THE ASYMPTOTIC GROWTH RATE OF RANDOM FIBONACCI TYPE SEQUENCES

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ABSTRACT

Estimating the growth rate of random Fibonacci-type sequences is both challenging and fascinating. In this paper, by using ergodic theory, we prove a new result in this area. Let a denote an infinite sequence of natural numbers $\{a_1, a_2, \cdots\}$ and define a random Fibonacci-type sequence by $f_{-1} = 0, f_0 = 1, a_0 = 0$, and

$$f_k = 2^{a_k} f_{k-1} + 2^{a_{k-1}} f_{k-2}$$

for $k \geq 1$. Then, for almost all such infinite sequences a, we have

$$\lim_{n\to\infty} \frac{1}{n} \ln f_n = 1.30022988 \cdots .$$

1. INTRODUCTION

It is well-known that the Fibonacci numbers, F_n, are given by Binet's formula:

$$F_n = \frac{1}{\sqrt{5}} \left(\frac{1+\sqrt{5}}{2} \right)^n - \frac{1}{\sqrt{5}} \left(\frac{1-\sqrt{5}}{2} \right)^n . \tag{1}$$

With (1), we can compute the *asymptotic growth rate* of the sequence $\{F_k\}$, which is given by

$$\lim_{n\to\infty} \frac{1}{n} \ln F_n = \ln \left(\frac{1+\sqrt{5}}{2} \right) = 0.4812 \cdots .$$

In the case of *random Fibonacci type sequences*, defined by (with fixed f_1 and f_2)

$$f_k = a(k)\ f_{k-1} + b(k)\ f_{k-2},$$

where $a(k)$ and $b(k)$ are random coefficients, the quest for the asymptotic growth rate will be much more difficult, if not impossible.

Recently, Viswanath made a surprising breakthrough in [26]. He considered the random Fibonacci sequences defined by $f_1 = f_2 = 1$ and

$$f_k = \pm f_{k-1} \pm f_{k-2}, \tag{2}$$

where the signs are chosen independently and with equal probabilities. He proved the following remarkable result

Theorem 1: (Viswanath, 2000) *The asymptotic growth rate of the random Fibonacci sequences defined in (2) is given by*

$$\lim_{n\to\infty} \frac{1}{n} \ln f_n = \ln(1.13198824\cdots) = 0.12397559\cdots \tag{3}$$

with probability 1.

For an interesting introduction, see Peterson's article [18]. The hard part of Viswanath's work is the computation of a fractal measure. That involves a large system of equations with dimension in the order of millions. Since then, some authors ([4, 29]) have generalized this result in various directions.

Viswanath's method is not the only way to estimate the growth rate of random Fibonacci type sequences. Write $x \in [0, 1)$ as

$$x = \cfrac{2^{-a_1}}{1 + \cfrac{2^{-a_2}}{1 + \cfrac{2^{-a_3}}{1 + \cdots}}}, \tag{4}$$

where $a_k = a_k(x) \in \{0, 1, 2, \cdots\}$ (see Section 2). Note that, in terms of the following compact notation (which we will use in the rest of this paper)

$$\cfrac{A_1}{B_1 + \cfrac{A_2}{B_2 + \cdots}} \equiv \frac{A_1|}{|B_1} + \frac{A_2|}{|B_2} + \cdots, \tag{5}$$

the above can be written as

$$x = \frac{2^{-a_1}|}{|\ 1} + \frac{2^{-a_2}|}{|\ 1} + \frac{2^{-a_3}|}{|\ 1} + \cdots .$$

Our main result is the following theorem

Theorem 2: *For each $x \in [0, 1)$, we associate with it an infinite sequence of natural numbers $\{a_1, a_2, \cdots\}$ through (4). Consider the random Fibonacci type sequences, $\{f_k\}$, defined by $f_{-1} = 0$, $f_0 = 1$, $a_0 = 0$, and*

$$f_k = 2^{a_k} f_{k-1} + 2^{a_{k-1}} f_{k-2}, \tag{6}$$

for $k \geq 1$. Then we have, for almost all x,

$$\lim_{n\to\infty} \frac{1}{n} \ln f_n = 1.30022988\cdots . \tag{7}$$

In fact, this is a generalization of the following theorem, which is proved by Lévy [11]:

