ON SUMS OF CERTAIN PRODUCTS OF LUCAS NUMBERS

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(Submitted April 2004)

ABSTRACT

New results about certain sums $S_n(k)$ of products of the Lucas numbers are derived. These sums are related to the generating function of the k-th powers of the Fibonacci numbers. The sums for $S_n(k)$ are expressed by the binomial and the Fibonomial coefficients. Proofs of these formulas are based on a special inverse formula.

1. INTRODUCTION

Generating functions are very helpful in finding many important relations for sequences of integers. Many of these identities for the Fibonacci numbers F_n and Lucas numbers L_n were found by simple manipulation of the various generating functions. Our approach to the problem is rather different. This paper is devoted to certain generalizations of the well–known formulas for the Fibonacci and Lucas numbers (see [8] pp. 179–183), for example

$$\sum_{i=0}^{n}(-1)^i L_{n-2i} = 2F_{n+1} \ . \tag{1}$$

In the past much attention has been focused on the generating function $f_k(x) = \sum_{n=0}^{\infty} F_n^k x^n$ for the k-th powers of F_n. In [4] Riordan found the general recurrence for $f_k(x)$ considering the initial obsolete conditions $F_0 = F_1 = 1$. We can rewrite his result with initial conditions $F_0 = 0$, $F_1 = 1$ as

$$(1 - L_k x + (-1)^k x^2) f_k(x) = x + kx \sum_{i=1}^{\lfloor k/2 \rfloor} A_{ki} f_{k-2i}(x(-1)^i) \ ,$$

where $\lfloor \frac{k}{2} \rfloor$ is the integer part of $\frac{k}{2}$ and A_{ki} are integers given by the equality $A_{ki} = \frac{a_{ki}}{i}$. Riordan showed that the numbers a_{ki} satisfy the relation

$$\frac{1}{(1-x-x^2)^i} = \sum_{k=2i}^{\infty} a_{ki} x^{k-2i} \ .$$

Recently Dujella in [2] discovered a more elegant way to compute a_{ki}

$$a_{ki} = \sum_{i=1}^{\lfloor k/2 \rfloor} \binom{k-m-1}{m-1} \binom{m}{i} \frac{1}{m}$$

and published a bijective proof of Riordan's theorem using the Morse code interpretation.

Carlitz in [1] and Horadam in [3] generalized Riordan's result and found similar recurrences for the generating functions of different types of generalized Fibonacci numbers. They found closed form for the polynomial $N_k(x)$ in the numerator and the polynomial $D_k(x)$ in the denominator of the generating functions.

As a special case of Horadam's result it is possible to get the following formula for the generating function of an odd integer powers of Fibonacci numbers

$$f_k(x) = \frac{\sum_{i=0}^{k} \sum_{j=0}^{i} (-1)^{\frac{j(j+1)}{2}} \begin{bmatrix} k+1 \\ j \end{bmatrix} F_{i-j}^{k}\, x^i}{\sum_{i=0}^{k+1} (-1)^{\frac{i(i+1)}{2}} \begin{bmatrix} k+1 \\ i \end{bmatrix} x^i}\ , \tag{2}$$

where $\begin{bmatrix} n \\ k \end{bmatrix}$ are the so-called Fibonomial coefficients defined by

$$\begin{bmatrix} n \\ k \end{bmatrix} = \frac{F_n F_{n-1} \cdots F_{n-k+1}}{F_1 F_2 \cdots F_k}\ , \quad \begin{bmatrix} n \\ 0 \end{bmatrix} = 1\ .$$

Applying Carlitz' approach, Shannon obtained in [6] some special results for the numerator and the denominator in the expression of the generating function $f_k(x)$. Later Stănică extended in [7] Horadam's results giving also some new formulas for weighted cases.

It is easy to obtain for any odd integer k that

$$f_k(x) = 5^{-\frac{k-1}{2}} \sum_{j=0}^{\frac{k-1}{2}} \binom{k}{j} \frac{F_{k-2j}\,x}{1-(-1)^j L_{k-2j}\,x - x^2} \tag{3}$$

after simplification of one of Shannon's results.

