GOLDEN PROPORTIONS IN HIGHER DIMENSIONS

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ABSTRACT. The *golden ratio* $\phi = (1+\sqrt{5})/2$ appears in numerous contexts in the literature. A study is made to generalize ϕ to dimension n. Novel results are obtained by generalizing three different characterizations of ϕ to higher dimension.

1. INTRODUCTION

In this article we generalize to dimension n three equivalent definitions of the golden ratio $\phi = (1+\sqrt{5})/2$, which is the positive root of the quadratic equation

$$\phi^2 = \phi + 1.$$

Moreover, the sequence $1, 1, 2, 3, 5, 8, 13, \ldots$ in which each term is the sum of the preceding two terms is the familiar *Fibonacci sequence*. It is known that the nth term a_n is

$$a_n = \frac{\phi^n - (1-\phi)^n}{\sqrt{5}} = \frac{\phi^n - (-\phi)^{-n}}{\sqrt{5}}.$$

The following three characterizations of ϕ are known in the literature, and will be generalized in the following three sections, respectively.

(1) *Golden rectangle:* A rectangle is said to be *golden* if the ratio of length to width is ϕ (see figure 1). Starting with a golden rectangle R_1 of dimensions $(1, \phi)$, cut off a square of side 1, there remains a golden rectangle R_2 of dimensions $(1, \phi - 1) = (1, \frac{1}{\phi})$. We emphasize that R_2 is similar to R_1. Repeating this process generates a nested sequence of golden rectangles $R_1 \supset R_2 \supset \cdots \supset R_n \supset \cdots$

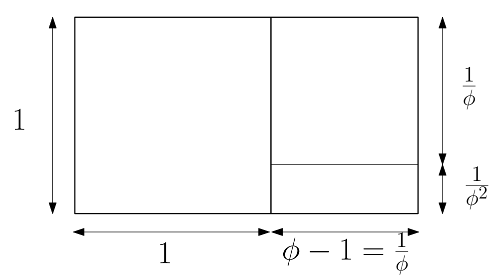

FIGURE 1. A golden rectangle.

(2) *Golden segment subdivision:* Given a line segment AC, there is a unique point B between A and C such that

$$\frac{AC}{AB} = \frac{AB}{BC}$$

It turns out that this ratio is the golden ratio ϕ.

FIGURE 2. A golden segment subdivision: $\frac{AC}{AB} = \frac{AB}{BC} = \phi$.

(3) *An alternative characterization of ϕ:* It is shown in [2] that ϕ is the unique number $r > 1$ satisfying the equation

$$\int_0^\infty \frac{dx}{(1+x^r)^r} = 1.$$

Remark. Other definitions of ϕ exist in the literature, e.g., continued fraction, series representation, etc.

2. n-Dimensional Golden Boxes

An *n-dimensional rectangular box R* (or simply an n-box) is defined to be

$$R = I_1 \times \cdots \times I_n,$$

where I_k is a closed interval in $\mathbb{R}$ of the form

$$I_k = [a_k, b_k].$$

The dimensions of R are defined to be

$$\dim(R) = (l_1, \ldots, l_n),$$

where l_k is the length of the interval I_k. We assume that

$$l_1 \leq \cdots \leq l_n.$$

In order to generalize the golden rectangle, we consider an n-box R and apply $n-1$ cuts with hyperplanes parallel to the faces (see figure 3). This subdivides R into 2^{n-1} smaller boxes. We assume that we make such cuts in a way that one of the boxes is a cube with side length l_1. We call the cutting a *regular cutting*, and refer to the cube of side length l_1 as the *residue* of the *regular cutting*. The n-box opposing the residue of the regular cutting is called the *result* of the regular cutting (see Figure 3).

Definition 2.1. *An n-box R is said to satisfy the golden cutting property if upon applying a regular cutting to it, the result of such a cutting is an n-box similar to R.*

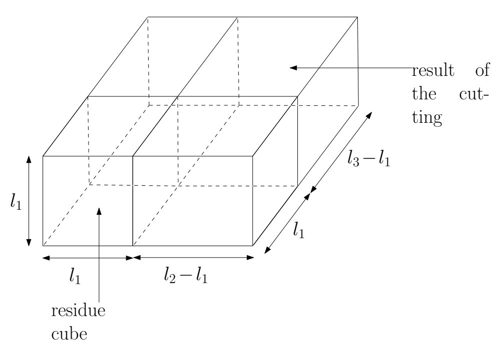

FIGURE 3. A regular cutting of a 3-box.