Theorem 3: *(Levy, 1929) For each $x \in [0, 1)$, we associate with it an infinite sequence $\{b_1, b_2, \cdots\}$ ($b_k = 1, 2, \cdots$) through the regular continued fraction representation of x; i.e.,*

$$x = \frac{1\ |}{|\ b_1} + \frac{1\ |}{|\ b_2} + \cdots .$$

Consider the random Fibonacci type sequences, $\{Q_k\}$, defined by $Q_{-1} = 0$, $Q_0 = 1$, and

$$Q_k = b_k Q_{k-1} + Q_{k-2} \tag{8}$$

for $k \geq 1$. Then we have, for almost all x,

$$\lim_{n\to\infty} \frac{1}{n} \ln Q_n = \frac{\pi^2}{12\ \ln 2} = 1.186569110\cdots . \tag{9}$$

See also Khintchin [8, 9]. Later, the same theorem is proved using *ergodic theory*; e.g., see [1, 3, 13, 19, 21]. See also Kac [7]. We will prove Theorem 2 by following the same strategy that uses ergodic theory.

A question may be raised at this point: what is ergodic theory and why is it able to compute the asymptotic growth rate of random recurrences like that of (6) and (8)?

The concept of ergodicity was originated in physics in the nineteenth century. Given a system that evolves dynamically, suppose one would like to study the behavior of a certain physical quantity P of this system. The "ergodic hypothesis" asserts that, under certain conditions, the time average of P should be the same as the phase space average of P. For example, consider a sealed box with a permeable partition that divides the box into two equal chambers. Then the "ergodic hypothesis" asserts that an air molecule should spend half of its time in each chamber. In the early part of the twentieth century, Birkhoff, von Neumann, Khintchin and others developed this into a mathematical theory which is now known as the *ergodic theory*. For a physicist's introduction, see [24] . Ergodic theory is closely related to the theory of chaos, see [2, 6, 16]. For introductions (in the form of book) to ergodic theory, consult [1, 3, 13, 17, 19, 23, 27]. See also Young's lecture [30]. Kac [7] gave a very readable account that connects the physicist's concept of the ergodic hypothesis and the mathematical foundation of ergodic theory.

With the above understood, we can look into the reason why ergodic theory can be used to compute the asymptotic growth rate of random recurrences. Random sequences, like that of (6) or (8), can be thought of being generated by certain "dynamical systems" (in a non-technical sense). By standard arguments (see Section 2 below), we can associate these sequences with continued fractions, like that of (4). This induces "dynamics" on continued fractions of which ergodic theory allows us to assert the asymptotic properties.

It should be remarked that Theorem 3 is hard to generalize: to compute explicitly the asymptotic growth rate, one has to know the analytical closed-form of the *invariant measure* (see Section 2 below) associated with the random recurrence. In the case of Theorem 2, since we found the invariant measure involved, therefore we are able to compute the asymptotic growth rate of the f_n defined in (6).

The outline of this paper is as follows. In Section 2, we fix some notations and introduce the *generalized Gauss map* associated with the random Fibonacci type sequence (6). Then in Section 3, we prove Theorem 2. The main ingredient of this proof is the ergodicity of the generalized Gauss map. Since the proof of ergodicity is rather long and complicated, therefore, in order to minimize digression, we assume this property in Section 3 and prove it in Section 4. Section 5 is our concluding remarks.

2. THE GENERALIZED GAUSS MAP

The goal of this section is to define the generalized Gauss map. In brief, this map is the machinery that generates random recurrences. To proceed, let us introduce a certain continued fraction representation of all $x \in [0,1)$.

Lemma 1: *For all* $x \in [0,1)$*, we have*

$$x = \frac{2^{-a_1}|}{|\ 1} + \frac{2^{-a_2}|}{|\ 1} + \cdots \equiv [a_1, a_2, \cdots], \tag{10}$$

where $a_k \in \{0, 1, 2, \cdots\}$.