As k is an odd positive integer, the denominator $D_{k+1}(x)$ is a polynomial of even degree $k+1$ and the relation

$$D_{k+1}(x) = \prod_{j=0}^{\frac{k-1}{2}} (1 - (-1)^j L_{k-2j}\,x - x^2)$$

holds. Thus, the coefficients of the powers of x in $D_{k+1}(x)$ include the following sums of products of the Lucas numbers

$$\sum_{i_n=0}^{\frac{k-1}{2}} \sum_{i_{n-1}=i_n+1}^{\frac{k-1}{2}} \cdots \sum_{i_{n-2}=i_{n-1}+1}^{\frac{k-1}{2}} (-1)^{i_1+i_2+\cdots+i_n} \prod_{j=1}^{n} L_{k-2\,i_j}\ .$$

Combining (2) and (3) we will find some new results about these sums with the help of the Fibonomial coefficients.

2. THE MAIN RESULT

Define the sequence $\{S_n(k)\}_{n=0}^{\infty}$ for any odd positive integer k in the following way:

$$S_0(k) = 1\ , \quad S_1(k) = \sum_{i_1=0}^{\frac{k-1}{2}} (-1)^{i_1} L_{k-2\,i_1}$$

and

$$S_n(k) = \sum_{i_n=0}^{\frac{k-1}{2}} \sum_{i_{n-1}=i_n+1}^{\frac{k-1}{2}} \cdots \sum_{i_1=i_2+1}^{\frac{k-1}{2}} (-1)^{i_1+i_2+\cdots+i_n} \prod_{j=1}^{n} L_{k-2\,i_j} \tag{4}$$

for any positive integer $n > 1$.

The main result about these sums is given in the next theorem.

Theorem 1: *Let k be any odd positive integer and n be any positive integer. Then*

$$S_{2n-1}(k) = \sum_{i=1}^{n} (-1)^{(i-1)(2i+1)} \left(\binom{\frac{k+3}{2} - n - i}{n-i} + \binom{\frac{k+1}{2} - n - i}{n-i-1} \right) \begin{bmatrix} k+1 \\ 2i-1 \end{bmatrix}$$

and

$$S_{2(n-1)}(k) = \sum_{i=1}^{n} (-1)^{(i+1)(2i-1)} \left(\binom{\frac{k+5}{2} - n - i}{n-i} + \binom{\frac{k+3}{2} - n - i}{n-i-1} \right) \begin{bmatrix} k+1 \\ 2(i-1) \end{bmatrix} .$$

3. THE PRELIMINARY RESULTS

Let $\{G_n\}$ be a generalized Fibonacci sequence, which obeys the recurrence relation $G_{n+2} = G_n + G_{n+1}$ with arbitrary seeds G_0 and G_1. This leads to the generalized Binet formula

$$G_n = A\alpha^n + B\beta^n, \qquad \text{where} \qquad \alpha = (1+\sqrt{5})/2, \beta = (1-\sqrt{5})/2 \; .$$

There are many identities for the generalized Fibonacci numbers G_n (see e.g. [8]). We will need the following special identities which are generalizations of (1).

Theorem 2: *Let a, $p \neq 0$, q be arbitrary integers and n be a nonnegative integer. Then*

$$\sum_{i=a}^{n} G_{pi+q} = \frac{G_{p(n+1)+q} + (-1)^{p+1} G_{pn+q} + (-1)^p G_{p(a-1)+q} - G_{pa+q}}{1 + (-1)^p - L_p} \tag{5}$$

and

$$\sum_{i=a}^{n} (-1)^i G_{pi+q} = \tag{6}$$

$$= \frac{(-1)^n G_{p(n+1)+q} + (-1)^{n+p} G_{pn+q} + (-1)^a G_{pa+q} + (-1)^{a+p} G_{p(a-1)+q}}{1 + (-1)^p + L_p} \; .$$

Proof: Using the generalized Binet formula we get immediately (5) and (6). □

Theorem 3: *Let a, $p \neq 0$, q be arbitrary integers and n be a nonnegative integer. Then*

$$\begin{aligned}\sum_{i=a}^{n} iG_{pi+q} &= \frac{nG_{p(n+2)+q} - G_{p(n-1)+q} - (n+1+2n(-1)^p)G_{p(n+1)+q}}{(1+(-1)^p - L_p)^2} \\ &+ \frac{(n+2(-1)^p(n+1))G_{pn+q}(n+1)}{(1+(-1)^p - L_p)^2} \\ &+ \frac{aG_{p(a-2)+q} - (a-1)G_{p(a+1)+q}}{(1+(-1)^p - L_p)^2} \\ &+ \frac{(a+2(a-1)(-1)^p)G_{pa+q} - (a-1+2a(-1)^p)G_{p(a-1)+q}}{(1+(-1)^p - L_p)^2}\end{aligned} \tag{7}$$

