# FIBONACCI MATRICES AND MODULAR FORMS

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Abstract. We show a hitherto undiscovered connection between Fibonacci matrices and modular forms, using standard properties of $F_n$ to prove a convergence result for infinite series. No prior knowledge of modular forms is required.

## 1. Introduction

In this paper, we examine a Fibonacci matrix that arises naturally in the theory of modular forms, and we show how the asymptotic properties of $F_n$ can be used to prove a convergence theorem for infinite series. This application was described in several talks [8, 10] but has never appeared in written form. No prior experience with the theory of modular forms will be assumed.

A Fibonacci matrix is a square matrix $A$ with the following properties: for each integer $k$, every entry of $A^k$ is equal in absolute value to some $F_n$; and every $F_n$ appears as an entry in some $A^k$. The best-known Fibonacci matrix [4] is

$$Q = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}.$$

However, the subject of our study is its less renowned fraternal twin, which will be defined in Section 3.

Modular forms are complex functions with a sort of invariance on the modular group or one of its subgroups. For example, if $f(z+1) = c_1 f(z)$ and $f(-1/z) = c_2 z^k f(z)$ for all complex $z$ with positive imaginary part, then $f$ is said to be a "modular form of weight $k$ on the full group." Here $k$ is a fixed parameter, usually taken to be real. (We are ignoring certain analytic requirements for the sake of brevity, as they have no bearing at all on the results of this paper.)

On the other hand, one could replace the transformation $z \to z+1$ with some other integral translation and obtain a form whose invariance (the ten-dollar word for this is *modularity*) takes place on a subgroup instead.

Just as the translation $z \to z+t$ and the inversion $z \to -1/z$ generate a group of transformations, so too do the corresponding constants $c_1$, $c_2$ give rise to a system of multipliers on the group of matrices representing those transformations. Typically, each linear fractional transformation is represented by a $2 \times 2$ matrix $M$ of determinant 1, so the multiplier can be written as $v(M)$. For example, if we choose to write the map $z \to -1/z$ in the form

$$z \to \frac{0z-1}{1z+0},$$

then the relevant matrix is

$$T = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix},$$

hence $\upsilon(T) = c_2$. We distinguish between $\upsilon(M)$ and $\upsilon(-M)$, though $M$ and $-M$ represent the same linear fractional transformation.

For fixed real weight and multiplier system, under the classical assumption $|\upsilon| \equiv 1$, the space of entire modular forms on the full modular group is finite-dimensional. (A form is *entire* if it satisfies certain analytic and growth conditions. See [5] for details.) In that case, as well as for certain other groups, the standard method for constructing a basis involves the use of Poincaré series, expressions that look like this:

$$g(z) = \sum \frac{\exp\{2\pi i(\nu+\varepsilon)(Mz)/t\}}{\upsilon(M)(cz+d)^k}. \tag{1.1}$$

Here $\nu$ is an integer, $cz+d$ is the denominator of the linear fractional transformation defined by the matrix $M$, $t$ is the minimal positive translation in the group, and $\varepsilon$ is defined by $c_1 = \exp(2\pi i\varepsilon)$ with $0 \le \operatorname{Re}(\varepsilon) < 1$ (so that $0 \le \varepsilon < 1$ when we assume that $|\upsilon| \equiv 1$). The sum is taken over all "lower rows" of the group; that is, only one matrix with each lower row $c, d$ is included. It is easy to check that the function is well-defined. Under the above assumptions (fixed real weight, etc.), a standard result [6] states that by varying the parameter $\nu$ in (1.1), one obtains a basis for the space of entire modular forms so long as $k$ is real and greater than 2.

For complex weight, one finds precisely the opposite situation. Whereas for real weight, the right-hand side of (1.1) is absolutely convergent (hence modular), and the absolute series is uniformly convergent on compact subsets of the complex upper half-plane (hence $g$ is analytic there), for nonreal weight the absolute series is not even *pointwise* convergent at any point! This despite the fact that nontrivial forms of nonreal weight do exist [9, 12].

## 2. Enter $F_n$

The preceding divergence result was proved for the full modular group and for one important subgroup in [11, 12], in each case by establishing divergence on a cyclic subgroup. What of other groups? It turns out that if we apply the same approach to functions satisfying $f(z+3) = c_1 f(z)$, this method leads immediately to Fibonacci matrices of order 2! (Fibonacci matrices of higher orders have also been the subject of continued scrutiny, for example in [2, 3, 7].)

