COUPLED SECOND–ORDER RECURRENCES

Michael D. Hirschhorn
School of Mathematics, UNSW, Sydney 2052, Australia
(Submitted June 2003)

ABSTRACT

We give explicit solutions to two longstanding problems on coupled second–order recurrences.

1. INTRODUCTION

The authors of [2] pose two problems on coupled second–order recurrence relations, stated below, and say "This problem was formulated in 1986, but up to the moment it is open." (See [1].) The purpose of this note is to present and derive solutions to the recurrences.

2. STATEMENT OF RESULTS

The recurrences we solve are

$$\begin{aligned} a_{n+2} &= pb_{n+1} + qb_n, \\ b_{n+2} &= ra_{n+1} + sa_n \end{aligned} \tag{1}$$

and

$$\begin{aligned} a_{n+2} &= pa_{n+1} + qb_n, \\ b_{n+2} &= rb_{n+1} + sa_n. \end{aligned} \tag{2}$$

The solution of (1) is for $n > 0$,

$$\begin{aligned} a_n = a_0 &\sum_{\substack{(n-4)/4 \le k \le (n-3)/2 \\ l_2+l_4=n-(2k+3) \\ l_1+l_2=k+1 \\ l_3+l_4=k}} \binom{k+1}{l_2}\binom{k}{l_4} p^{l_1} q^{l_2} r^{l_3} s^{l_4+1} \\ &+ a_1 \sum_{\substack{(n-1)/4 \le k \le (n-1)/2 \\ l_2+l_4=n-(2k+1) \\ l_1+l_2=k \\ l_3+l_4=k}} \binom{k}{l_2}\binom{k}{l_4} p^{l_1} q^{l_2} r^{l_3} s^{l_4} \\ &+ b_0 \sum_{\substack{(n-2)/4 \le k \le (n-2)/2 \\ l_2+l_4=n-(2k+2) \\ l_1+l_2=k \\ l_3+l_4=k}} \binom{k}{l_2}\binom{k}{l_4} p^{l_1} q^{l_2+1} r^{l_3} s^{l_4} \\ &+ b_1 \sum_{\substack{(n-2)/4 \le k \le (n-2)/2 \\ l_2+l_4=n-(2k+2) \\ l_1+l_2=k+1 \\ l_3+l_4=k}} \binom{k+1}{l_2}\binom{k}{l_4} p^{l_1} q^{l_2} r^{l_3} s^{l_4}, \end{aligned} \tag{3}$$

$$
\begin{aligned}
b_n = a_0 &\sum_{\substack{(n-2)/4\le k\le (n-2)/2\\ l_2+l_4=n-(2k+2)\\ l_1+l_2=k\\ l_3+l_4=k}} \binom{k}{l_2}\binom{k}{l_4} p^{l_1}q^{l_2}r^{l_3}s^{l_4+1}\\
+ a_1 &\sum_{\substack{(n-3)/4\le k\le (n-2)/2\\ l_2+l_4=n-(2k+2)\\ l_1+l_2=k\\ l_3+l_4=k+1}} \binom{k}{l_2}\binom{k+1}{l_4} p^{l_1}q^{l_2}r^{l_3}s^{l_4}\\
+ b_0 &\sum_{\substack{(n-4)/4\le k\le (n-3)/2\\ l_2+l_4=n-(2k+3)\\ l_1+l_2=k\\ l_3+l_4=k+1}} \binom{k}{l_2}\binom{k+1}{l_4} p^{l_1}q^{l_2+1}r^{l_3}s^{l_4}\\
+ b_1 &\sum_{\substack{(n-1)/4\le k\le (n-1)/2\\ l_2+l_4=n-(2k+1)\\ l_1+l_2=k\\ l_3+l_4=k}} \binom{k}{l_2}\binom{k}{l_4} p^{l_1}q^{l_2}r^{l_3}s^{l_4},
\end{aligned}
$$

while the solution of (2) is

$$
\begin{aligned}
a_n = a_0 &\sum_{4k+l+m=n} \binom{k+l-1}{l}\binom{k+m-1}{m} p^l q^k r^m s^k\\
+ a_1 &\sum_{4k+l+m=n-1} \binom{k+l}{l}\binom{k+m-1}{m} p^l q^k r^m s^k\\
+ b_0 &\sum_{4k+l+m=n-2} \binom{k+l}{l}\binom{k+m-1}{m} p^l q^{k+1} r^m s^k\\
+ b_1 &\sum_{4k+l+m=n-3} \binom{k+l}{l}\binom{k+m}{m} p^l q^{k+1} r^m s^k,
\end{aligned}
\tag{4}
$$

$$
\begin{aligned}
b_n = a_0 &\sum_{4k+l+m=n-2} \binom{k+l-1}{l}\binom{k+m}{m} p^l q^k r^m s^{k+1}\\
+ a_1 &\sum_{4k+l+m=n-3} \binom{k+l}{l}\binom{k+m}{m} p^l q^k r^m s^{k+1}\\
+ b_0 &\sum_{4k+l+m=n} \binom{k+l-1}{l}\binom{k+m-1}{m} p^l q^k r^m s^k\\
+ b_1 &\sum_{4k+l+m=n-1} \binom{k+l-1}{l}\binom{k+m}{m} p^l q^k r^m s^k.
\end{aligned}
$$