and

$$\begin{aligned}\sum_{i=a}^{n}(-1)^{i-1} iG_{pi+q} &= \frac{n(-1)^{n+1}G_{p(n+2)+q} - (n+1+2n(-1)^p)(-1)^n G_{p(n+1)+q}}{(1+(-1)^p + L_p)^2} \\ &+ \frac{(-1)^{n+1}\left((n+2(-1)^p(n+1))G_{pn+q} + (n+1)G_{p(n-1)+q}\right)}{(1+(-1)^p + L_p)^2} \\ &- \frac{(a-1)(-1)^a G_{p(a+1)+q} + (a+2(a-1)(-1)^p)(-1)^a G_{pa+q}}{(1+(-1)^p + L_p)^2} \\ &- \frac{(a-1+2a(-1)^p)(-1)^a G_{p(a-1)+q} + a(-1)^a G_{p(a-2)+q}}{(1+(-1)^p + L_p)^2}\ ,\end{aligned}$$

which we will denote by (8).

Proof: These identities can be proved in a similar way as Theorem 2 but now using the identity

$$\sum_{i=a}^{n} ix^{i-1} = \frac{nx^{n+1} - (n+1)x^n - (a-1)x^a + ax^{a-1}}{(x-1)^2}\ ,$$

which is formed by differentiating the formula for the sum of a geometrical progression. □

Lemma 1: *Let n be any positive integer. Then $S_n(k) = 0$ for each odd positive integer $k < 2n-1$.*

Proof: Rewriting relation (4) in the form

$$S_n(k) = \sum_{\substack{i_1, i_2, \ldots, i_n \\ 0 \le i_n < i_{n-1} < \cdots < i_1 \le \frac{k-1}{2}}} (-1)^{i_1+i_2+\cdots+i_n} \prod_{j=1}^{n} L_{k-2\,i_j}$$

the assertion easily follows from the condition

$$0 \le i_n < i_{n-1} < \cdots < i_1 \le \frac{k-1}{2}$$

which does not hold for any values $i_1, i_2, \ldots, i_n$ if $\frac{k-1}{2} < n-1$. □

Lemma 2: *Let k be any odd positive integer and n be any positive integer. Then*

$$(i) \qquad \sum_{i=1}^{n} \binom{n+i-\frac{k+5}{2}}{n-i} S_{2i-1}(k) = 0 \quad \textit{for} \quad n \geq \frac{k+3}{2}$$

and

$$(ii) \quad \sum_{i=1}^{n} \binom{n+i-\frac{k+7}{2}}{n-i} S_{2(i-1)}(k) = 0 \quad \textit{for} \quad n \geq \frac{k+5}{2} \ .$$

Proof: We show the proof of (i). Case (ii) can be proved by the analogous procedure. Let l be any even positive integer. Thus, each positive integer $n \geq \frac{k+3}{2}$ can be written in the form $n = \frac{k+3+l}{2}$. Then for the sum in (i) the following holds

$$\sum_{i=1}^{\frac{k+l+3}{2}} \binom{i+\frac{l-2}{2}}{\frac{k+l+3}{2}-i} S_{2i-1}(k) = P(k,l) + Q(k,l) \ ,$$

where

$$P(k,l) = \sum_{i=1}^{\lfloor \frac{k+3}{4} \rfloor} \binom{i+\frac{l-2}{2}}{\frac{k+l+3}{2}-i} S_{2i-1}(k)$$

and

$$Q(k,l) = \sum_{i=\lfloor \frac{k+3}{4} \rfloor+1}^{\frac{k+l+3}{2}} \binom{i+\frac{l-2}{2}}{\frac{k+l+3}{2}-i} S_{2i-1}(k)$$

$$= \sum_{p=0}^{\frac{k+l+1}{2}-\lfloor \frac{k+3}{4} \rfloor} \binom{\lfloor \frac{k+3}{4} \rfloor + \frac{l}{2} + p}{\frac{k+l+1}{2} - \lfloor \frac{k+3}{4} \rfloor - p} S_{2\lfloor \frac{k+3}{4} \rfloor+1+2p}(k) \ .$$

It is easily seen that $\binom{i+\frac{l-2}{2}}{\frac{k+l+3}{2}-i} = 0$ for $i < \frac{k+5}{2}$ and therefore $P(k,l) = 0$ for any k, l. Since for any nonnegative integer p the equality $S_{2\lfloor \frac{k+3}{4} \rfloor+1+2p}(k) = 0$ is implied by Lemma 1, it follows that $Q(k,l) = 0$. □

4. THE SPECIAL CASES FOR SMALL n

We now consider the integers $S_n(k)$ for values $n = 1$, 2 and 3.

