ON THE D(4)-TRIPLE $\{\mathbf{F_{2k}}, \mathbf{F_{2k+6}}, \mathbf{4F_{2k+4}}\}$

ALAN FILIPIN, BO HE, AND ALAIN TOGBÉ

Abstract. Let k be a positive integer. In this paper we study the $D(4)$-quadruples

$$\{F_{2k}, F_{2k+6}, 4F_{2k+4}, d\},$$

where F_k is a kth Fibonacci number. We prove that if d is a positive integer such that the product of any two distinct elements of the set increased by 4 is a perfect square, then $d = 4F_{2k+2}F_{2k+3}F_{2k+5}$. Therefore, we prove the uniqueness of the extension of another $D(4)$-triple involving Fibonacci numbers.

1. Introduction

A Diophantine m-tuple with the property $D(n)$ or a $D(n)$-m-tuple (or a P_n-set of size m) is a set of m distinct positive integers $\{a_1, \ldots, a_m\}$ such that $a_ia_j + n$ is a perfect square, where $n \neq 0$ is an integer. The first Diophantine quadruple was found by Fermat who proved that the set $\{1, 3, 8, 120\}$ is a $D(1)$-quadruple. Moreover, Baker and Davenport [1] proved that the set $\{1, 3, 8, 120\}$ cannot be extended to a $D(1)$-quintuple. Authors also considered n as a parametric expression. For example see [14].

The problem of extendibility of P_n-sets is of big interest (for details see `http://web.math.hr/~duje/dtuples.html`). Several results of the generalization of the result of Baker and Davenport are obtained. In 1997, Dujella [5] proved that the Diophantine triples of the form $\{k-1, k+1, 4k\}$, for $k \geq 2$, cannot be extended to a Diophantine quintuple. The Baker-Davenport result corresponds to $k = 2$. In 1998, Dujella and Pethő [8] proved that the Diophantine pair $\{1, 3\}$ cannot be extended to a Diophantine quintuple. It is known that there does not exist a $D(1)$-sextuple and there are only finitely many $D(1)$-quintuples [6, 16, 17]. In 2005, Dujella and Ramasamy [7, Conjecture 1], conjectured that there does not exist a $D(4)$-quintuple. A stronger version of this conjecture is the following.

Conjecture. *There does not exist a $D(4)$-quintuple. Moreover, if $\{a, b, c, d\}$ is a $D(4)$-quadruple such that $a < b < c < d$, then*

$$d = a + b + c + \frac{1}{2}(abc + rst),$$

where r, s, t are positive integers defined by

$$ab + 4 = r^2, ac + 4 = s^2, bc + 4 = t^2.$$

If we denote $d_+ = a + b + c + \frac{1}{2}(abc + rst)$, then $\{a, b, c, d_+\}$ is a $D(4)$-quadruple called a regular $D(4)$-quadruple. We also define the number $d_- = a + b + c + \frac{1}{2}(abc - rst)$. If $d_- \neq 0$, then $\{a, b, c, d_-\}$ is also a $D(4)$-quadruple, but $d_- < c$.

The first result of nonextendibility of $D(4)$-m-tuples was proven by Mohanty and Ramasamy [19]. They proved that $D(4)$-quadruple $\{1, 5, 12, 96\}$ cannot be extended to a D(4)-quintuple. Later Kedlaya [18], proved that if $\{1, 5, 12, d\}$ is a $D(4)$-quadruple, then $d = 96$.

One generalization of this result was given by Dujella and Ramasamy in [7] where they proved Conjecture 1 for a parametric family of $D(4)$-quadruples. Precisely, they proved that if k and d are positive integers and $\{F_{2k}, 5F_{2k}, 4F_{2k+2}, d\}$ is a $D(4)$-quadruple, then $d = 4L_{2k}F_{4k+2}$ where F_k and L_k are Fibonacci and Lucas numbers. A second generalization was given by Fujita in [15]. He proved that if $k \geq 3$ is an integer and $\{k-2, k+2, 4k, d\}$ is a $D(4)$-quadruple, then $d = k^3 - 4k$. All these results support Conjecture 1. The first author studied the size of a $D(4)$-m-tuple. He proved that there does not exist a $D(4)$-sextuple and that there are only finitely many $D(4)$-quintuples [11, 9, 10, 12].

The aim of this paper is to consider the $D(4)$-triple involving Fibonacci numbers

$$\{F_{2k}, F_{2k+6}, 4F_{2k+4}\}$$

and to prove the following result.

