DEDEKIND SUMS AND SOME GENERALIZED FIBONACCI AND LUCAS SEQUENCES

KARL DILCHER AND JEFFREY L. MEYER

Abstract. We derive evaluations of the classical Dedekind sum $s(h,k)$ for certain classes of generalized Fibonacci and Lucas numbers that had previously not been considered. As particular cases we obtain explicit formulas for $s(p^n-1, p^{n+1}-1)$ for integers $p \geq 2$ and $n \geq 1$, and for $s(p^n+1, p^{n+1}+1)$, with $p \geq 2$ even.

1. Introduction

The classical Dedekind sum is defined by

$$s(d,c) = \sum_{j=1}^{c} \left(\left(\frac{j}{c}\right)\right)\left(\left(\frac{dj}{c}\right)\right),$$

with

$$((x)) = \begin{cases} 0, & \text{if } x \in \mathbb{Z}, \\ x - [x] - \frac{1}{2}, & \text{otherwise.} \end{cases}$$

Because of important applications, mainly in number theory, the Dedekind sum has been studied extensively by many authors in a variety of contexts. See Rademacher and Grosswald [4] for a bibliography. The most important result about Dedekind sums, first proved by Dedekind himself [2], is the reciprocity law. There are many different proofs in the literature, including four in [4].

Theorem 1 (Reciprocity Law). *If $(h,k)=1$ and $h,k>0$, then*

$$s(k,h) + s(h,k) = \frac{h^2+k^2+1-3hk}{12hk}. \tag{1.1}$$

From the definition and with the help of this reciprocity law one can easily obtain special values of the Dedekind sum, among them

$$s(1,k) = \frac{(k-1)(k-2)}{12k}, \tag{1.2}$$

and, if k is odd,

$$s(2,k) = \frac{(k-1)(k-5)}{24k} \tag{1.3}$$

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(see, e.g., [1, p. 62]), with numerous extensions that can be found in [1, p. 73] and more recently in [6]. A related property is the fact that the only *integer* value taken by $s(h,k)$ is zero, and

$$s(h,k) = 0 \qquad \text{if and only if} \qquad h^2 + 1 \equiv 0 \pmod{k}.$$

The second author [3] used this and the reciprocity law (1.1) to show that $s(h,k) = s(k,h)$ if and only if $h = F_{2n+1}$ and $k = F_{2n+3}$ for positive integers n, where F_n is the nth Fibonacci number.

This indicates that Fibonacci numbers play a special role in the evaluation of the Dedekind sum. Indeed, special values of the Dedekind sum can be found in [1, p. 72] and [7], with extensions to Lucas and generalized Lucas numbers in [5], and further extensions in [8].

It is the purpose of this paper to deal with several classes of generalized Fibonacci and Lucas sequences not covered by the results in [8]. In Section 2 we introduce the sequences to be considered, and in Section 3 we prove our evaluations. We conclude this paper with some easy consequences in Section 4.

2. The Sequences

The sequences under consideration here are as follows. For $p, q \in \mathbb{Z}$ we define the generalized Fibonacci and Lucas sequences, respectively, by

$$u_0 = 0, \quad u_1 = 1, \quad u_{n+1} = pu_n - qu_{n-1}$$

and

$$v_0 = 2, \quad v_1 = p, \quad v_{n+1} = pv_n - qv_{n-1}.$$

If we set $\Delta = p^2 - 4q$, $\alpha = (p + \sqrt{\Delta})/2$, and $\beta = (p - \sqrt{\Delta})/2$, these sequences have the standard Binet forms, for $n \geq 0$,

$$u_n = \frac{\alpha^n - \beta^n}{\alpha - \beta} \quad \text{and} \quad v_n = \alpha^n + \beta^n. \tag{2.1}$$

The classical Fibonacci and Lucas sequences are obtained when $p = 1$ and $q = -1$. In [5] and [8], the class of sequences investigated involve $|q| = 1$. In this paper we consider certain cases involving arbitrary values of q; in particular we address the sequences

$$u_{n+1} = pu_n + (p+1)u_{n-1}, \qquad v_{n+1} = pv_n + (p+1)v_{n-1} \tag{2.2}$$

and

$$u_{n+1} = (p+1)u_n - pu_{n-1}, \qquad v_{n+1} = (p+1)v_n - pv_{n-1}, \tag{2.3}$$

with initial values as above. Because the characteristic polynomials of these sequences are reducible, we get the following first-order recurrences.