3. DERIVATION OF (3)

We have

$$a_{n+2} = pb_{n+1} + qb_n,$$
$$b_{n+2} = ra_{n+1} + sa_n.$$

If we set

$$A = \sum_{n\geq 0} a_n t^n, \quad B = \sum_{n\geq 0} b_n t^n,$$

then

$$A - ptB - qt^2B = a_0 + (a_1 - pb_0)t,$$
$$B - rtA - st^2A = b_0 + (b_1 - ra_0)t,$$

or,

$$\begin{pmatrix} 1 & -pt - qt^2 \\ -rt - st^2 & 1 \end{pmatrix} \begin{pmatrix} A \\ B \end{pmatrix} = \begin{pmatrix} a_0 + (a_1 - pb_0)t \\ b_0 + (b_1 - ra_0)t \end{pmatrix}.$$

It follows that

$$\begin{pmatrix} A \\ B \end{pmatrix} = \frac{1}{1-(pt+qt^2)(rt+st^2)} \begin{pmatrix} 1 & pt + qt^2 \\ rt + st^2 & 1 \end{pmatrix} \begin{pmatrix} a_0 + (a_1 - pb_0)t \\ b_0 + (b_1 - ra_0)t \end{pmatrix}$$
$$= \frac{1}{1-(pt+qt^2)(rt+st^2)} \begin{pmatrix} 1 & pt + qt^2 \\ rt + st^2 & 1 \end{pmatrix} \begin{pmatrix} a_0 + a_1t - b_0pt \\ -ra_0t + b_0 + b_1t \end{pmatrix}$$
$$= \frac{1}{1-(pt+qt^2)(rt+st^2)} \begin{pmatrix} a_0(1 - prt^2 - qrt^3) + a_1t + b_0qt^2 + b_1(pt^2 + qt^3) \\ a_0st^2 + a_1(rt^2 + st^3) + b_0(1 - prt^2 - pst^3) + b_1t \end{pmatrix}.$$

It follows that

$$\sum_{n\geq 0} a_n t^n = a_0 \frac{1 - prt - qrt^2}{1-(pt+qt^2)(rt+st^2)} + a_1 \frac{t}{1-(pt+qt^2)(rt+st^2)}$$
$$+ b_0 \frac{qt^2}{1-(pt+qt^2)(rt+st^2)} + b_1 \frac{pt^2 + qt^3}{1-(pt+qt^2)(rt+st^2)}$$

$$= a_0 \left(1 + \frac{st^3(p+qt)}{1 - t^2(p+qt)(r+st)}\right) + a_1 \frac{t}{1 - t^2(p+qt)(r+st)}$$
$$+ b_0 \frac{qt^2}{1 - t^2(p+qt)(r+st)} + b_1 \frac{t^2(p+qt)}{1 - t^2(p+qt)(r+st)}$$

$$= a_0\left(1 + st^3(p+qt)\sum_{k\ge 0} t^{2k}(p+qt)^k(r+st)^k\right)$$

$$+ a_1 t\sum_{k\ge 0} t^{2k}(p+qt)^k(r+st)^k$$

$$+ b_0 qt^2\sum_{k\ge 0} t^{2k}(p+qt)^k(r+st)^k$$

$$+ b_1 t^2(p+qt)\sum_{k\ge 0} t^{2k}(p+qt)^k(r+st)^k$$

$$= a_0\left(1 + \sum_{k\ge 0} st^{2k+3}\sum_{l_1+l_2=k+1}\binom{k+1}{l_2}p^{l_1}q^{l_2}t^{l_2}\sum_{l_3+l_4=k}\binom{k}{l_4}r^{l_3}s^{l_4}t^{l_4}\right)$$

$$+ a_1\sum_{k\ge 0} t^{2k+1}\sum_{l_1+l_2=k}\binom{k}{l_2}p^{l_1}q^{l_2}t^{l_2}\sum_{l_3+l_4=k}\binom{k}{l_4}r^{l_3}s^{l_4}t^{l_4}$$

$$+ b_0\sum_{k\ge 0} qt^{2k+2}\sum_{l_1+l_2=k}\binom{k}{l_2}p^{l_1}q^{l_2}t^{l_2}\sum_{l_3+l_4=k}\binom{k}{l_4}r^{l_3}s^{l_4}t^{l_4}$$

$$+ b_1\sum_{k\ge 0} t^{2k+2}\sum_{l_1+l_2=k+1}\binom{k+1}{l_2}p^{l_1}q^{l_2}t^{l_2}\sum_{l_3+l_4=k}\binom{k}{l_4}r^{l_3}s^{l_4}t^{l_4}.$$

