KIMBERLING'S $\lfloor n^2\alpha\rfloor - n\lfloor n\alpha\rfloor$ FUNCTION

RUSSELL JAY HENDEL

Abstract. Kimberling defines the function $\kappa(n) = \lfloor n^2\alpha\rfloor - n\lfloor n\alpha\rfloor$, and presents conjectures and open problems. We present three main theorems. The theorems provide quick, effectively computable, lower bounds on $\kappa(n)$ which are useful in proving that certain values do not lie in the range of κ. Our main contribution is describing the behavior of $\kappa(n)$ within an almost negligible error using the differences of the indices in the Zeckendorf representation of n. We list 4 open problems connected with κ.

1. Notation and Main Results

Throughout this paper if n is a positive integer we let

$$n = \sum_{i\in I} F_i, I = \{i_1, i_2, \dots, i_m\}, 2 \le i_1 < i_2 \dots < i_m, i_{j+1} - i_j \ge 2, 1 \le j \le m-1, \qquad (1.1)$$

be the Zeckendorf representation of n with I an index set. We call m the *weight* of n. Recall the important convention for Zeckendorf index sets that

$$i_1 \ge 2, i_j - i_k \ge 2(j-k) \ge 2, 1 \le k < j \le m. \qquad (1.2)$$

It is convenient to have notation for the shifting of indices in the Zeckendorf representation of n. With notations as in (1.1), define

$$\hat{n} = \sum_{i\in I} F_{i+1}.$$

Kimberling [2] introduces the function

$$\kappa(n) = \lfloor n^2\alpha\rfloor - n\lfloor n\alpha\rfloor.$$

He presents two open problems: (a) Prove that 3 is not in the range of κ. (b) Find a closed-form formula describing the complement (over the positive integers) of the range of the function κ on the positive integers.

While preparing this manuscript Kimberling communicated to me that Behrend successfully proved that 3 is not in the range of κ [1]. Our goal in this paper is to develop a *general* theory about the range of κ. Since $\kappa(n)$ has discontinuity jumps this is a non-trivial problem which requires development of new methods. Our main tool will be use of the Zeckendorf representation with a consideration of cases depending on the parity of i_1, the minimal index in the Zeckendorf representation. This approach allows explicit computations with extremely small error terms. We next present two main theorems on the values of $\kappa(n)$.

Theorem 1.1. *Define*

$$A(I) = \sum_{i \in I} \frac{(-1)^{i+1}}{\sqrt{5}}, \tag{1.3}$$

$$B(I) = \sum_{i \in I} \frac{\beta^{2i}}{\sqrt{5}}, \tag{1.4}$$

$$C(I) = \sum_{\substack{i,j \in I \\ j>i}} (-1)^{i+1} \frac{\beta^{j-i}}{\sqrt{5}}, \quad \textit{and} \tag{1.5}$$

$$D(I) = \sum_{\substack{i,j \in I \\ j>i}} \frac{\beta^{j+i}}{\sqrt{5}}. \tag{1.6}$$

Let

$$c = A(I) + B(I) + 2C(I) + 2D(I). \tag{1.7}$$

(a) If $i_1 \equiv 0 \pmod 2$ *then*

$$\kappa(n) = n + \sum_{\substack{i,j \in I \\ j>i}} (-1)^{i+1} F_{j-i} + \lfloor c \rfloor.$$

(b) If $i_1 \equiv 1 \pmod 2$ *then*

$$\kappa(n) = \sum_{\substack{i,j \in I \\ j>i}} (-1)^{i+1} F_{j-i} + \lfloor c \rfloor.$$

Theorem 1.2.

$$\begin{cases} -m \le \lfloor c \rfloor \le m-2, & \textit{if } i_1 \equiv 0 \pmod 2; \\ -(m-1) \le \lfloor c \rfloor \le (m-1), & \textit{if } i_1 \equiv 1 \pmod 2. \end{cases}$$

These bounds are best possible.

An outline of the rest of this paper is the following. In Section 2 we prove Theorem 1.1. In Section 3 we show that Theorem 1.1 implies Theorem 1.2. In Section 4 we develop general computational tools and prove a third theorem. These three theorems allow one to prove that certain values do not lie in the range of κ. In Section 5 we illustrate application of the computational tools.

