**: Set $p = 3, 5, 7, 11$ in Theorem 2, when $n \geq 0$, we have

$$\begin{aligned}
&E_{2n+2} \equiv (-1)^n \pmod 3,\\
&E_{2n+2} - E_{2n+4} \equiv (-1)^n \pmod 5,\\
&E_{2n+2} - E_{2n+4} + E_{2n+6} \equiv (-1)^n \pmod 7,\\
&E_{2n+2} - E_{2n+4} + E_{2n+6} - E_{2n+8} + E_{2n+10} \equiv (-1)^n \pmod{11}.
\end{aligned}$$

## 2. PROOF OF THE THEOREMS

**Lemma**: $\sum_{k=0}^{2n}(-1)^k \cos\,(2n-2k)x = \sec\,x \cos(2n+1)x$.

**Proof** : Define $S(t) = \sum_{n=0}^{\infty} t^n \sin\,nx$ and $C(t) = \sum_{n=0}^{\infty} t^n \cos\,nx$. It follows from De Moivre's Theorem (see [1]) that, for $|t| < 1$,

$$C(t) + iS(t) = \sum_{n=0}^{\infty} t^n (\cos\,x + i\sin\,x)^n = \frac{1}{1 - t\,\cos\,x - it\,\sin\,x} = \frac{1 - t\,\cos\,x + it\,\sin\,x}{1 - 2t\,\cos\,x + t^2}.$$

Therefore

$$S(t) = \sum_{n=0}^{\infty} t^n \sin nx = \frac{t \sin x}{1 - 2t \cos\ x + t^2},\ |t| < 1,$$

and

$$C(t) = \sum_{n=0}^{\infty} t^n \cos nx = \frac{1 - t \cos x}{1 - 2t \cos x + t^2},\ |t| < 1.$$

Then

$$C_0(t) = \sum_{n=0}^{\infty} t^n \cos(2n+1)x = \frac{1}{2\sqrt{t}} \left[C(t) - C(-\sqrt{t})\right] = \frac{(1-t)\cos x}{(1+t)^2 - 4t\cos^2 x},$$

and

$$C_e(t) = \sum_{n=0}^{\infty} t^n \cos 2nx = \frac{1}{2} \left[C(t) + C(-\sqrt{t})\right] = \frac{1 - 2t\cos^2 x + t}{(1+t)^2 - 4t\cos^2 x}.$$

It follows that for $|t| < 1$,

$$\begin{aligned}
\sum_{n=0}^{\infty} t^n \sum_{k=0}^{2n} (-1)^k \cos(2n-2k)x &= \sum_{n=0}^{\infty} t^n \left[2\sum_{k=0}^{n} (-1)^k \cos(2n-2k)x - (-1)^n\right]\\
&= \frac{2C_e(t)}{1+t} - \frac{1}{1+t}\\
&= C_0(t) \sec x,
\end{aligned}$$

which completes the proof immediately. □

**Proof of Theorem 1**: According to the Lemma and (1), we find

$$\begin{aligned}\sum_{n=0}^{\infty} E_{2n}\frac{x^{2n}}{(2n)!} &= \sec x\\ &= \sec(2k+1)x\sum_{i=0}^{2k}(-1)^i\cos(2k-2i)x\\ &= \sum_{j=0}^{\infty}(2k+1)^{2j}E_{2j}\frac{x^{2j}}{(2j)!}\sum_{i=0}^{2k}(-1)^i\sum_{n=0}^{\infty}(-1)^n(2k-2i)^{2n}\frac{x^{2n}}{(2n)!}\\ &= \sum_{n=0}^{\infty}\sum_{j=0}^{n}\binom{2n}{2j}(2k+1)^{2j}E_{2j}\sum_{i=0}^{2k}(-1)^i(-1)^{n-j}(2k-2i)^{2n-2j}\frac{x^{2n}}{(2n)!},\end{aligned}$$