Theorem 4: *Let k be any odd positive integer. Then*

$$(i) \quad S_1(k) = \sum_{i_1=0}^{\frac{k-1}{2}} (-1)^{i_1} L_{k-2i_1} = F_{k+1},$$

$$(ii) \quad S_2(k) = \sum_{i_2=0}^{\frac{k-1}{2}} \sum_{i_1=i_2+1}^{\frac{k-1}{2}} (-1)^{i_1+i_2} L_{k-2i_2} L_{k-2i_1} = \frac{k+1}{2} - F_{k+1}F_k,$$

$$(iii) \quad S_3(k) = \sum_{i_3=0}^{\frac{k-1}{2}} \sum_{i_2=i_3+1}^{\frac{k-1}{2}} \sum_{i_1=i_2+1}^{\frac{k-1}{2}} (-1)^{i_3+i_2+i_1} L_{k-2i_3} L_{k-2i_2} L_{k-2i_1}$$
$$= \frac{k-1}{2} F_{k+1} - \frac{1}{2} F_{k+1} F_k F_{k-1} \ .$$

Proof: All cases can be proved with a suitable choice of the parameters in Theorem 2 and Theorem 3:

(i) Putting $n = \frac{k-1}{2}$, $p = -2$, $q = k$ and $a = 0$ in (5) we have immediately

$$S_1(k) = \sum_{i_1=0}^{\frac{k-1}{2}} (-1)^{i_1} L_{k-2i_1} = F_{k+1} \ .$$

(ii) If we take $n = \frac{k-1}{2}$, $p = -4$, $q = 2k-1$, $a = 0$ and use the identities $L_n = F_{n+2} - F_{n-2}$ and $L_{n+m} + (-1)^m L_{n-m} = 5 F_m F_n$ (see [8], (17b)), we get, using (5),

$$\sum_{i=0}^{\frac{k-1}{2}} F_{2(k-2i)-1} = \frac{1}{5}(F_{2k+3} - F_{2k-1} + 1) = \frac{1}{5}(L_{2k+1} + 1) = F_{k+1} F_k.$$

Setting $n = \frac{k-1}{2}$, $p = -2$, $q = k$ we obtain from (6)

$$\sum_{i=a}^{\frac{k-1}{2}} (-1)^i L_{k-2i} = (-1)^a F_{k-2a+1} \ ,$$

where a is a nonnegative integer.
Finally using the identity $L_n F_{n-1} = F_{2n-1} + (-1)^n$ (see [8], (15b)) we have

$$S_2(k) = \sum_{i_2=0}^{\frac{k-1}{2}} \sum_{i_1=i_2+1}^{\frac{k-1}{2}} (-1)^{i_1+i_2} L_{k-2i_2} L_{k-2i_1} = -\sum_{i_2=0}^{\frac{k-1}{2}} L_{k-2i_2} F_{(k-2i_2)-1}$$
$$= \sum_{i_2=0}^{\frac{k-1}{2}} (1 - F_{2(k-2i_2)-1}) = \frac{k+1}{2} - F_{k+1} F_k.$$

(iii) First, we derive several identities which are necessary to prove this case. Setting $n = \frac{k-1}{2}$, $p = -4$, $q = 2k-1$, $a = j+1$ and using the well–known identity $L_n = F_{n+2} - F_{n-2}$ we obtain from (5)

$$\sum_{i=j+1}^{\frac{k-1}{2}} F_{2(k-2i)-1} = \frac{1}{5}(F_{2(k-2j)-1} - F_{2(k-2j)-5} + 1) = \frac{1}{5}(L_{2(k-2j)-3} + 1) \ .$$

For $a = 0$, $p = -6$, $q = 3(k-1)$ and $n = \frac{k-1}{2}$ we get from (6), using the relation $L_{m-3} + L_{m+3} = 10 F_{3m}$,

$$\sum_{j=0}^{\frac{k-1}{2}} (-1)^j L_{3(k-2j-1)} = (-1)^{\frac{k-1}{2}} + \frac{1}{20}(L_{3(k-1)} + L_{3(k+1)}) = (-1)^{\frac{k-1}{2}} + \frac{1}{2} F_{3k} \ .$$

For $a = 0$, $p = -2$, $q = k - 3$ and $n = \frac{k-1}{2}$ the identity

$$\sum_{j=0}^{\frac{k-1}{2}} (-1)^j L_{k-2j-3} = 2(-1)^{\frac{k-1}{2}} + F_{k-2}$$

follows from (6).