Define $T$ as before,

$$S_\lambda = \begin{pmatrix} 1 & \lambda \\ 0 & 1 \end{pmatrix},$$

and $S = S_1$. For integer $\lambda$, we have $S_\lambda = S^\lambda$. The proof of Poincaré series divergence in [11, 12] for $\lambda = 1, 2$ relies on showing that a particular subseries diverges, namely the sum on the cyclic group generated by $S^2T$. The current proof for $\lambda = 3$ will use the cyclic group $\langle S^3T \rangle$ instead, which will entail additional complications involving $F_n$.

Observe that for $n \in \mathbb{Z}$,

$$\left(S^2T\right)^n = \begin{pmatrix} 2 & -1 \\ 1 & 0 \end{pmatrix}^n = \begin{pmatrix} n+1 & -n \\ n & -(n-1) \end{pmatrix},$$

while for $n \in \mathbb{Z}^+$,

$$\left(S^3T\right)^n = \begin{pmatrix} 3 & -1 \\ 1 & 0 \end{pmatrix}^n = \begin{pmatrix} F_{2(n+1)} & -F_{2n} \\ F_{2n} & -F_{2(n-1)} \end{pmatrix}.$$

(Note the amusing identity: $(-1)^{j+1}\left(S^3T\right)^n_{ij} = F_{2(-1)^{j+1}(S^2T)^n_{ij}}$ for $n \in \mathbb{Z}^+$.) For a negative integer $n$, likewise,

$$\left(S^3T\right)^n = \begin{pmatrix} -F_{-2(n+1)} & F_{-2n} \\ -F_{-2n} & F_{-2(n-1)} \end{pmatrix} = \begin{pmatrix} F_{2(n+1)} & -F_{2n} \\ F_{2n} & -F_{2(n-1)} \end{pmatrix}.$$

Hence $S_3T$ is *almost* a Fibonacci matrix: it is true that its powers contain only $\{\pm F_n\}$ as entries, but on the other hand, not every $F_n$ appears in some power of $S_3T$. What explains the fact that half of the Fibonacci numbers have gone missing?

## 3. Where Have All the $F_n$'s Gone?

The key is that $S_3T$ has a "square root" that will fill in the missing terms of the Fibonacci sequence. While it's easy enough to find all square roots of this particular matrix directly, here's a theorem that will accomplish the same end while also applying more generally.

**Theorem.** *Let* $A = \left(\begin{smallmatrix} \alpha & \beta \\ \gamma & \delta \end{smallmatrix}\right) \in M_2(\mathbb{C})$*, the set of* $2 \times 2$ *matrices with complex entries. If* $B \in M_2(\mathbb{C})$ *with* $B^2 = A \neq cI$*, then*

$$B = \pm \begin{pmatrix} \sqrt{\alpha - \delta + t^2} & \dfrac{\beta}{t + \sqrt{\alpha - \delta + t^2}} \\ \dfrac{\gamma}{t + \sqrt{\alpha - \delta + t^2}} & t \end{pmatrix},$$

*where* $t$ *satisfies the polynomial equation*

$$\begin{pmatrix} t^4 & t^2 & 1 \end{pmatrix} \begin{pmatrix} \chi^2 - 4\Delta \\ 8\delta\Delta - 2\chi\left(\delta^2 + \Delta\right) \\ \left(\delta^2 - \Delta\right)^2 \end{pmatrix} = 0,$$

*which has degree at most 4. Here* $\Delta$ *and* $\chi$ *are the determinant and trace, respectively, of* $A$*, and the radical symbol should be interpreted according to the convention* $0 \leq \arg\sqrt{w} < \pi$.

The proof is mechanical, so we leave it as an exercise (in algebra or faith). It should be pointed out that the theorem does *not* guarantee the existence of a square root for every matrix. Rather, in the event that a square root exists, the theorem specifies what form it will take.

**Corollary 1.** *Let* $\lambda \in \mathbb{C}$ *and* $B \in M_2(\mathbb{C})$*. If* $B^2 = S_\lambda T$*, then*

$$B = \frac{1}{\pm\sqrt{\lambda+2}} \begin{pmatrix} \lambda+1 & -1 \\ 1 & 1 \end{pmatrix} \quad or \quad \frac{1}{\pm\sqrt{\lambda-2}} \begin{pmatrix} \lambda-1 & -1 \\ 1 & -1 \end{pmatrix}.$$

*That is,* $S_\lambda T$ *has two square roots when* $\lambda = \pm 2$*, and four square roots otherwise.*

**Corollary 2.** $(S_3T)^{1/2} = \pm\dfrac{1}{\sqrt{5}} \begin{pmatrix} 4 & -1 \\ 1 & 1 \end{pmatrix}, \pm \begin{pmatrix} 2 & -1 \\ 1 & -1 \end{pmatrix}.$

Interestingly, no value of $(S_3T)^{1/2}$ is an element of the group $\langle S_3, T\rangle$.