Example 1.3. *Theorems 1.1 and 1.2 give very good estimates of* $\kappa(n)$ *with almost negligible error. For example, if* $I = \{900, 903, 905, 907\}$ *then* $m = 4$, *and although* $\kappa(n)$ *is a 189-digit number the theorems still give a quickly computable value for* $\kappa(n)$ *with an error of only* $\lfloor c \rfloor = m - 2 = 2$. *Similarly, if* $I = \{900, 902, 904, 906\}$ *then* $\lfloor c \rfloor = -m = -4$. *Similarly, if* $I = \{901, 904, 906, 908\}$ *or* $I = \{901, 903, 905, 907\}$ *then* $\lfloor c \rfloor$ *equals -3 or 3, respectively. These numerical examples show that the bounds for* $\lfloor c \rfloor$ *are best possible. These examples are further developed in Examples 3.1 and 4.3 below.*

2. Proof of Theorem 1.1

We begin with an elementary proposition summarizing all needed identities. These identities follow routinely by the Binet form, the formula for the sum of geometric series and the identity $1 - |\beta^2| = |\beta|$. Throughout the proposition n, m are assumed to be positive integers.

Proposition 2.1.

$$\beta^n < 0, \text{ if } n \equiv 1 \pmod 2; \quad \beta^n > 0, \text{ if } n \equiv 0 \pmod 2. \tag{2.1}$$

$$\alpha F_n = F_{n+1} - \beta^n. \tag{2.2}$$

$$\sum_{\substack{n \geq n_0 \geq 2 \\ n \equiv n_0 \pmod 2}} |\beta|^n = |\beta|^{n_0 - 1}. \tag{2.3}$$

$$F_n \beta^m = (-1)^m F_{n-m} + F_m \beta^n, \tag{2.4}$$

$$1 + 2|\beta| = \sqrt{5}. \tag{2.5}$$

The next proposition has intrinsic interest in its own right since it gives a closed formula for $\lfloor n\alpha \rfloor$ and $n\lfloor n\alpha \rfloor$ in terms of Zeckendorf representations.

Proposition 2.2.

$$n\lfloor n\alpha \rfloor = \begin{cases} n\hat{n}, & \text{if } i_1 \equiv 1 \pmod 2, \\ n\hat{n} - n, & \text{if } i_1 \equiv 0 \pmod 2. \end{cases} \tag{2.6}$$

Proof.

$$\left.\begin{aligned} \alpha n &= \alpha \sum_{i \in I} F_i, \text{ by (1.1)}, \\ &= \sum_{i \in I} F_{i+1} - \sum_{i \in I} \beta^i, \text{ by (2.2)}, \\ &= \hat{n} - \sum_{i \in I} \beta^i. \end{aligned}\right\} \tag{2.7}$$

Hence, by (1.2), (2.3), and (2.1), $\lfloor n\alpha \rfloor$ equals $\hat{n}$ or $\hat{n} - 1$ depending on whether i_1 is odd or even, respectively. Equation (2.6) follows immediately. □

We can now prove Theorem 1.1.

Proof. By (1.1) and (2.7) we immediately have

$$n^2\alpha = n(n\alpha) = n\hat{n} - \sum_{i \in I} F_i \sum_{j \in I} \beta^j. \tag{2.8}$$

Since $n\alpha > \lfloor n\alpha \rfloor$, we have $n^2\alpha - n\lfloor n\alpha \rfloor > 0$. Hence, by (2.8) and (2.6) we have

$$n^2\alpha - n\lfloor n\alpha \rfloor = \begin{cases} n - \sum_{i \in I} F_i \sum_{j \in I} \beta^j, & \text{if } i_1 \equiv 0 \pmod 2, \\ -\sum_{i \in I} F_i \sum_{j \in I} \beta^j, & \text{if } i_1 \equiv 1 \pmod 2. \end{cases} \tag{2.9}$$

Therefore, to prove Theorem 1.1 it suffices to show that

$$-\sum_{i \in I} F_i \sum_{j \in I} \beta^j = \sum_{\substack{i,j \in I \\ j > i}} (-1)^{i+1} F_{j-i} + A(I) + B(I) + 2C(I) + 2D(I). \tag{2.10}$$

To accomplish this we view the summands in the product of the two sums on the left side of (2.10) as lying in a square whose rows are labeled by F_i and whose columns are labeled by β^j. We sum the diagonal, upper and lower triangles of this square separately.

The Diagonal. By the Binet form, (1.3), (1.4) and the identity $\alpha\beta = -1$, we have

$$-\sum_{i \in I} F_i \beta^i = -\sum_{i \in I} \frac{\alpha^i - \beta^i}{\sqrt{5}} \beta^i = A(I) + B(I). \tag{2.11}$$

The Upper Triangle. Again, by the Binet form, (1.5), (1.6) and the identity $\alpha\beta = -1$, we have

$$-\sum_{\substack{i,j \in I \\ j>i}} F_i \beta^j = -\sum_{\substack{i,j \in I \\ j>i}} \frac{\alpha^i - \beta^i}{\sqrt{5}} \beta^i \beta^{j-i} = C(I) + D(I). \tag{2.12}$$

The Lower Triangle. By (2.4) and (2.12) we have

$$-\sum_{\substack{i,j \in I \\ j>i}} F_j \beta^i = \sum_{\substack{i,j \in I \\ j>i}} (-1)^{i+1} F_{j-i} - \sum_{\substack{i,j \in I \\ j>i}} F_i \beta^j = \sum_{\substack{i,j \in I \\ j>i}} (-1)^{i+1} F_{j-i} + C(I) + D(I). \tag{2.13}$$

Equations (2.11)–(2.13) yield (2.10), completing the proof of Theorem 1.1. □

3. Proof of Theorem 1.2

We must deal with four cases depending on the parity of i_1 and whether we are estimating upper or lower bounds. For purposes of exposition we estimate lower bounds for the case $i_1 \equiv 1 \pmod 2$, the proof of the other three cases being similar.