Remark: One can think of the a_k as the "digits" of x.

Proof: For any $x \in [0,1)$, we have

$$\frac{1}{2^{a_1+1}} < x \le \frac{1}{2^{a_1}},$$

where $a_1 \in \{0, 1, 2, \cdots\}$. This implies, for some $p \in [0,1)$,

$$x = (1-p)\ 2^{-a_1} + \frac{p}{2}\ 2^{-a_1} = (1 - \frac{p}{2})\ 2^{-a_1}.$$

Define $x_1 \in [0,1)$ by $x_1 = p/(2-p)$, we can write x as

$$x = \frac{2^{-a_1}}{1 + x_1}.$$

Since $x_1 \in [0,1)$, we can repeat the same iteration and obtain

$$x = \cfrac{2^{-a_1}}{1 + \cfrac{2^{-a_2}}{1 + \cdots}} = \frac{2^{-a_1}|}{|\ 1} + \frac{2^{-a_2}|}{|\ 1} + \cdots . \quad \square$$

Remark: If x is irrational, then x has an infinite expansion in the form of (10).

This continued fraction representation of x is closely related to the random Fibonacci type sequence in (6). First, we fix an x and this fixes a sequence of natural numbers, $\{a_k\}$, through Lemma 1. Note that, a_k are functions of x; i.e., $a_k = a_k(x)$. Next, we define the following sequences:

Definition 1: *For all* $x = [a_1, a_2, \cdots] \in [0,1)$*, define*

$$\begin{aligned} g_k &= 2^{a_k} g_{k-1} + 2^{a_{k-1}} g_{k-2}, \quad k \ge 2 \\ f_k &= 2^{a_k} f_{k-1} + 2^{a_{k-1}} f_{k-2}, \quad k \ge 1 \end{aligned} \tag{11}$$

where $g_0 = 0,\ g_1 = 1,\ f_{-1} = 0$ *and* $f_0 = 1$.

Note that f_k and g_k are integer-valued functions of x. Also, the second equation of (11) is the defining equation for the random Fibonacci type sequences in (6). All these machinery are tied up together by the following identity: for $t \in [0,1]$, we have

$$\frac{g_k + t\ 2^{a_k} g_{k-1}}{f_k + t\ 2^{a_k} f_{k-1}} = \frac{2^{-a_1}|}{|\ 1} + \frac{2^{-a_2}|}{|\ 1} + \cdots + \frac{2^{-a_k}|}{|1+t}. \tag{12}$$

This can be proved by standard induction arguments (see e.g., Niven's book [15]).

With these understood, we define the *generalized Gauss map*:

Definition 2: *For all* $x \in [0,1)$*, the generalized Gauss map* $T : [0,1) \to [0,1)$ *is defined as follows: for* $x = 0$*,* $T\,0 \equiv 0$*; for* $x \neq 0$*, we have, using the notation in (10),*

$$Tx = T[a_1, a_2, a_3, \cdots] \equiv [a_2, a_3, a_4, \cdots]. \tag{13}$$

There is a beautiful way to define T alternatively, as pointed out by the referee; see equation (37) and the last section in this paper. One can think of T as a *shift map*, as it shifts the digits of x. Note that the *original* Gauss map is defined as follows. Write $x \in [0,1)$ as a regular continued fraction; i.e.,

$$x = \frac{1\mid}{\mid b_1} + \frac{1\mid}{\mid b_2} + \cdots \equiv [b_1, b_2, \cdots]_G,$$

where $b_k \in \{1, 2, \cdots\}$. The Gauss map $T_G : [0,1) \to [0,1)$ is defined as follows: for $x = 0$, $T_G\,0 \equiv 0$; for $x \neq 0$, we have $T_G\,x = T_G\,[b_1, b_2, b_3, \cdots]_G \equiv [b_2, b_3, b_4, \cdots]_G$.