*Remark.* While each $1 \times 1$ matrix has one or two complex square roots, a $2 \times 2$ matrix can have two or four of them, or infinitely many (as in the case of the identity matrix and its scalar multiples), or none at all. In any event both $\left(\begin{smallmatrix} 2 & -1 \\ 1 & -1 \end{smallmatrix}\right)$ and its opposite are true Fibonacci matrices, for

$$\begin{pmatrix} 2 & -1 \\ 1 & -1 \end{pmatrix}^n = \begin{pmatrix} F_{n+2} & -F_n \\ F_n & -F_{n-2} \end{pmatrix},$$

and in this way the "missing" $F_n$ are restored. Essentially, we have used a matrix argument to interpolate the Fibonacci recursion.

Like $Q$, this matrix can be used to prove a variety of identities.

## 4. Main Result: Application to Poincaré Series

We show that Poincaré series on the group $G_3 = \langle S_3, T\rangle$ are absolutely divergent for all nonreal weights. Recall that

$$(S_3T)^n = \begin{cases} \begin{pmatrix} F_{2n+2} & -F_{2n} \\ F_{2n} & -F_{2n-2} \end{pmatrix}, \ n > 0, \\ \\ \begin{pmatrix} -F_{-(2n+2)} & F_{-2n} \\ -F_{-2n} & F_{-(2n-2)} \end{pmatrix}, \ n < 0, \end{cases}$$

and the lower rows of these matrices are distinct. (The same relation can be written as a single expression if we permit the use of negative subscripts; but the piecewise point of view will better serve our purposes.)

The absolute series of interest here is

$$\sum := \sum_{M \in \langle S^3\rangle\backslash G_3} \left| \frac{\exp\left\{2\pi i\left(\nu+\varepsilon\right)\left(Mz\right)/3\right\}}{\upsilon\left(M\right)\left(cz+d\right)^k} \right|,$$

which indicates that the sum is taken over all lower rows of the group $G_3$ as described earlier. We have:

$$\begin{aligned} \sum &> \sum_{M \in \langle S^3T\rangle} \left| \frac{\exp\left\{2\pi i\left(\nu+\varepsilon\right)\left(Mz\right)/3\right\}}{\upsilon\left(M\right)\left(cz+d\right)^k} \right| \\ &> \sum_{n=1}^{\infty} \left\{ \left| \frac{\exp\left\{\frac{2\pi i}{3}\left(\nu+\varepsilon\right)\left(S^3T\right)^n z\right\}}{\upsilon\left(\left(S^3T\right)^n\right)\left(F_{2n}z - F_{2n-2}\right)^k} \right| + \left| \frac{\exp\left\{\frac{2\pi i}{3}\left(\nu+\varepsilon\right)\left(S^3T\right)^{-n} z\right\}}{\upsilon\left(\left(S^3T\right)^{-n}\right)\left(-F_{2n}z + F_{2n+2}\right)^k} \right| \right\} \\ &= \sum_{n=1}^{\infty} \left\{ \frac{\exp\left\{\operatorname{Re}\left(\frac{2\pi i}{3}\left(\nu+\varepsilon\right)\frac{F_{2n+2}z-F_{2n}}{F_{2n}z-F_{2n-2}}\right)\right\}}{\left|\upsilon\left(\left(S^3T\right)^n\right)\right| \left|\left(F_{2n}z - F_{2n-2}\right)^k\right|} + \frac{\exp\left\{\operatorname{Re}\left(\frac{2\pi i}{3}\left(\nu+\varepsilon\right)\frac{-F_{2n-2}z+F_{2n}}{-F_{2n}z+F_{2n+2}}\right)\right\}}{\left|\upsilon\left(\left(S^3T\right)^{-n}\right)\right| \left|\left(-F_{2n}z + F_{2n+2}\right)^k\right|} \right\} \\ &= \sum_{n=1}^{\infty} \left\{ \frac{\exp\left\{\frac{-2\pi}{3}\operatorname{Im}\left(\left(\nu+\varepsilon\right)\frac{F_{2n+2}z-F_{2n}}{F_{2n}z-F_{2n-2}}\right)\right\}}{\left|\upsilon\left(\left(S^3T\right)^n\right)\right| \left|\left(F_{2n}z - F_{2n-2}\right)^k\right|} + \frac{\exp\left\{\frac{-2\pi}{3}\operatorname{Im}\left(\left(\nu+\varepsilon\right)\frac{-F_{2n-2}z+F_{2n}}{-F_{2n}z+F_{2n+2}}\right)\right\}}{\left|\upsilon\left(\left(S^3T\right)^{-n}\right)\right| \left|\left(-F_{2n}z + F_{2n+2}\right)^k\right|} \right\}. \end{aligned}$$