Setting $a = 0$, $p = -2$, $q = k$ and $n = \frac{k-1}{2}$ we get from (8) using the identities $L_k + L_{k+2} = 5F_{k+1}$ and $F_{k-1} + F_{k+1} = L_k$ (see [8], (5) and (6))

$$\sum_{i=0}^{\frac{k-1}{2}} (-1)^i i L_{k-2i} = \frac{1}{5}((-1)^{\frac{k-1}{2}} - L_k) \ .$$

From (17a) in [8] we obtain the following special case

$$L_{k-2j} L_{2(k-2j)-3} = L_{3(k-2j-1)} - L_{k-2j-3} \ .$$

The previous identities enable us to finish the proof of the third case:

$$\begin{aligned}
S_3(k) &= \sum_{i_3=0}^{\frac{k-1}{2}} \sum_{i_2=i_3+1}^{\frac{k-1}{2}} \sum_{i_1=i_2+1}^{\frac{k-1}{2}} (-1)^{i_3+i_2+i_1} L_{k-2i_3} L_{k-2i_2} L_{k-2i_1} \\
&= \sum_{i_3=0}^{\frac{k-1}{2}} \sum_{i_2=i_3+1}^{\frac{k-1}{2}} (-1)^{i_3+1} L_{k-2i_3} L_{k-2i_2} F_{(k-2i_2)-1} \\
&= \sum_{i_3=0}^{\frac{k-1}{2}} \sum_{i_2=i_3+1}^{\frac{k-1}{2}} (-1)^{i_3+1} L_{k-2i_3} (F_{2(k-2i_2)-1} - 1) \\
&= \sum_{i_3=0}^{\frac{k-1}{2}} (-1)^{i_3+1} L_{k-2i_3} \left(\frac{1}{5}(L_{2(k-2i_3)-3} + 1) - \frac{k-1}{2} + i_3 \right) \\
&= \left(\frac{k-1}{2} - \frac{1}{5} \right) \sum_{i_3=0}^{\frac{k-1}{2}} (-1)^{i_3} L_{k-2i_3} - \sum_{i_3=0}^{\frac{k-1}{2}} (-1)^{i_3} i_3 L_{k-2i_3} \\
&- \frac{1}{5} \sum_{i_3=0}^{\frac{k-1}{2}} (-1)^{i_3} L_{3(k-2i_3-1)} + \frac{1}{5} \sum_{i_3=0}^{\frac{k-1}{2}} (-1)^{i_3} L_{k-2i_3-3} \\
&= \frac{k-1}{2} F_{k+1} + \frac{1}{10}(2F_k - F_{3k}) = \frac{k-1}{2} F_{k+1} - \frac{1}{2} F_{k+1} F_k F_{k-1}. \quad \square
\end{aligned}$$

Remark: *It is known that $L_{-m} = (-1)^m L_m$. If we assume that n in (1) is an odd number, then*

$$\sum_{i=0}^{n} (-1)^i L_{n-2i} = \sum_{i=0}^{\frac{n-1}{2}} (-1)^i L_{n-2i} + \sum_{i=\frac{n+1}{2}}^{n} (-1)^i L_{n-2i} = 2 \sum_{i=0}^{\frac{n-1}{2}} (-1)^i L_{n-2i} = 2F_{n+1} \ ,$$

which shows a relation between case (i) in Theorem 4 and (1).

The previous method of evaluation of the integers $S_n(k)$ can be used similarly also for $n > 3$. Again we only need the identities (5), (6), (7) and (8) for suitable values of the parameters a, k, p, q. But its concrete realization would be more complicated.

5. THE PROOF OF THE MAIN THEOREM

Relations (2) and (3) which hold for any odd positive integer k lead to

$$D_{k+1}(x) = \prod_{j=0}^{\frac{k-1}{2}} (1 - (-1)^j L_{k-2j} x - x^2) = \sum_{i=0}^{k+1} d_{k+1,i}\, x^i \ ,$$

where $d_{k+1,i} = (-1)^{\frac{i(i+1)}{2}} \begin{bmatrix} k+1 \\ i \end{bmatrix}$.