Lemma 1. *For the sequences given by* (2.2) *we have, respectively,*

$$u_{n+1} = (p+1)u_n + (-1)^n$$

and

$$v_{n+1} = (p+1)v_n + (-1)^{n+1}(p+2).$$

Lemma 2. *For the sequences given by* (2.3) *we have, respectively,*

$$u_{n+1} = pu_n + 1$$

and

$$v_{n+1} = pv_n - (p-1).$$

The proofs of these lemmas are straightforward and follow directly from the Binet forms in (2.1).

3. The Evaluations

To establish our evaluations we require a few well-known elementary properties of the Dedekind sum, which we summarize in the following lemmas.

Lemma 3. *For integers h, k, h_1, h_2, and q we have the following.*

(a) $s(-h,k) = -s(h,k)$ *and* $s(h,-k) = s(h,k)$.
(b) *If* $h_1 \equiv h_2 \pmod{k}$, *then* $s(h_1,k) = s(h_2,k)$.
(c) $s(qh,qk) = s(h,k)$.

The next lemma can be found in [1, p. 73] as Exercise 13.

Lemma 4. *If* $h,k,r \geq 1$, $(h,k)=1$, *and* $k \equiv r \pmod{h}$, *then*

$$s(h,k) = \frac{h^2+k^2+1-(12s(r,h)+3)\,hk}{12hk}.$$

We consider evaluations of the Dedekind sums at consecutive terms of the sequences under consideration. The proofs are similar to each other; hence we omit many of the calculations in later proofs.

Theorem 2. *Let the sequence $\{u_n\}$ be defined by the first part of (2.2), with $p \in \mathbb{N}$. Then we have*

$$s(u_{2n+1},u_{2n+2}) = \frac{(p+1)^2u_{2n}\big((p+1)u_{2n}+p-3\big)+p^2-3p+2}{12u_{2n+2}} \tag{3.1}$$

and

$$s(u_{2n},u_{2n+1}) = \frac{(p+1)^2u_{2n-1}\big(p+3-(p+1)u_{2n-1}\big)-p^2-3p-2}{12u_{2n+1}}. \tag{3.2}$$

Proof. We prove the evaluation (3.1); the proof of (3.2) is similar. By the Reciprocity Law, we have

$$s(u_{2n+1},u_{2n+2}) = \frac{u_{2n+1}^2+u_{2n+2}^2+1-3u_{2n+1}u_{2n+2}}{12u_{2n+1}u_{2n+2}} - s(u_{2n+2},u_{2n+1}).$$

Now, by the definition of u_n, Lemma 3(b), Lemma 1, Lemma 3(a), and Lemma 3(c), we have that

$$\begin{aligned}
s(u_{2n+2},u_{2n+1}) &= s(pu_{2n+1}+(p+1)u_{2n},u_{2n+1}) \\
&= s\left((p+1)u_{2n},u_{2n+1}\right) \\
&= s(u_{2n+1}-1,u_{2n+1}) \\
&= s(-1,u_{2n+1}) \\
&= -\,s(1,u_{2n+1}) \\
&= -\,\frac{(u_{2n+1}-1)(u_{2n+1}-2)}{12u_{2n+1}}.
\end{aligned}$$