The formula for a_n follows. The formula for b_n can be obtained similarly. □

4. DERIVATION OF (4)

We have

$$\begin{aligned} a_{n+2} &= pa_{n+1} + qb_n,\\ b_{n+2} &= rb_{n+1} + sa_n. \end{aligned}$$

If we set

$$A = \sum_{n\ge 0} a_n t^n, \quad B = \sum_{n\ge 0} b_n t^n,$$

then

$$\begin{aligned} A - ptA - qt^2B &= a_0 + (a_1 - pa_0)t,\\ B - rtB - st^2A &= b_0 + (b_1 - rb_0)t, \end{aligned}$$

or,

$$\begin{pmatrix} 1-pt & -qt^2 \\ -st^2 & 1-rt \end{pmatrix} \begin{pmatrix} A \\ B \end{pmatrix} = \begin{pmatrix} a_0+(a_1-pa_0)t \\ b_0+(b_1-rb_0)t \end{pmatrix}.$$

It follows that

$$\begin{pmatrix} A \\ B \end{pmatrix} = \frac{1}{(1-pt)(1-rt)-qst^4} \begin{pmatrix} 1-rt & qt^2 \\ st^2 & 1-pt \end{pmatrix} \begin{pmatrix} a_0(1-pt)+a_1t \\ b_0(1-rt)+b_1t \end{pmatrix}$$
$$= \frac{1}{(1-pt)(1-rt)-qst^4} \begin{pmatrix} a_0(1-pt)(1-rt)+a_1t(1-rt)+b_0qt^2(1-rt)+b_1qt^3 \\ a_0st^2(1-pt)+a_1st^3+b_0(1-pt)(1-rt)+b_1t(1-pt) \end{pmatrix}.$$

It follows that

$$\begin{aligned}
\sum_{n\geq 0} a_nt^n &= a_0\frac{(1-pt)(1-rt)}{(1-pt)(1-rt)-qst^4} + a_1\frac{t(1-rt)}{(1-pt)(1-rt)-qst^4} \\
&+ b_0\frac{qt^2(1-rt)}{(1-pt)(1-rt)-qst^4} + b_1\frac{qt^3}{(1-pt)(1-rt)-qst^4} \\
&= a_0\frac{1}{1-\frac{qst^4}{(1-pt)(1-rt)}} + a_1\frac{t}{1-pt}\frac{1}{1-\frac{qst^4}{(1-pt)(1-rt)}} \\
&+ b_0\frac{qt^2}{1-pt}\frac{1}{1-\frac{qst^4}{(1-pt)(1-rt)}} + b_1\frac{qt^3}{(1-pt)(1-rt)}\frac{1}{1-\frac{qst^4}{(1-pt)(1-rt)}} \\
&= a_0\sum_{k\geq 0}\frac{q^ks^kt^{4k}}{(1-pt)^k(1-rt)^k} + a_1\sum_{k\geq 0}\frac{q^ks^kt^{4k+1}}{(1-pt)^{k+1}(1-rt)^k} \\
&+ b_0\sum_{k\geq 0}\frac{q^{k+1}s^kt^{4k+2}}{(1-pt)^{k+1}(1-rt)^k} + b_1\sum_{k\geq 0}\frac{q^{k+1}s^kt^{4k+3}}{(1-pt)^{k+1}(1-rt)^{k+1}} \\
&= a_0\sum_{k\geq 0}q^ks^kt^{4k}\sum_{l\geq 0}\binom{k+l-1}{l}p^lt^l\sum_{m\geq 0}\binom{k+m-1}{m}r^mt^m \\
&+ a_1\sum_{k\geq 0}q^ks^kt^{4k+1}\sum_{l\geq 0}\binom{k+l}{l}p^lt^l\sum_{m\geq 0}\binom{k+m-1}{m}r^mt^m \\
&+ b_0\sum_{k\geq 0}q^{k+1}s^kt^{4k+2}\sum_{l\geq 0}\binom{k+l}{l}p^lt^l\sum_{m\geq 0}\binom{k+m-1}{m}r^mt^m \\
&+ b_1\sum_{k\geq 0}q^{k+1}s^kt^{4k+3}\sum_{l\geq 0}\binom{k+l}{l}p^lt^l\sum_{m\geq 0}\binom{k+m}{m}r^mt^m.
\end{aligned}$$

The formula for a_n follows. The formula for b_n can be obtained similarly. □

ADDITIONAL REMARKS

In the special case $r = p$, $s = q$, simpler formulas can be obtained for a_n and b_n by following the same method. But in that case also, it is easy to factorise the quartics that occur in the denominators and obtain formulas for a_n and b_n that make it plain how fast they grow.

REFERENCES

[1] K. Atanassov. "On a Second New Generalization of the Fibonacci Sequence." *The Fibonacci Quarterly* **24** (1986): 362-365.
[2] K. Atanassov, V. Atanassova, A. Shannon and J. Turner. *New Visual Perspectives on Fibonacci Numbers*, World Scientific, 2002.

AMS Classification Numbers: 11B39, 11B37

✠ ✠ ✠