The following remarkable property of the generalized Gauss map (13) is the key to the proof of Theorem 2:

Theorem 4: *Given an integrable function* f *on the unit interval, then, for almost all* x*, we have*

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} f(T^k x) = \int_0^1 \rho(x) f(x)\; dx. \tag{14}$$

Here, $\rho(x)$*, the probability density of an invariant measure (see (31)), is given by*

$$\rho(x) = \frac{\rho_0}{(x+1)(x+2)}, \tag{15}$$

where $\rho_0^{-1} = \ln(4/3)$.

Note that, equation (14) is the same as saying that the "time" average of f (i.e., the l.h.s. of (14)) is the same as the "phase space" average of f (i.e., the r.h.s. of (14)). The normalization ρ_0 in (15) is chosen such that $\int_0^1 \rho(x)\; dx = 1$. Note that $\rho(x)$ satisfies the following equation: for $x \in [0,1]$, we have

$$\rho(x) = \sum_{k=0}^{\infty} \frac{1}{2^k(1+x)^2} \rho\left(\frac{1}{2^k(1+x)}\right).$$

This can be understood as the fact that $\rho(x)$ is the eigenfunction of eigenvalue 1 of the Frobenius-Perron operator; e.g., see [5, 22]. This will be explored elsewhere.

The integrability in Theorem 4 is defined with respect to the Lebesgue measure, or an equivalent (invariant) measure (the *generalized Gauss measure*), which will be introduced in Section 4. In the case of the original Gauss map, we have a similar theorem, with T replaced by T_G, and $\rho(x)$ replaced by $\rho_G(x) = 1/(x+1)$. Note that this Gauss map (and its ergodicity) is closely related to the *Gauss-Kusmin-Lévy* problem; see [1, 5, 10, 13, 20, 22, 25].

With the generalized Gauss map (13), we can derive a formula (cf. (20)) which will be useful in the subsequent sections. Using the T map, we can write x as

$$x = \frac{2^{-a_1}}{1+Tx} = \cdots = \frac{2^{-a_1}|}{|\ 1} + \frac{2^{-a_2}|}{|\ 1} + \cdots + \frac{2^{-a_n}\ \ |}{|1+T^n x}. \tag{16}$$

Note that we have used the iteration procedure used in the proof of Lemma 1. Equations (16) and (12) imply

$$x = \frac{g_n(x) + (T^n x)\ 2^{a_n(x)} g_{n-1}(x)}{f_n(x) + (T^n x)\ 2^{a_n(x)} f_{n-1}(x)}. \tag{17}$$

For subsequent sections, we define the following:

Definition 3: *For $n \geq 1$,*

$$A_n(x) \equiv g_n(x) + (T^n x)\ 2^{a_n(x)} g_{n-1}(x) \tag{18}$$

$$B_n(x) \equiv f_n(x) + (T^n x)\ 2^{a_n(x)} f_{n-1}(x). \tag{19}$$

Of course, $A_n(x)$ and $B_n(x)$ are simply the numerator and the denominator of the quotient in (17); i.e., $x = A_n(x)/B_n(x)$. Note that, in the same manner, we have $Tx = A_{n-1}(Tx)/B_{n-1}(Tx)$. This is because, the expansion of Tx is obtained by removing the top level of x; i.e.,

$$Tx = \frac{2^{-a_2}|}{|\ 1} + \frac{2^{-a_3}|}{|\ 1} + \cdots + \frac{2^{-a_n}\ \ |}{|1+T^n x},$$

and there are $n-1$ levels in this expansion. In particular, we have

$$B_{n-1}(Tx) = A_n(x). \tag{20}$$

To see this, we compare the numerator in the first quotient and the numerator in the last quotients in the following equation:

$$x = \frac{A_n(x)}{B_n(x)} = \frac{2^{-a_1}}{1+Tx} = \frac{2^{-a_1}}{1 + A_{n-1}(Tx)/B_{n-1}(Tx)} = \frac{B_{n-1}(Tx)}{2^{a_1}\left[A_{n-1}(Tx) + B_{n-1}(Tx)\right]}.$$

In order to minimize digression, we will assume and apply Theorem 4 in the next section. We will come back for a proof of this theorem in Section 4. As we will see, Theorem 4 is a consequence of the *ergodicity* of the generalized Gauss map.