Theorem 1. *If d is a positive integer such that the product of any two distinct elements of the set*

$$\{F_{2k}, F_{2k+6}, 4F_{2k+4}, d\} \tag{1}$$

increased by 4 is a perfect square, then

$$d = 4F_{2k+2}F_{2k+3}F_{2k+5}. \tag{2}$$

The organization of this paper is as follows. In Section 2, we recall some useful results obtained by Dujella-Ramasamy and the first author. We adapt them to our case. Moreover, we use congruences and Fibonacci number properties to obtain a gap principle. In Section 3, we use linear forms in logarithms and the Baker-Davenport reduction method to prove Theorem 1. It is good to specify that recently, using this method the first and the third authors [13] proved the uniqueness of an extension of a parametric family of $D(4)$-triples $\{k+2, 4k, 9k+6\}$. The motivation of this paper is to show the difference when we consider an exponential family. The main difference when we consider an exponential family is that the constants grow so quickly that we don't need to use the results from hypergeometric method (Bennett's Theorem) like in a polynomial case and the use of linear forms in logarithms is enough. For example, we don't have a lemma similar to Lemma 5 in [13]. The family we consider in the present paper appeared for the first time in [3]. In the last section we discuss other families of $D(4)$-triples involving Fibonacci, Pell, and Pell-Lucas numbers.

2. Preliminary Results

Let r, s, t be positive integers defined by

$$ab + 4 = r^2, \quad ac + 4 = s^2, \quad bc + 4 = t^2. \tag{3}$$

To extend the Diophantine $D(4)$-triple $\{a, b, c\}$ to a Diophantine $D(4)$-quadruple $\{a, b, c, d\}$, we have to solve the system

$$ad + 4 = x^2, \quad bd + 4 = y^2, \quad cd + 4 = z^2. \tag{4}$$

One can eliminate d to obtain the following system of Pellian equations

$$az^2 - cx^2 = 4(a - c), \tag{5}$$
$$bz^2 - cy^2 = 4(b - c). \tag{6}$$

Using Lemma 1 of [9], one can see that there exists a solution $\left(z_0^{(i)}, x_0^{(i)}\right)$ of (5) such that $z = v_m^{(i)}$, where

$$v_0^{(i)} = z_0^{(i)}, \quad v_1^{(i)} = \frac{1}{2}\left(sz_0^{(i)} + cx_0^{(i)}\right), \quad v_{m+2}^{(i)} = sv_{m+1}^{(i)} - v_m^{(i)},$$

and $|z_0^{(i)}| < \sqrt{\frac{c\sqrt{c}}{\sqrt{a}}}$. Similarly, there exists a solution $\left(z_1^{(i)}, y_1^{(i)}\right)$ of (6) such that $z = w_n^{(j)}$, where

$$w_0^{(i)} = z_1^{(j)}, \quad w_1^{(j)} = \frac{1}{2}(tz_1^{(j)} + cy_1^{(j)}), \quad w_{n+2}^{(j)} = tw_{n+1}^{(j)} - w_n^{(j)},$$

and $|z_1^{(j)}| < \sqrt{\frac{c\sqrt{c}}{\sqrt{b}}}$.

The initial terms $z_0^{(i)}$ and $z_1^{(j)}$ are completely determined using Lemma 9 in [11] that we recall here.

Lemma 1.

(1) *If the equation* $v_{2m} = w_{2n}$ *has a solution, then* $z_0 = z_1$. *Moreover,* $|z_0| = 2$, *or* $|z_0| = \frac{1}{2}(cr - st)$, *or* $|z_0| < 1.608a^{-\frac{5}{14}}c^{\frac{9}{14}}$.
(2) *If the equation* $v_{2m+1} = w_{2n}$ *has a solution, then* $|z_0| = t$, $|z_1| = \frac{1}{2}(cr - st)$, $z_0z_1 < 0$.
(3) *If the equation* $v_{2m} = w_{2n+1}$ *has a solution, then* $|z_1| = s$, $|z_0| = \frac{1}{2}(cr - st)$, $z_0z_1 < 0$.
(4) *If the equation* $v_{2m+1} = w_{2n+1}$ *has a solution, then* $|z_0| = t$, $|z_1| = s$, $z_0 \cdot z_1 > 0$.