After multiplication of all factors in $D_{k+1}(x)$ it follows that

$$d_{k+1,0} = S_0(k) = 1 \ , \quad d_{k+1,i} = \sum_{l=0}^{\lfloor \frac{i}{2} \rfloor} \binom{\frac{k+1}{2} - (i-2l)}{l} (-1)^{i+l} S_{i-2l}(k) \ ,$$

where $i = 1, 2, \ldots, k+1$.

We can rewrite the last identity into the following two relations for any positive integer n

$$d_{k+1,2n-1} = - \sum_{i=1}^{n} \binom{n+i-\frac{k+5}{2}}{n-i} S_{2i-1}(k) \ , \tag{9}$$

$$d_{k+1,2(n-1)} = \sum_{i=1}^{n} \binom{n+i-\frac{k+7}{2}}{n-i} S_{2(i-1)}(k) \ , \tag{10}$$

with respect to Lemma 2 and the well–known formula $\binom{r}{m} = (-1)^m \binom{m-r-1}{m}$.

Proof of Theorem 1: We have to invert identities (9) and (10) to obtain explicit formulas for the sums $S_n(k)$. We use the inversion theorem from [5] (see (23), p. 74). Thus,

$$a_n = \sum_{i=1}^{n} \binom{n+i+p}{n-i} b_i$$

holds if and only if

$$b_n = \sum_{i=1}^{n} (-1)^{i+n} \left(\binom{2n+p}{n-i} - \binom{2n+p}{n-i-1} \right) a_i \ , \tag{11}$$

where p is any integer.

To prove the first equality in Theorem 1 we set $a_n = d_{k+1,2n-1}$, $b_i = -S_{2i-1}(k)$ and $p = -\frac{k+5}{2}$ in (11). Then identity

$$\begin{aligned} S_{2n-1}(k) &= \sum_{i=1}^{n} (-1)^{n-i+1} \left(\binom{2n-\frac{k+5}{2}}{n-i} - \binom{2n-\frac{k+5}{2}}{n-i-1} \right) d_{k+1,2i-1} \\ &= \sum_{i=1}^{n} (-1)^{1} \left(\binom{\frac{k+3}{2}-n-i}{n-i} + \binom{\frac{k+1}{2}-n-i}{n-i-1} \right) d_{k+1,2i-1} \end{aligned}$$

holds. Thus,

$$S_{2n-1}(k) = \sum_{i=1}^{n}(-1)^{(i-1)(2i+1)} \left(\binom{\frac{k+3}{2}-n-i}{n-i} + \binom{\frac{k+1}{2}-n-i}{n-i-1} \right) \begin{bmatrix} k+1 \\ 2i-1 \end{bmatrix} .$$

Similarly setting $a_n = d_{k+1,2(n-1)}$, $b_i = S_{2(i-1)}(k)$ and $p = -\frac{k+7}{2}$ in (11) we get

$$\begin{aligned} S_{2(n-1)}(k) =& \sum_{i=1}^{n}(-1)^{n-i} \left(\binom{2n-\frac{k+7}{2}}{n-i} - \binom{2n-\frac{k+7}{2}}{n-i-1} \right) d_{k+1,2(i-1)} \\ =& \sum_{i=1}^{n} \left(\binom{\frac{k+5}{2}-n-i}{n-i} + \binom{\frac{k+3}{2}-n-i}{n-i-1} \right) d_{k+1,2(i-1)} \\ =& \sum_{i=1}^{n}(-1)^{(i+1)(2i-1)} \left(\binom{\frac{k+5}{2}-n-i}{n-i} + \binom{\frac{k+3}{2}-n-i}{n-i-1} \right) \begin{bmatrix} k+1 \\ 2(i-1) \end{bmatrix} . \quad \square \end{aligned}$$

6. CONCLUSION

The effectiveness of the formulas from Theorem 1 for the computation of $S_n(k)$ is shown by the following fact. Using the standard PC we have found that the computation of $S_{12}(51)$ by relation (4) took 26.5 minutes approximately and by Theorem 1 less than a second only.

ACKNOWLEDGMENTS

The paper was supported by the research project MSM 184400 002.

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AMS Classification Numbers: 11B39, 05A15, 05A19

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