3. PROOF OF THEOREM 2

In this section, following the strategy in chapter 24 of Schweiger's book [22], we prove Theorem 2 by applying Theorem 4. See also [1, 2].

• **Step 1**
We need to establish the following formula

$$\prod_{k=0}^{n-1} T^k x = \frac{1}{f_n + (T^n x)\ 2^{a_n} f_{n-1}} = \frac{1}{B_n(x)}. \tag{21}$$

Indeed, write the product on the left by A_k and B_k defined in the last section:

$$\prod_{k=0}^{n-1} T^k x = \frac{A_n(x)}{B_n(x)} \frac{A_{n-1}(Tx)}{B_{n-1}(Tx)} \cdots \frac{A_1(T^{n-1}x)}{B_1(T^{n-1}x)}.$$

Using (20) repeatedly, we have most of the factors canceled, except the denominator of the first factor (i.e. $B_n(x)$), and the numerator of the last factor (i.e. $A_1(T^{n-1}x)$). This gives the desired result as $A_1(T^{n-1}x) = 1$.

• **Step 2**
Equation (21) and the fact that $0 \leq T^n x \leq 1$ imply

$$\frac{1}{2f_n} \leq \prod_{k=0}^{n-1} T^k x \leq \frac{1}{f_n}. \tag{22}$$

Note that, in the upper inequality, we have used (21) and the fact that, for $k \geq 0$, $f_k > 0$ (this can be shown by induction). For the lower inequality, we have used the fact that

$$f_n + (T^n x) 2^{a_n} f_{n-1} \leq f_n + 2^{a_n} f_{n-1} \leq f_n + (2^{a_n} f_{n-1} + 2^{a_{n-1}} f_{n-2}) = 2 f_n.$$

The log of inequality (22) will be useful in the next and final step:

$$-\frac{\ln f_n}{n} - \frac{2}{n} \leq \frac{1}{n} \sum_{k=0}^{n-1} \ln T^k x \leq -\frac{\ln f_n}{n}. \tag{23}$$

• **Step 3**
Taking the limit $n \to \infty$ in (23) and using Theorem 4 (with $f(x) = \ln x$) , we have a.e. ("almost everywhere")

$$\lim_{n\to\infty} \frac{1}{n} \ln f_n = -\lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} \ln\, T^k x = -\rho_0 \int_0^1 \frac{\ln x}{(x+1)(x+2)}\ dx. \tag{24}$$

Since, for $x \in [0,1]$,

$$0 \leq \frac{1}{(x+1)(x+2)} \leq -\frac{1}{3}x + \frac{1}{2},$$

the last integral in (24) can be shown to be bounded:

$$\int_0^1 \frac{-\ln x}{(x+1)(x+2)} dx \leq \int_0^1 (-\ln x) \left(-\frac{1}{3}x + \frac{1}{2}\right) dx = \frac{5}{12}.$$

Numerical integration gives

$$\lim_{n\to\infty} \frac{1}{n} \ln f_n = 1.30022988\cdots.$$

This completes the proof of Theorem 2. □

4. THE ERGODICITY OF THE GENERALIZED GAUSS MAP

It is time to prove Theorem 4. To this end, all we need to do is to establish the ergodicity of the generalized Gauss map. Then Theorem 4 follows by applying the *ergodic theorem* (see [1, 3, 13, 17, 23, 27]). We will proceed as follows. First, in Section 4.1, we introduce the concept of *cylinders*. It is a special way to partition the unit interval so that the subsequent computations may be simplified. Next, in Section 4.2, we introduce the *generalized Gauss measure*; cf. the probability density $\rho(x)$ in (15). With these two major ingredients defined, we proceed to prove Theorem 4 in Section 4.3.