In the present paper, we consider

$$a = F_{2k}, \quad b = F_{2k+6}, \quad c = 4F_{2k+4},$$

and using equations (3) we get

$$r = F_{2k+3}, \quad s = 2F_{2k+2}, \quad t = 2F_{2k+5}.$$

If the second or fourth items of Lemma 1 holds, then

$$bc < bc + 4 = t^2 = |z_0|^2 < \frac{c\sqrt{c}}{\sqrt{a}}.$$

So we have $ab^2 < c$. This implies

$$F_{2k}F_{2k+6}^2 < 4F_{2k+4}.$$

From Binet's formula $F_k = \left(\alpha^k - \bar{\alpha}^k\right)/\sqrt{5}$ with $\alpha = (1+\sqrt{5})/2$, $\bar{\alpha} = (1-\sqrt{5})/2$, we obtain $\alpha^{2k-1}/\sqrt{5} < F_{2k} < \alpha^{2k}/\sqrt{5}$. This and the above inequality give

$$\frac{\alpha^{2k-1}}{\sqrt{5}}\left(\frac{\alpha^{2k+5}}{\sqrt{5}}\right)^2 < \frac{4\alpha^{2k+4}}{\sqrt{5}}.$$

Therefore we get $\alpha^{4k+5} < 20$. This contradicts the fact that $\alpha > 1.618$ and $k \geq 1$.

Similarly, if the third item of Lemma 1 holds, we obtain

$$ac < ac + 4 = s^2 = |z_1|^2 < \frac{c\sqrt{c}}{\sqrt{b}}.$$

This yields

$$\left(\frac{\alpha^{2k-1}}{\sqrt{5}}\right)^2 \frac{\alpha^{2k+5}}{\sqrt{5}} < F_{2k}^2F_{2k+6} = a^2b < c = 4F_{2k+4} < \frac{4\alpha^{2k+4}}{\sqrt{5}}.$$

Then by the result $\alpha^{4k-1} < 20$, we have $k < 2$. If $k = 1$, then $a = 1$, $b = 21$, $c = 32$, and $d = 780$. It has been checked that the $D(4)$-triple $\{1, 21, 32\}$ can be extended in a unique way to the $D(4)$-quadruple $\{1, 21, 32, 780\}$ [9]. In that paper the first author has already used the fact that all $D(4)$-triples $\{a, b, c\}$ such that $a < b < c$ and $ab^2c < 10^7$, have a unique extension to a $D(4)$-quadruple, and it was checked using the Baker-Davenport reduction.

Therefore, we only consider the first item of Lemma 1 with $x_0 = 2$, $y_0 = 2$, and $|z_0| = (cr - st)/2 = 2(F_{2k+3}F_{2k+4} - F_{2k+2}F_{2k+5}) = 2$. In fact, let us define $d_0 = (z_0^2 - 4)/c$ in the third possibility of the first item. If $|z_0| > 2$, then $d_0 < z_0^2/c < 1.608^2 a^{-5/7} c^{9/7}/c < c$. Thus, according to the proof of the above lemma in [11], $\{a, b, c, d_0\}$ is an irregular Diophantine quadruple. Also by [10, Proposition 1], if $\{a, b, c, d\}$ is an irregular Diophantine quadruple with $a < b < c < d$, then $d > 0.173c^{6.5}a^{5.5}$ or $d > 0.087c^{3.5}a^{2.5}$. Therefore we have $c > 0.173b^{6.5}a^{5.5}$ or $c > 0.087b^{3.5}a^{2.5}$. When $k \geq 1$, we get a contradiction.

Therefore, we need to solve the system of Pell equations

$$F_{2k}\, z^2 - 4F_{2k+4}\, x^2 = 4(F_{2k} - 4F_{2k+4}), \tag{7}$$

$$F_{2k+6}\, z^2 - 4F_{2k+4}\, y^2 = 4(F_{2k+6} - 4F_{2k+4}), \tag{8}$$

with $x_0 = y_1 = 2$ and $z_0 = z_1 = \pm 2$, for integer $k \geq 1$. It is equivalent to solve the sequence equation

$$z = v_{2m} = w_{2n}. \tag{9}$$

In fact, the sequences $\{v_m\}_{m \geq 0}$ and $\{w_n\}_{n \geq 0}$ are defined by

$$v_0 = \pm 2, \quad v_1 = \pm 2F_{2k+2} + 4F_{2k+4}, \quad v_{m+2} = 2F_{2k+2}v_{m+1} - v_m,$$

$$w_0 = \pm 2, \quad w_1 = \pm 2F_{2k+5} + 4F_{2k+4}, \quad w_{n+2} = 2F_{2k+5}w_{n+1} - w_n.$$

In order to get a gap principle between indices m, n and k, we recall the following lemma.