Then we get

$$\begin{aligned}
s(u_{2n+1}, u_{2n+2}) &= \frac{u_{2n+1}^2 + u_{2n+2}^2 + 1 - 3u_{2n+1}u_{2n+2}}{12u_{2n+1}u_{2n+2}} + \frac{(u_{2n+1}-1)(u_{2n+1}-2)}{12u_{2n+1}} \\
&= \frac{u_{2n+1}^2 + u_{2n+2}^2 + 1 - 6u_{2n+1}u_{2n+2} + u_{2n+1}^2 u_{2n+2} + 2u_{2n+2}}{12u_{2n+1}u_{2n+2}} \\
&= \frac{(u_{2n+2} - pu_{2n+1})^2 + (1-p^2)u_{2n+1}^2 + 1}{12u_{2n+1}u_{2n+2}} \\
&\quad + \frac{(2p-6)u_{2n+1}u_{2n+2} + u_{2n+1}^2 u_{2n+2} + 2u_{2n+2}}{12u_{2n+1}u_{2n+2}},
\end{aligned}$$

where, prior to the last step, we subtract and add $2pu_{2n+2}u_{2n+1} - p^2u_{2n+1}^2$ in the numerator. After some simplification, including several applications of Lemma 1, we arrive at the desired result. □

Theorem 3. *Let the sequence $\{v_n\}$ be defined by the second part of (2.2), where $p \in \mathbb{N}$ and p is odd. Then we have*

$$s(v_{2n+1}, v_{2n+2}) = -\frac{(p+1)\left(v_{2n+1}\left(v_{2n+1} - \frac{(p+1)(p+7)}{2}\right) - \frac{p^2+9p+12}{2}\right)}{12(p+2)v_{2n+2}}$$

and

$$s(v_{2n}, v_{2n+1}) = \frac{(p+1)v_{2n}\left(v_{2n} + \frac{p^2-4p-17}{2}\right) - \frac{(p+3)(p^2-5p-12)}{2}}{12(p+2)v_{2n+1}}.$$

Proof. The proofs of these results follow the same steps as the proof of Theorem 2, but in this case we note that the initial calculation involves the evaluation of $s(p+2, v_{2n+1})$. This is accomplished with Lemma 4 and the evaluation of $s(2, p+2)$ from the identity (1.3). □

The proofs of Theorems 4 and 5 are also analogous to that of Theorem 2 and are omitted.

Theorem 4. *Let $\{u_n\}$ be defined by the first part of (2.3), where $p \in \mathbb{N}$. Then we have*

$$s(u_n, u_{n+1}) = -\frac{pu_n(u_n - p)}{12u_{n+1}}.$$

Theorem 5. *Let $\{v_n\}$ be defined by the second part of (2.3), where $p \in \mathbb{N}$ and p is even. Then we have*

$$s(v_n, v_{n+1}) = \frac{p\Big(v_n\left(p(p-6) + 2v_n\right) - (p^2 - 7p + 4)\Big)}{24(p-1)v_{n+1}}.$$

4. Some Consequences

In this final brief section we show how Theorems 4 and 5, together with the Binet formula, easily provide some special evaluations involving powers of integers.

Corollary 1. *For $p \geq 2$ and $n \in \mathbb{N}$, we have*

$$s\left(p^n - 1, p^{n+1} - 1\right) = -\frac{p\left(p^n - 1\right)\left(p^n - p^2 + p - 1\right)}{12\left(p^{n+1} - 1\right)(p-1)}.$$

Proof. The Binet form (2.1) for the sequence u_n is $u_n = (p^n - 1)/(p-1)$. In light of this and Lemma 3(c), we have that

$$\begin{aligned} s(u_n, u_{n+1}) &= s\left(\frac{p^n - 1}{p-1}, \frac{p^{n+1}-1}{p-1}\right) \\ &= s\left(p^n - 1, p^{n+1} - 1\right). \end{aligned}$$

We use the evaluation from Theorem 4 and the Binet formula again to get the desired result. □

Corollary 2. *For even $p \geq 2$ and $n \in \mathbb{N}$, we have*

$$s\left(p^n + 1, p^{n+1} + 1\right) = \frac{p\big(p^n\left(2p^n + p^2 - 6p + 4\right) + p - 2\big)}{24(p-1)\left(p^{n+1} + 1\right)}.$$

Proof. This is a direct application of the Binet form $v_n = p^n + 1$ and Theorem 5. □

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Department of Mathematics and Statistics, Dalhousie University, Halifax, Nova Scotia, B3H 3J5, Canada
E-mail address: dilcher@mathstat.dal.ca

Department of Mathematics, Syracuse University, Syracuse, NY 13144
E-mail address: jlmeye01@syr.edu