Ordinarily, evaluating an expression of the form $\upsilon\left(M_1M_2\right)$ requires the use of a complicated "consistency condition", by virtue of which $\upsilon$ is consistent with the invariance of its accompanying modular form. Fortunately, in this particular case (according to calculations in [12]) we have instead simple multiplicativity:

$$\begin{aligned} \upsilon\left(\left(S^3T\right)^n\right) &= \upsilon\left(S^3\right)^n \upsilon\left(T\right)^n = \left(c_1c_2\right)^n, \\ \upsilon\left(\left(S^3T\right)^{-n}\right) &= \upsilon\left(S^3\right)^{-n} \upsilon\left(T\right)^{-n} = \left(c_1c_2\right)^{-n}, \end{aligned}$$

where $c_1 = \upsilon\left(S^3\right)$ and $c_2 = \upsilon\left(T\right)$. Also $c_1 = e^{2\pi i\varepsilon}$, and the only possible values for $c_2$ are $\pm i^{-k}$ or $\pm i^{-k+1}$. Thus $\left|\upsilon\left(\left(S^3T\right)^n\right)\right| = \left|e^{2\pi i\varepsilon}i^{-k}\right|^n = e^{-2\pi n\ \mathrm{Im}(\varepsilon)}\left|e^{-k\log i}\right|^n = e^{-2\pi n\ \mathrm{Im}(\varepsilon)}\left|e^{-ki\pi/2}\right|^n = e^{-2\pi n\ \mathrm{Im}(\varepsilon)+\mathrm{Im}(k)n\pi/2}$ for $n \in \mathbb{Z}^+$, and likewise $\left|\upsilon\left(\left(S^3T\right)^{-n}\right)\right| = e^{2\pi n\ \mathrm{Im}(\varepsilon)-\ \mathrm{Im}(k)n\pi/2}$. For convenience, let $\xi = e^{2\pi\ \mathrm{Im}(\varepsilon-k/4)}$ so that $\left|\upsilon\left(\left(S^3T\right)^n\right)\right| = \xi^{-n}$ and $\left|\upsilon\left(\left(S^3T\right)^{-n}\right)\right| = \xi^n$. As readers of the *The Fibonacci Quarterly* are well aware, $\frac{F_{j+1}}{F_j} \to \alpha = \frac{\sqrt{5}+1}{2}$, the golden ratio, as $j \to \infty$. Since $\lim\limits_{j\to\infty}\frac{F_{j+2}}{F_j} = \alpha^2$, we have $\frac{F_{2n+2}z-F_{2n}}{F_{2n}z-F_{2n-2}} = \frac{F_{2n}}{F_{2n-2}}\frac{(F_{2n+2}/F_{2n})z-1}{(F_{2n}/F_{2n-2})z-1} \to \alpha^2$ for fixed $z$. Similarly, $\frac{-F_{2n-2}z+F_{2n}}{-F_{2n}z+F_{2n+2}} \to \alpha^{-2}$. Thus, we are interested in convergence of the series

$$\sum_{n=1}^{\infty}\left\{\frac{\exp\left\{-2\pi\alpha^2\ \mathrm{Im}\left(\nu+\varepsilon\right)/3\right\}}{\xi^{-n}\left|\left(F_{2n}z-F_{2n-2}\right)^k\right|} + \frac{\exp\left\{-2\pi\alpha^{-2}\ \mathrm{Im}\left(\nu+\varepsilon\right)/3\right\}}{\xi^{n}\left|\left(-F_{2n}z+F_{2n+2}\right)^k\right|}\right\},$$

where $\xi = \exp\left\{2\pi\ \mathrm{Im}\left(\varepsilon-k/4\right)\right\}$.