Lemma 2. *We have*

$$v_{2m} \equiv z_0 + \frac{1}{2}c(az_0m^2 + sx_0m) \pmod{c^2},$$

$$w_{2n} \equiv z_1 + \frac{1}{2}c(bz_1n^2 + ty_1n) \pmod{c^2}.$$

Proof. See [7, Lemma 3]. □

For the relations of indices m and n, we have the following lemma.

Lemma 3. *If $v_{2m} = w_{2n}$, then $n \leq m \leq 2n$.*

Proof. By [11, Lemma 5], if $v_m = w_n$, then $n - 1 \leq m \leq 2n + 1$. In our even case we have $2n - 1 \leq 2m \leq 4n + 1$. The result is obtained. □

Using Lemma 2, we have

$$\pm am^2 + sm \equiv \pm bn^2 + tn \pmod{c}. \tag{10}$$

In our case, it is

$$\pm F_{2k} \cdot m^2 + 2F_{2k+2} \cdot m \equiv \pm F_{2k+6} \cdot n^2 + 2F_{2k+5} \cdot n \pmod{F_{2k+4}}.$$

From the recursive relation $F_{k+2} = F_{k+1} + F_k$, one can show that $F_{k+4} = 3F_{k+2} - F_k$. Thus we have $F_{2k} \equiv 3F_{2k+2} \pmod{F_{2k+4}}$ and

$$F_{2k+6} = F_{2k+5} + F_{2k+4} \equiv F_{2k+5} = F_{2k+4} + F_{2k+3} \equiv F_{2k+3} \pmod{F_{2k+4}},$$

$$F_{2k+3} = F_{2k+4} - F_{2k+2} \equiv -F_{2k+2} \pmod{F_{2k+4}}.$$

We combine the above results to obtain

$$\pm F_{2k+2}(3m^2 \mp 2m + n^2 \mp 2n) \equiv 0 \pmod{F_{2k+4}}.$$

As $\gcd(F_{2k+2}, F_{2k+4}) = \gcd(F_{2k+2}, F_{2k+2} + F_{2k+3}) = \gcd(F_{2k+2}, F_{2k+3}) = 1$, we have

$$3m^2 \mp 2m + n^2 \mp 2n \equiv 0 \pmod{c/4}.$$

Since m and n are positive, by Lemma 3, we have

$$3m^2 \mp 2m + n^2 \mp 2n \geq 3m^2 - 2m + n^2 - 2n \geq 3n^2 - 4n + n^2 - 2n = 4n^2 - 6n > 0$$

for $n \geq 2$. This and the above congruence imply

$$3m^2 + 2m + n^2 + 2n \geq 3m^2 \mp 2m + n^2 \mp 2n \geq c/4.$$

Again, using Lemma 3 we obtain $4(m^2 + m) > c/4$. This leads to the following result.

Lemma 4. *Assume that $v_{2m} = w_{2n}$ with $m, n \geq 2$, then*

$$m \geq \frac{\sqrt{c}}{4} - \frac{1}{2}.$$

3. Proof of Theorem 1

In this section, we prove Theorem 1 using Baker's method. So let us consider the following algebraic numbers

$$\alpha_1 = \frac{s + \sqrt{ac}}{2} \quad \text{and} \quad \alpha_2 = \frac{t + \sqrt{bc}}{2}.$$

From equations (7) and (8), we deduce

$$v_{2m} = \frac{1}{2\sqrt{a}}\left((z_0\sqrt{a} + x_0\sqrt{c})\alpha_1^{2m} + (z_0\sqrt{a} + x_0\sqrt{c})\alpha_1^{-2m}\right)$$

and

$$w_{2n} = \frac{1}{2\sqrt{b}}\left((z_1\sqrt{b} + y_1\sqrt{c})\alpha_2^{2n} + (z_1\sqrt{b} + y_1\sqrt{c})\alpha_2^{-2n}\right),$$

respectively. Notice $x_0 = y_1 = 2$ and $z_0 = z_1 = \pm 2$. Solving equations (7) and (8) is equivalent to solving $z = v_{2m} = w_{2n}$ with $m, n \neq 0$. So we have [11, Lemma 10]

$$0 < \Lambda := 2m \log \alpha_1 - 2n \log \alpha_2 + \log \alpha_3 < 2ac\alpha_1^{-4m}, \tag{11}$$

where

$$\alpha_3 = \frac{\sqrt{b}(\sqrt{c} \pm \sqrt{a})}{\sqrt{a}(\sqrt{c} \pm \sqrt{b})}.$$