4.1 Cylinders

Fix non-negative integers $a_1, \cdots, a_k$. Let $t \in [0, 1)$ and define $\psi_{a_1\cdots a_k}$ by

$$\psi_{a_1\cdots a_k}(t) \equiv \frac{2^{-a_1}|}{|\ \ 1} + \frac{2^{-a_2}|}{|\ \ 1} + \cdots + \frac{2^{-a_k}|}{|1+t} = \frac{g_k + t\ 2^{a_k} g_{k-1}}{f_k + t\ 2^{a_k} f_{k-1}}. \tag{25}$$

For the second equality, we have used (12). The *cylinder* (or *fundamental interval*) of rank k is defined by $\triangle_{a_1\cdots a_k} = \{\psi_{a_1\cdots a_k}(t);\ t \in [0, 1)\}$. A crucial property is that the lengths of these cylinders shrink to zero: let λ denote the Lebesque measure and we have

Lemma 2:

$$\lim_{k\to\infty} \lambda(\triangle_{a_1\cdots a_k}) = 0. \tag{26}$$

Remarks: The implication of this lemma is as follows. Let $\mathcal{B}^{(k)}$ denote the σ-algebra generated by the cylinders of order k and $\mathcal{F}$ denote the σ-algebra of the Borel sets. Then Lemma 2 implies

$$\bigvee_{k=1}^{\infty} \mathcal{B}^{(k)} = \mathcal{F}; \tag{27}$$

i.e., the class of all cylinders generates the σ-algebra $\mathcal{F}$ of Borel sets. For a proof, see p. 48 of [22]. This fact allows us to reduce many calculations in the subsequent sections to be done in terms of cylinders. We now turn to the proof of Lemma 2.

Proof: All we need to do is to establish, for a constant, D_*, which is independent of k,

$$\lambda(\triangle_{a_1\cdots a_k}) \le \frac{D_*}{2^k}. \tag{28}$$

In particular, this implies Lemma 2.

Indeed, by direct computation, we can show that

$$\lambda(\triangle_{a_1\cdots a_k}) = |\psi_{a_1\cdots a_k}(1) - \psi_{a_1\cdots a_k}(0)| = \frac{\prod_{i=1}^k 2^{a_i}}{f_k(f_k + 2^{a_k} f_{k-1})} \le \frac{\prod_{i=1}^k 2^{a_i}}{f_k^2}. \tag{29}$$

To obtain the desired bound, we consider first the case of an odd k. To this end, since $f_k \geq f_{k-1}$, which can be shown by induction, we have

$$f_i = 2^{a_i} f_{i-1} + 2^{a_{i-1}} f_{i-2} \geq (2^{a_i} + 2^{a_{i-1}}) f_{i-2}.$$

This implies

$$f_k \geq f_1 \prod_{m=3,5,\cdots,k} (2^{a_m} + 2^{a_{m-1}}). \tag{30}$$

By combining (29) and (30), we obtain

$$\begin{aligned}
\lambda(\triangle_{a_1\cdots a_k}) &\leq \frac{2^{a_1}}{f_1^2} \prod_{m=3,5,\cdots,k} \frac{2^{a_m}\ 2^{a_{m-1}}}{(2^{a_m} + 2^{a_{m-1}})^2} \\
&= \frac{2^{a_1}}{f_1^2} \prod_{m=3,5,\cdots,k} \left(2 + \frac{2^{a_m}}{2^{a_{m-1}}} + \frac{2^{a_{m-1}}}{2^{a_m}}\right)^{-1} \\
&\leq \frac{2^{a_1}}{f_1^2} \prod_{m=3,5,\cdots,k} \frac{1}{2^2} \\
&\leq \frac{1}{2^{k-1}} = \frac{2}{2^k}.
\end{aligned}$$

Note that $f_1 = 2^{a_1}$ and therefore the prefactor $2^{a_1}/f_1^2 = 1/f_1 \leq 1$. For the inequality in the third line, we set $x = 2^{a_m}/2^{a_{m-1}}$ and used the fact that, for $x > 0$, $x + x^{-1} \geq 2$. This proves the case of k being odd. The case of an even k can be shown in a similar manner. Therefore we establish (28). □

4.2 The Generalized Gauss Measure

In this section, we define the generalized Gauss measure and study some of its crucial properties.

Definition 4: *The generalized Gauss measure is given by*

$$P(A) = \rho_0 \int_A \frac{dx}{(x+1)(x+2)}, \tag{31}$$

where $A \in \mathcal{F}$ and ρ_0 is given in Theorem 4.