therefore,

$$\begin{aligned}E_{2n} &= \sum_{j=0}^{n}(-1)^{n-j}\binom{2n}{2j}(2k+1)^{2j}E_{2j}\sum_{i=0}^{2k}(-1)^i(2k-2i)^{2n-2j}\\ &= (2k+1)^{2n}E_{2n}+2(-1)^k\sum_{j=0}^{n-1}(-1)^{n-j}\binom{2n}{2j}(2k+1)^{2j}E_{2j}\sum_{i=1}^{k}(-1)^i(2i)^{2n-2j}\\ &= (2k+1)^{2n}E_{2n}+2(-1)^{n+k}E_0\sum_{i=1}^{k}(-1)^i(2i)^{2n}\\ &+ 2(-1)^k\sum_{j=1}^{n-1}(-1)^{n-j}\binom{2n}{2j}(2k+1)^{2j}E_{2j}\sum_{i=1}^{k}(-1)^i(2i)^{2n-2j}\\ &= (2k+1)^{2n}E_{2n}+(-1)^{n+k}2^{2n+1}\sum_{i=1}^{k}(-1)^i i^{2n}\\ &+ 2(-1)^k\sum_{j=1}^{n-1}(-1)^{n-j}\binom{2n}{2j}(2k+1)^{2j}E_{2j}\sum_{i=1}^{k}(-1)^i(2i)^{2n-2j}. \qquad (7)\end{aligned}$$

By (7), we immediately obtain (3).

This completes the proof of Theorem 1. □

**Proof of Corollary 1.1**: Setting $n = k = (p-1)/2$ in Theorem 1, we immediately obtain (4).

**Proof of Corollary 1.2**: Setting $p = 2m+1$ in Theorem 1, we have

$$\begin{aligned}E_{2n} &\equiv (-1)^{n+(p-1)/2}2^{2n+1}\sum_{i=1}^{(p-1)/2}(-1)^i i^{2n} \pmod{p^2}\\ &\equiv (-1)^{n+(p-1)/2}2^{2n+1}\sum_{i=1}^{(p-1)/2}(-1)^i i^{2n} \pmod{p}, \qquad (8)\end{aligned}$$

from which we obtain

$$E_{2n+k(p-1)} \equiv (-1)^{n+(k+1)\frac{p-1}{2}} 2^{2n+k(p-1)+1} \sum_{i=1}^{(p-1)/2} (-1)^i i^{2n+k(p-1)} \pmod p. \tag{9}$$

By Fermat's Little Theorem, we have

$$(2i)^{p-i} \equiv 1 \pmod p \ (1 \le i \le (p-1)/2). \tag{10}$$

By (9) and (10), we get

$$E_{2n+k(p-1)} \equiv (-1)^{n+(k+1)\frac{p-1}{2}} 2^{2n+1} \sum_{i=1}^{(p-1)/2} (-1)^i i^{2n} \equiv (-1)^{\frac{k(p-1)}{2}} E_{2n} \pmod p.$$

This proves Corollary 1.2. □

**Proof of Theorem 2**: By (8), we have

$$\sum_{j=1}^{(p-1)/2} (-1)^{n+j} E_{2n+2j} \equiv (-1)^{\frac{p-1}{2}} \sum_{j=1}^{(p-1)/2} 2^{2n+2j+1} \sum_{i=1}^{(p-1)/2} (-1)^i i^{2n+2j}$$

$$\equiv 2(p-1)^{2n} \sum_{j=1}^{(p-1)/2} (p-1)^{2j} + (-1)^{\frac{p-1}{2}} 2^{2n+1} \sum_{i=1}^{(p-3)/2} (-1)^i i^{2n} \sum_{j=1}^{(p-1)/2} (2i)^{2j}$$

$$\equiv -1 + (-1)^{\frac{p-1}{2}} 2^{2n+3} \sum_{i=1}^{(p-3)/2} (-1)^i i^{2n+2} \left( \frac{(2i)^{p-1}-1}{(2i)^2-1} \right) \pmod p. \tag{11}$$

By Fermat's Little Theorem, we have

$$\left( \frac{(2i)^{p-1}-1}{(2i)^2-1} \right) \equiv 0 \pmod p \ (1 \le i \le (p-3)/2). \tag{12}$$

By (11) and (12), we obtain

$$\sum_{j=1}^{(p-1)/2} (-1)^{n+j} E_{2n+2j} \equiv -1 \pmod p.$$

This completes the proof of Theorem 2. □

## ACKNOWLEDGMENT

The author would like to thank the anonymous referee for valuable comments, including the proof of the Lemma, which improved the presentation of the original paper.

This work is supported by the Natural Science Foundation of Guangdong Province (021072).

## REFERENCES

[1] M. Abramowitz and A. Stegun. *Handbook of Mathematical Functions.* National Bureau of Standards, Washington, DC, 1964.
[2] T. M. Apostol. *Introduction to Analytic Number Theory.* Springer-Verlag, New York, 1976.
[3] Wenpeng Zhang. "Some Identities Involving the Euler and the Central Factorial Numbers." *The Fibonacci Quarterly* **36.2** (1998): 154-57.

AMS Classification Numbers: 11B68

✠ ✠ ✠