It follows that

$$\log|\Lambda| < -4m \log \alpha_1 + \log(2ac) < (2 - 4m)\log \alpha_1. \tag{12}$$

In [9], using Baker's method (in fact, applying Baker-Wüstholz's Theorem, [2]), the first author proved that

$$\frac{2m}{\log(2m+1)} < 6.543 \cdot 10^{15} \log^2 c. \tag{13}$$

When $m, n \geq 2$, this and Lemma 4 imply

$$c^{1/2} - 2 < 2 \cdot 6.543 \cdot 10^{15} \left(\log^2 c\right)\left(\log(0.5c^{1/2})\right).$$

We obtain $c < 4.29 \cdot 10^{43}$. Since $c = 4F_{2k+4} = 4 \cdot \frac{\alpha^{2k+4} - \bar{\alpha}^{2k+4}}{\sqrt{5}}$, we have

$$1.618^{2k+4} < \left(\frac{1+\sqrt{5}}{2}\right)^{2k+4} = \alpha^{2k+4} < \bar{\alpha}^{2k+4} + \frac{\sqrt{5}}{4} \cdot 4.29 \cdot 10^{43} < 2.4 \cdot 10^{43}.$$

Hence, we have $k < 102$. Moreover, using (13) and the upper bound to c, we obtain $m < 1.7 \cdot 10^{21}$.

In order to deal with the remaining cases $1 \le k \le 101$, we will use a Diophantine approximation algorithm called the Baker-Davenport reduction method. The following lemma is a slight modification of the original version of the Baker-Davenport reduction method. [8, Lemma 5a].

Lemma 5. *Assume that M is a positive integer. Let P/Q be the convergent of the continued fraction expansion of κ such that $Q > 6M$ and let*

$$\eta = \|\mu Q\| - M \cdot \|\kappa Q\|,$$

where $\|\cdot\|$ denotes the distance from the nearest integer. If $\eta > 0$, then there is no solution of the inequality

$$0 < m\kappa - n + \mu < EB^{-m}$$

in integers m and n with

$$\frac{\log(EQ/\eta)}{\log B} \le m \le M.$$

We apply Lemma 5 with

$$\kappa = \frac{\log \alpha_1}{\log \alpha_2}, \quad \mu = \frac{\log \alpha_3}{2 \log \alpha_2}, \quad E = \frac{ac}{\log \alpha_2}, \quad B = \alpha_1^4,$$

and $M = 1.7 \cdot 10^{21}$.

The program was developed in PARI/GP running with 400 digits. For the computations, if the first convergent such that $q > 6M$ does not satisfy the condition $\eta > 0$, then we use the next convergent until we find the one that satisfies the conditions.

• If $z_0 = z_1 = 2$, then we used the second convergent only if $k = 24, 29, 30, 47$. In all cases we obtained $m \le 8$. We took $M = 8$ and ran the program again to obtain $m \le 1$. We ran the program in one minute to get the results.

• With $z_0 = z_1 = -2$, we used the first convergent in 12 cases. For the other cases, we obtained convergents of higher orders and the worst case is when $k = 99$ where we have the 145th convergent. The use of 400 digits is enough to ensure the accuracy of the computations. In all cases, we obtained $m \le 7$. We again ran the program with $M = 7$ and obtained $m \le 1$. The computations were done in one minute.

Since $n \le m \le 1$, we put $m = n = 1$ in equation (9) ($m = n = 0$ gives the trivial solution $d = 0$). When $z = v_0 = w_0 = 2$, we have $v_2 = 2F_{2k+2}(2F_{2k+2} + 4F_{2k+4}) - 2$ and $w_2 = 2F_{2k+5}(2F_{2k+5} + 4F_{2k+4}) - 2$, such that $v_2 < w_2$. When $z = v_0 = w_0 = -2$, by

$-2F_{2k+2} + 4F_{2k+4} = 2F_{2k+3} + 2F_{2k+4} = 2F_{2k+5}$, we have

$$\begin{aligned} v_2 &= 2F_{2k+2}(-2F_{2k+2} + 4F_{2k+4}) + 2 \\ &= 2F_{2k+2} \cdot 2F_{2k+5} + 2 \\ &= (-2F_{2k+5} + 4F_{2k+4}) \cdot 2F_{2k+5} + 2 = w_2. \end{aligned}$$