By using the inequality

$$\frac{1}{6} \leq \frac{1}{(x+1)(x+2)} \leq \frac{1}{2}, \quad x \in [0,1]$$

we obtain

$$\frac{\lambda(A)}{6\ \ln(4/3)} \leq P(A) \leq \frac{\lambda(A)}{2\ \ln(4/3)}. \tag{32}$$

Note that P and λ are absolutely continuous with respect to each other (they have the same set of measure zero). This implies that if the sequence $a_1(x), a_2(x), \cdots$ has a certain property a.e. with respect to P, it will also have the same property a.e. with respect to λ.

An important property of P is that it is *preserved* by the generalized Gauss map T ($measure - preserving$); i.e., we have $P(T^{-1}A) = P(A)$ for every $A \in \mathcal{F}$. To establish this, we need

Lemma 3: *For $t > 0$, we have $P\left(T^{-1}[0,t]\right) = P\left([0,t]\right)$.*

Proof: Let us set $\gamma = 1/2$ and note that

$$T^{-1}[0,t] = \{x : \ 0 \le Tx \le t\} = \bigcup_{k=0}^{\infty} \left[\frac{\gamma^k}{1+t}, \ \gamma^k\right].$$

With these understood, we proceed to compute $P(T^{-1}[0,t])$:

$$\begin{aligned} P(T^{-1}[0,t]) &= \rho_0 \sum_{k=0}^{\infty} \int_{\frac{\gamma^k}{1+t}}^{\gamma^k} \frac{dx}{(x+1)(x+2)} \\ &= \rho_0 \sum_{k=0}^{\infty} \ln\left(\frac{\gamma^k+1}{\frac{\gamma^k}{1+t}+1}\right) - \ln\left(\frac{\gamma^{k+1}+1}{\frac{\gamma^{k+1}}{1+t}+1}\right) \\ &= \rho_0 \ \ln\left(\frac{2}{\frac{1}{1+t}+1}\right) = \rho_0 \left[\ln 2 + \ln\left(\frac{t+1}{t+2}\right)\right]. \end{aligned}$$

Note that the infinite sum is a telescoping sum and only the first term survives. Next, we proceed to compute $P([0,t])$:

$$P([0,t]) = \rho_0 \int_0^t \frac{dx}{(x+1)(x+2)} = \rho_0 \left[\ln\left(\frac{t+1}{t+2}\right) - \ln\frac{1}{2}\right].$$

This proves the lemma. □

Remark: let $\mathcal{F}_0$ be the sub-algebra of disjoint unions of intervals contained in $[0,1]$. Lemma 3 implies that $P(T^{-1}A) = P(A)$ for every $A \in \mathcal{F}_0$. By Proposition 2.1 in p. 27 of [13], this can be extended to every $A \in \mathcal{F}$. This shows that T is measure-preserving.

4.3 Proof of Theorem 4

The crux of the issue is the following theorem:

Theorem 5: *T is ergodic with respect to P.*

Proof: To prove this, we follow standard procedure (e.g., see [1]): we need to show that if $A \in \mathcal{F}$ is such that $T^{-1}A = A$ and $P(A) > 0$, then $P(A) = 1$.

Fix $a_1, \cdots, a_k$. First, we show that, for all $A \in \mathcal{F}$, we have

$$\frac{1}{2}\lambda(A) \le \lambda\left(T^{-k}A | \triangle_{a_1 \cdots a_k}\right). \tag{33}$$

Here, $\lambda(A|B) = \lambda(A \cap B)/\lambda(B)$.