Definition 2.2. *A rectangular n-box is said to be golden if the following proportionality holds*

$$\begin{pmatrix} l_2 - l_1 \\ l_3 - l_1 \\ \vdots \\ l_n - l_1 \\ l_1 \end{pmatrix} = \lambda \begin{pmatrix} l_1 \\ l_2 \\ l_3 \\ \vdots \\ l_n \end{pmatrix}$$

for some $\lambda > 0$.

Remark. A golden n-box clearly satisfies the golden cutting property. For the remainder of the article, we shall only focus on golden n-boxes. We also note that a golden 2-box is a golden rectangle.

This is equivalent to the following eigenvalue problem

$$\begin{pmatrix} -1 & 1 & 0 & \cdots & 0 & 0 \\ -1 & 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ -1 & 0 & 0 & \cdots & 0 & 1 \\ 1 & 0 & 0 & \cdots & 0 & 0 \end{pmatrix} \begin{pmatrix} l_1 \\ l_2 \\ \vdots \\ l_{n-1} \\ l_n \end{pmatrix} = \lambda \begin{pmatrix} l_1 \\ l_2 \\ \vdots \\ l_{n-1} \\ l_n \end{pmatrix}. \tag{2.1}$$

The characteristic equation of this problem is

$$\sum_{i=1}^{n} \lambda^i = 1. \tag{2.2}$$

Consider the function

$$p(x) = \sum_{i=1}^{n} x^i - 1,$$

defined on the interval $[0, \infty)$. Since $p'(x) > 0$ on this interval, $p(x)$ is strictly increasing there. Since $p(0) = -1$, $p(1) = n - 1$, then by the intermediate value theorem, the equation $p(x) = 0$ has a unique solution in $[0, \infty)$ that lies in the interval $(0, 1)$. Denote this solution by λ.

Solving the eigenvalue problem (2.1) for this λ using backward substitution (starting from the bottom equation upwards), we obtain

$$\begin{pmatrix} l_1 \\ l_2 \\ \vdots \\ l_{n-1} \\ l_n \end{pmatrix} = l_1 \begin{pmatrix} 1 \\ 1 + \lambda \\ \vdots \\ 1 + \lambda + \lambda^2 + \cdots + \lambda^{n-2} \\ 1 + \lambda + \lambda^2 + \cdots + \lambda^{n-2} + \lambda^{n-1} \end{pmatrix}. \tag{2.3}$$

We define the *golden n-proportions* to be the vector

$$(1, 1 + \lambda, \ldots, 1 + \lambda + \cdots + \lambda^{n-1}) \tag{2.4}$$

up to a positive factor.

Remark. For $n = 2$ we obtain $\lambda = \frac{1}{\phi} \approx 0.6180$, and we recover the golden 2-proportions $(1, 1.6180)$ of the golden rectangle. Also, for $n = 3$ we obtain $\lambda \approx 0.5437$ using Cardano's formula, and we obtain the golden 3-proportions $(1, 1.5437, 1.8393)$.

Setting
$$r = \frac{1}{\lambda},$$
the characteristic equation becomes
$$r^n = r^{n-1} + r^{n-2} + \cdots + r + 1. \tag{2.5}$$
This is the n-step Fibonacci equation, related to the n-step Fibonacci recurrence relation
$$a_{m+n} = a_m + a_{m+1} + \cdots + a_{m+n-1}. \tag{2.6}$$
For $n = 2$, we recover the usual Fibonacci recurrence relation.

3. An Equation with a Generalized Golden Solution

The following theorem is a generalization of the result in [2].

Theorem 3.1. *There is a unique $r > 1$ such that*
$$\int_0^\infty \frac{dx}{(1 + x^{\frac{r^{n-1}}{1+r+\cdots+r^{n-2}}})^r} = 1. \tag{3.1}$$
Moreover, such an r satisfies
$$r^n = 1 + r + \cdots + r^{n-1}. \tag{3.2}$$

Proof. For the proof we need the following.

Lemma 3.2. *The equation*
$$\int_0^\infty \frac{dx}{(1 + x^{\frac{r^{n-1}}{1+r+\cdots+r^{n-2}}})^{1+\frac{1}{r}+\cdots+\frac{1}{r^{n-1}}}} = 1 \tag{3.3}$$
for all $r > 1$.