Proof. Since i_1 is assumed odd we have, using the notation of (1.1),

$$i_1 \geq 3, i_2 \geq 5. \tag{3.1}$$

By (1.3) and the assumption of oddness of i_1, we have

$$A(I) \geq \frac{1 - (m-1)}{\sqrt{5}} \geq -\frac{m-2}{\sqrt{5}}. \tag{3.2}$$

By (1.4), (2.3), and (3.1) we have

$$|B(I)| \leq \frac{1}{\sqrt{5}} \sum_{i \geq i_1} |\beta|^{2i} = \frac{|\beta|^{2i_1 - 1}}{\sqrt{5}} \leq \frac{|\beta|^5}{\sqrt{5}}. \tag{3.3}$$

Similarly, by (1.6), (2.3), (1.2), and (3.1) we have

$$|D(I)| \leq \frac{1}{\sqrt{5}} \sum_{\substack{j \in I \\ j>i}} |\beta|^j \sum_{i \in I} |\beta|^i \leq \frac{1}{\sqrt{5}} \sum_{\substack{j \in I \\ j>i}} |\beta|^{j+i_1-1} \leq \frac{1}{\sqrt{5}} |\beta|^{i_2+i_1-2} \leq \frac{1}{\sqrt{5}} |\beta|^6. \tag{3.4}$$

By (2.3), (1.5), and (1.2) we have

$$|C(I)| \leq \frac{1}{\sqrt{5}} \sum_{j=2}^{m} \sum_{k=1}^{j-1} |\beta|^{i_j - i_k} \leq \frac{1}{\sqrt{5}} \sum_{j=2}^{m} \sum_{k=1}^{\infty} |\beta|^{2k} \leq \frac{1}{\sqrt{5}} \sum_{j=2}^{m} |\beta| \leq \frac{m-1}{\sqrt{5}} |\beta|. \tag{3.5}$$

By (3.2)–(3.5) and (2.5), a lower bound for $A(I) + B(I) + 2C(I) + 2D(I)$ is

$$-(m-2)\frac{1 + 2|\beta|}{\sqrt{5}} - \frac{2|\beta|^6 + |\beta|^5 + 2|\beta|}{\sqrt{5}} = -(m-2) - 0.642956\ldots.$$

Therefore, by (1.7), $\lfloor c \rfloor \geq -(m-1)$, as was to be shown.

To complete the proof of Theorem 1.2, we must show the bounds in Theorem 1.2 best possible. We accomplish this by generalizing the numerical illustrations presented in Example 1.3.

Example 3.1. *We generalize the numerical examples presented in Example 1.3 which show the bounds in Theorem 1.2 best possible.*

If i is odd, define $I_1(i) = \{i, i+2, i+4, i+6\}, i \geq 3$. It is easy to compute (for small values of i) that if $n(I_1(i)) = n$ is given by (1.1) *then* $\kappa(n) = \sum_{\substack{i,j \in I \\ j>i}} (-1)^{i+1} F_{j-i} + 3 = 17 + (m-1)$. *Similarly, if we define $I_2(i) = \{i, i+3, i+5, i+7\}, i \geq 2, i \equiv 1 \pmod 2$, then for small odd i,* $\kappa(n(I_2(i)) = \sum_{\substack{i,j \in I \\ j>i}} (-1)^{i+1} F_{j-i} - 3 = 15 - (m-1)$.

If $i \geq 4$ is even, then the computation for small i shows $\kappa(n(I_1(i)) - n(I_1(i) = \sum_{\substack{i,j \in I \\ j>i}} (-1)^{i+1} F_{j-i} - 4 = -17 - m$ *and* $\kappa(n(I_2(i)) - n(I_2(i) = \sum_{\substack{i,j \in I \\ j>i}} (-1)^{i+1} F_{j-i} + 2 = -15 + (m-2)$.