To this end, we first consider the interval $B = [x, y)$, where $0 \le x < y \le 1$ and we have

$$\lambda\left(T^{-k}B|\triangle_{a_1\cdots a_k}\right) = \frac{|\psi_{a_1\cdots a_k}(y) - \psi_{a_1\cdots a_k}(x)|}{|\psi_{a_1\cdots a_k}(1) - \psi_{a_1\cdots a_k}(0)|}$$

$$= |y - x| \underbrace{\frac{f_k(f_k + 2^{a_k} f_{k-1})}{(f_k + y\ 2^{a_k} f_{k-1})(f_k + x\ 2^{a_k} f_{k-1})}}_{h(x,y)}. \tag{34}$$

Again, (25) has been used. Since $h(1,1) \le h(x,y)$ and

$$h(1,1)^{-1} = \frac{f_k + 2^{a_k} f_{k-1}}{f_k} \le \frac{f_k + (2^{a_k} f_{k-1} + 2^{a_{k-1}} f_{k-2})}{f_k} = 2,$$

therefore, (34) implies

$$\frac{1}{2}\lambda(B) \le \lambda\left(T^{-k}B|\triangle_{a_1\cdots a_k}\right).$$

The last inequality also holds if B is a disjoint union of intervals, and therefore, it is also true for any $A \in \mathcal{F}$. This proves (33).

By (32) and (33), we have a similar inequality for $P(A)$:

$$\frac{1}{C}P(A) \le P\left(T^{-k}A|\triangle_{a_1\cdots a_k}\right). \tag{35}$$

Here, the constant C is given by $\frac{3}{\ln(4/3)}$. To complete the proof of Theorem 5, we proceed as follows. Suppose A is invariant under T (i.e., $T^{-1}A = A$) and $P(A) > 0$. Then inequality (35) implies $P(A) \le C\,P\left(A|\triangle_{a_1\cdots a_k}\right)$; i.e.,

$$P(A) \le C\ \frac{P\left(A \cap \triangle_{a_1\cdots a_k}\right)}{P(\triangle_{a_1\cdots a_k})}.$$

We multiply both sides by $P(\triangle_{a_1\cdots a_k})/P(A)$ and obtain $P(\triangle_{a_1\cdots a_k}) \le C\,P(\triangle_{a_1\cdots a_k}|A)$. This implies

$$P(E) \le C\,P(E|A) \tag{36}$$

for all finite disjoint unions E of cylinders; since the sets of cylinders generate $\mathcal{F}$, (36) is also true for any $E \in \mathcal{F}$. Setting $E = A^c$ (the complement of A), (36) implies $P(A^c) = 0$ and so $P(A) = 1$. This completes the proof of Theorem 5. $\square$

Finally, Theorem 4 follows immediately as the consequence of Theorem 5 and the ergodic theorem.

5. CONCLUDING REMARKS

In this paper, we have applied ergodic theory in computing the asymptotic growth rate of random Fibonacci type sequences. The key ingredients are the generalized Gauss map (13) and its ergodicity with respect to the generalized Gauss measure (31). The result obtained here can be used to estimate the effectiveness of continued fractions in the form of (10) (in the sense of Lochs [12]). This will be addressed in a forthcoming paper.

By imitating Mayer [14] and Wirsing [28], one can derive not just the constant but also the *rate* of convergence to the constant. Again, this will be addressed in a future paper.

Note added: The anonymous referee pointed out a beautiful generalization of the generalized Gauss map: Let $k = 2, 3, \cdots$, we have

$$Tx = \frac{k^{\left(\log_k \frac{1}{x}\right) \mod 1} - 1}{k - 1}. \tag{37}$$

Denote the density of the invariant measure by $\rho_k(x)$. Then, for $x \in (0, 1)$, one can show that (with C being a normalizing constant)

$$\rho_k(x) = \frac{C}{(1 + (k-1)x)(k + (k-1)x)},$$

as it is the eigenfunction of eigenvalue 1 of the corresponding Frobenius-Perron operator; i.e., it satisfies

$$\rho_k(x) = \sum_{a=0}^{\infty} \frac{k-1}{k^a(1 + (k-1)x)^2} \rho_k \left(\frac{1}{k^a(1 + (k-1)x)} \right).$$

The situation discussed in this paper corresponds to $k = 2$.

Note that (37) leads to a continued fraction expansion: following the notation in (5), this expansion will have $B_i = 1$ for $i \geq 1$; $A_1 = k^{-a_1}$ and $A_m = (k-1)k^{-a_m}$ for $m \geq 2$. It will be interesting to generalize Theorem 2 for the interval map (37).

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