Proof. Denote the integral on the left-hand side of (3.3) by $I(r)$. In order to evaluate $I(r)$, we make use of the substitution
$$u = (1 + x^{\frac{r^{n-1}}{1+r+\cdots+r^{n-2}}})^{-1}.$$

After making this substitution, we obtain:
$$\begin{aligned}
I(r) &= (\frac{1}{r} + \cdots + \frac{1}{r^{n-1}}) \int_0^1 \left(\frac{1}{u} - 1\right)^{\frac{1}{r}+\cdots+\frac{1}{r^{n-1}}-1} u^{\frac{1}{r}+\cdots+\frac{1}{r^{n-1}}-1} du \\
&= (\frac{1}{r} + \cdots + \frac{1}{r^{n-1}}) \int_0^1 (1-u)^{\frac{1}{r}+\cdots+\frac{1}{r^{n-1}}-1} du \\
&= -(1-u)^{\frac{1}{r}+\cdots+\frac{1}{r^{n-1}}}\Big|_0^1 \\
&= 1.
\end{aligned}$$

□

Hence, if r satisfies the n-step Fibonacci equation, it follows that r satisfies equation (3.1). This proves the existence part of the theorem. It remains to prove uniqueness. Using a similar argument as in [2], we consider numbers α, β with $1 < \alpha < r < \beta$. It then follows that

$$1 + \frac{1}{\alpha} + \cdots + \frac{1}{\alpha^{n-1}} > 1 + \frac{1}{r} + \cdots + \frac{1}{r^{n-1}} = r > 1 + \frac{1}{\beta} + \cdots + \frac{1}{\beta^{n-1}}.$$

It thus follows that

$$\begin{aligned} 1 + \frac{1}{\alpha} + \cdots + \frac{1}{\alpha^{n-1}} &> \alpha \\ 1 + \frac{1}{\beta} + \cdots + \frac{1}{\beta^{n-1}} &< \beta. \end{aligned}$$

Therefore,

$$\int_0^\infty \frac{dx}{\left(1 + x^{\frac{\beta^{n-1}}{1+\beta+\cdots+\beta^{n-2}}}\right)^{\beta}} < \int_0^\infty \frac{dx}{\left(1 + x^{\frac{\beta^{n-1}}{1+\beta+\cdots+\beta^{n-2}}}\right)^{1+\frac{1}{\beta}+\cdots+\frac{1}{\beta^{n-1}}}} = 1.$$

Likewise, we prove that

$$\int_0^\infty \frac{dx}{\left(1 + x^{\frac{\alpha^{n-1}}{1+\alpha+\cdots+\alpha^{n-2}}}\right)^{\alpha}} > 1.$$

This shows uniqueness, and the theorem is proved. □

Remark. We note that the particular value of r in Theorem 3.1 is equal to $1/\lambda$, where λ is the particular eigenvalue found in Section 2 that produces the golden n-proportions. Also, for $n = 2$, we recover the alternative definition of the usual golden ratio ϕ in [2].

4. Golden n-Subdivision of a Line Segment

Consider a line segment A. We will abuse the notation slightly and denote its length also by A. We subdivide A into n consecutive subsegments, $a_1, \ldots, a_n$ (of lengths also denoted by $a_1, \ldots, a_n$, respectively). We assume that

$$a_1 > \cdots > a_n.$$

We call such a subdivision an *n-subdivision* of A (see figure 4).

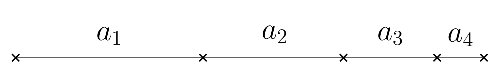

Figure 4. An example of a 4-subdivision of a line segment with $a_1 > a_2 > a_3 > a_4$.

Definition 4.1. *An n-subdivision is said to be golden if the following ratios are equal*

$$\frac{a_1}{a_2} = \frac{a_1 + a_2}{a_2 + a_3} = \cdots = \frac{\sum_{i=1}^{k} a_i}{\sum_{i=2}^{k+1} a_i} = \cdots = \frac{\sum_{i=1}^{n-1} a_i}{\sum_{i=2}^{n} a_i} = \frac{\sum_{i=1}^{n} a_i}{a_1}. \tag{4.1}$$

If we now let

$$\begin{aligned} l_1 &= a_1 \\ l_2 &= a_1 + a_2 \\ &\vdots \\ l_n &= a_1 + a_2 + \cdots + a_n, \end{aligned}$$

it is then easy to show that the equality of the ratios defining a golden n-subdivision is equivalent to the n-box of dimensions $(l_1, \ldots, l_n)$ being golden. It is worth noting that a 2-subdivision is golden if and only if

$$\frac{a_1}{a_2} = \frac{a_1 + a_2}{a_1}.$$

This ratio is of course the golden ratio ϕ.

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