When $k \notin \mathbb{Z}$, we cannot in general write $\left(z_1z_2\right)^k$ as $z_1^kz_2^k$. However, a positive real factor can be factored out, so that

$$\begin{aligned}\left(F_{2n}z-F_{2n-2}\right)^k &= F_{2n}^k\left(z-F_{2n-2}/F_{2n}\right)^k,\\ \left(-F_{2n}z+F_{2n+2}\right)^k &= F_{2n}^k\left(-z+F_{2n+2}/F_{2n}\right)^k.\end{aligned}$$

(Again, $z$ is fixed here since we are proving pointwise divergence.) Then convergence of the preceding series is equivalent to convergence of

$$\sum_{n=1}^{\infty}\xi^nF_{2n}^{-\mathrm{Re}(k)} + \sum_{n=1}^{\infty}\xi^{-n}F_{2n}^{-\mathrm{Re}(k)},$$

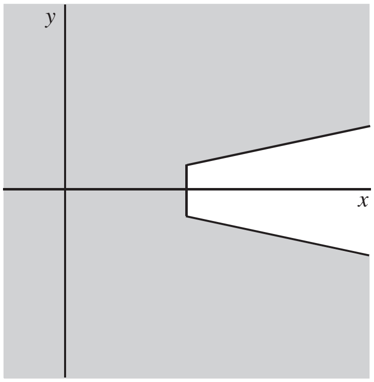
with $\xi = \exp\left\{2\pi\ \mathrm{Im}\left(\varepsilon-k/4\right)\right\}$. We know that $F_j \sim \alpha^j$, and so it suffices now to prove divergence of

$$\begin{aligned}\sum_{n=1}^{\infty}\xi^n\alpha^{-2n\mathrm{Re}(k)} + \sum_{n=1}^{\infty}\xi^{-n}\alpha^{-2n\mathrm{Re}(k)} &= \sum_{n=1}^{\infty}\left(\xi\alpha^{-2\mathrm{Re}(k)}\right)^n + \sum_{n=1}^{\infty}\left(\xi^{-1}\alpha^{-2\mathrm{Re}(k)}\right)^n\\ &= \sum\nolimits_1 + \sum\nolimits_2.\end{aligned}$$

Now, $\sum_1$ diverges if $\xi\alpha^{-2\mathrm{Re}(k)} \geq 1$, which is to say

$$\exp\left\{2\pi\ \mathrm{Im}\left(\varepsilon-k/4\right)\right\} \geq \exp\left(2\ \mathrm{Re}(k)\log\alpha\right),$$

i.e. $\pi\ \mathrm{Im}\left(\varepsilon-k/4\right) \geq \mathrm{Re}(k)\log\alpha$. Meanwhile $\sum_2$ diverges if $-\pi\ \mathrm{Im}\left(\varepsilon-k/4\right) \geq \mathrm{Re}(k)\log\alpha$. That implies divergence of $\sum > \sum_1 + \sum_2$ everywhere except *possibly* when $-\mathrm{Re}(k)\log\alpha < \pi\ \mathrm{Im}\left(\varepsilon-k/4\right) < \mathrm{Re}(k)\log\alpha$; and as in Section 1 we make the technical assumption $\mathrm{Re}(k) > 2$. (Convergence of the series (1) is not well understood for small positive weights; see [6, p. 36], and [1, p. 468].)




For example, if Im $\varepsilon = 0$, then we have excluded the shaded region $\left\{k = x + iy \mid x < 2 \text{ or } |y| > (4/\pi)\, x \log\left(\left(1+\sqrt{5}\right)/2\right)\right\}$ shown in the figure above. The unshaded region shows possible convergence, and other subgroups besides our cyclic group $\langle S_3 T\rangle$ can be used to establish divergence for these remaining nonreal values of $k$. In this way, it can be shown that the *only* weights where convergence takes place are real, i.e. the classical case. For another example, when Im $\varepsilon = 1$, each of the subgroups $\langle S_3T\rangle$, $\langle S_3^2T\rangle$, $\langle S_3^3T\rangle$, and so on, results in a different divergence region. Each of the aforementioned subgroups leads to a sequence of entries that resembles $F_n$ in its recurrence relation, asymptotic behavior, and Binet-style general formula; indeed, these sequences constitute an interesting generalization of the Fibonacci sequence. (Moreover, other relations between $S_\lambda T$ and $F_n$ are apparent if we relax the requirement that $\lambda \in \mathbb{Z}$. For example,

$$\left(S_{\sqrt{5}}T\right)^n = \begin{pmatrix} F_n + F_{n+2} & -F_n\sqrt{5} \\ F_n\sqrt{5} & -F_n - F_{n-2} \end{pmatrix} = \begin{pmatrix} L_{n+1} & -F_n\sqrt{5} \\ F_n\sqrt{5} & -L_{n-1} \end{pmatrix},$$

when $n \in 2\mathbb{Z}^+$.)

Thus, using standard properties of the Fibonacci sequence, we have proved divergence of the absolute Poincaré series on the group $G_3 = \langle S_3, T\rangle$ for nonreal weights.

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MSC2010: 11B39, 11F11, 15B36.

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