This implies

$$\begin{aligned} d = \frac{z^2 - 4}{c} &= \frac{(4F_{2k+2}F_{2k+5} + 2)^2 - 4}{4F_{2k+4}} \\ &= \frac{16F_{2k+2}^2F_{2k+5}^2 + 16F_{2k+2}F_{2k+5}}{4F_{2k+4}} \\ &= \frac{4F_{2k+2}F_{2k+5}(F_{2k+2}F_{2k+5} + 1)}{F_{2k+4}} \\ &= \frac{4F_{2k+2}F_{2k+5} \cdot F_{2k+3}F_{2k+4}}{F_{2k+4}} \\ &= 4F_{2k+2}F_{2k+3}F_{2k+5}. \end{aligned}$$

This completes the proof of Theorem 1.

4. Some Other Families of $D(4)$-triples

By Theorem 1, we prove the uniqueness of the extension of one specific exponential parametric family of $D(4)$-triples. There are many other similar families for which the method presented here can be applied. Some of those families can be found in [3, 4]. Using exactly this method, with only slightly different constants, we can prove the following theorem, where P_n and Q_n are Pell and Pell-Lucas numbers, respectively.

Theorem 2. *Let k and d be positive integers. Then,*

(i) *if $\{F_{2k}, F_{2k+6}, 4F_{2k+2}, d\}$ is a $D(4)$-quadruple, then $d = 4F_{2k+1}F_{2k+3}F_{2k+4}$;*
(ii) *if $\{P_{2k}, P_{2k+4}, 4P_{2k+2}, d\}$ is a $D(4)$-quadruple, then $d = 4P_{2k+1}P_{2k+2}P_{2k+3}$;*
(iii) *if $\{P_{2k}, P_{2k+4}, 8P_{2k+2}, d\}$ is a $D(4)$-quadruple, then $d = 4P_{2k+2}Q_{2k+1}Q_{2k+3}$.*

In fact, we apply the method to the above three families. First, for the triple $\{F_{2k}, F_{2k+6}, 4F_{2k+2}\}$, we consider $v_{2m} = w_{2n}$ with $z_0 = z_1 = \pm 2$. We get

$$m \geq \frac{\sqrt{c}}{4} - \frac{1}{2},$$

then $c < 4.29 \cdot 10^{43}$, $m < 1.7 \cdot 10^{21}$, and $k < 103$.

The last two families are connected because $c = a + b \pm 2r$ and here $r = P_{2k+2}$. Pell numbers are given by $P_0 = 0$, $P_1 = 1$, $P_n = 2P_{n-1} + P_{n-2}$, for $n \geq 2$ and Pell-Lucas numbers are defined by $Q_0 = 2$, $Q_2 = 2$, $Q_n = 2Q_{n-1} + Q_{n-2}$, for $n \geq 2$. So using this and congruence relations, and the fact that P_n and P_{n+1} are relatively prime, we get

$$m \geq \frac{\sqrt{c}}{4\sqrt{2}} - \frac{1}{2},$$

then $c < 1.9 \cdot 10^{44}$, $m < 1.7 \cdot 10^{21}$, and $k < 59$. Now, we can apply the Baker-Davenport reduction method to each these three families. We ran the program in one minute. Here are some comments on the computations done in one minute with a use of 400 digits:

• For the family (i), we used the first or the second convergents in all cases. We obtained $m \leq 10$. We again ran the program with $M = 10$ and obtained $m \leq 1$.

• For the family (ii), we got $m \leq 6$ with the use of the first or the second convergents in all cases. The program is run again with $M = 6$ to obtain $m \leq 1$.

• For the last family, we used the first or the second convergents in all cases to obtain $m \leq 5$. We again ran the program with $M = 5$ and obtained $m \leq 1$.

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Faculty of Civil Engineering, University of Zagreb, Fra Andrije Kačića-Miošića 26, 10000 Zagreb, Croatia
E-mail address: filipin@grad.hr

Department of Mathematics, ABa Teacher's College, Wenchuan, Sichuan 623000, P. R. China
E-mail address: bhe@live.cn

Mathematics Department, Purdue University North Central, 1401 S U.S. 421, Westville, IN 46391
E-mail address: atogbe@pnc.edu