Corollary 4.2 of Section 4 will allow us to prove that $\kappa(n(I_1(i))) \equiv 20$ *and* $\kappa(n(I_2(i))) \equiv 12$ *for all odd integers $i \geq 3$; we will also show that* $\kappa(n(I_1(i))) - n(I_1(i)) \equiv -21$ *and* $\kappa(n(I_2(i))) - n(I_2(i)) \equiv -13$ *for all even integers $i \geq 4$.*

This completes the proof of Theorem 1.2. □

4. Tools For Computing $\kappa(n)$

The goal of this section is to provide quick effectively computable tools for determining if a particular value lies in the range of κ. It would appear that the density of integers not in the range of κ is positive. For example, a quick numerical check shows that

$$\frac{\#\{i \leq n : (\exists m \leq 1{,}000{,}000)(i = \kappa(m))\}}{n} \in \{0.370, 0.50\}, \text{ for } n \in \{100k : 1 \leq k \leq 15\}$$

where # indicates cardinality. The theorems presented in this section allow us to reduce checking whether a value is in the range of κ to a few computations with relatively small numbers.

Motivated by the difference of indices occurring in the summands occurring on the right hand side of the equations in Theorem 1.1(a) and (b) we introduce some notation. Using (1.1) we define

$$J = J(I) = \{d_1, d_2, \ldots, d_m\}, \text{ with } d_j = i_j - i_1, 1 \leq j \leq m. \tag{4.1}$$

Notice the elementary fact that

$$i_j - i_k = d_j - d_k, 1 \leq k < j \leq m.$$

For integer i we use the notation $i + J = \{i, i + d_2, \ldots, i + d_m\}$. In particular, $i_1 + J = I$.

Theorem 4.1.

$$A(I) = (-1)^{i_1} A(J). \tag{4.2}$$

$$B(I) = \beta^{2i_1} B(J). \tag{4.3}$$

$$C(I) = (-1)^{i_1} \sum_{m \geq j > k \geq 1} \beta^{d_j - d_k}. \tag{4.4}$$

$$D(I) = \beta^{2i_1} D(J). \tag{4.5}$$

Proof. (1.3)–(1.6) and (4.1). □

Corollary 4.2. *Suppose* $i_1 \equiv k \pmod 2$*, with* $k = k(i_1) \in \{2,3\}$*. Then*

$$\lim_{\substack{i\to\infty \\ i\equiv k \pmod 2}} A(i+J)+B(i+J)+2C(i+J)+2D(i+J) = (-1)^k\big(A(J)+2C(J)\big).$$

Furthermore, the sequence $\Big\{A(k+2i+J)+B(k+2i+J)+2C(k+2i+J)+2D(k+2i+J)\Big\}_{i\geq 0}$ *is monotone.*

Example 4.3. *To illustrate the computational usefulness of Corollary 4.2 we revisit Examples 1.3 and 3.1. We have* $I = \{900, 903, 905, 907\}$*,* $J = \{0,3,5,7\}$*, and* $k = k(900) = 2$*. We compute* $(-1)^k\big(A(J)+2C(J)\big) = 2.0308$*, and* $A(k+J)+B(k+J)+2C(k+J)+2D(k+J) = 2.0575$*. Hence, by Corollary 4.2 we have for all non-negative integer* i*,*

$$2.0308 < A(k+2i+J)+B(k+2i+J)+2C(k+2i+J)+2D(k+2i+J) \leq 2.0575,$$

and therefore, by (1.7)*, we have*

$$\lfloor c \rfloor = 2.$$

But Theorem 1.1 states that if $n(i) = \sum_{l\in k+2i+J} F_l, i \geq 0$*, then*

$$\kappa(n(i)) - n(i) = \sum_{1\leq j<l\leq m} (-1)^{d_j+1}F_{d_l-d_j} + \lfloor c \rfloor = \sum_{1\leq j<l\leq m} (-1)^{d_j+1}F_{d_l-d_j} + 2.$$

It immediately follows that for all non-negative integer i*,*

$$\kappa(n(i)) = n(i) + \sum_{1\leq j<l\leq m} (-1)^{d_j+1}F_{d_l-d_j} + 2 = n(i) - 13.$$

So for example, $\kappa(53) = 40, \kappa(139) = 126, \kappa(364) = 351, \ldots$*.*

Using the notations of Example 3.1 we can show $\kappa(n(I_1(i)) \equiv 20, i \geq 3, i \equiv 1 \pmod 2$*,* $\kappa(n(I_2(i)) \equiv 12, i \geq 3, i \equiv 1 \pmod 2$*, and* $\kappa(n(I_1(i)) \equiv n(I_1(i)) - 21, i \geq 4, i \equiv 0 \pmod 2$*. Note the unusual feature that* $\kappa(n(I_1(2)) \equiv n(I_1(2)) - 20, i \geq 4$*, since* $c > -3$ *for* $i = 2$ *while* $c < -3$ *for* $i \geq 4, i \equiv 0 \pmod 2$*.*

The following theorem gives a lower bound useful in computing the value of $\kappa(n)$ using Theorem 1.1(a) and (b). We use the notations of (1.1).

Theorem 4.4. *(a) If* $i_1 \equiv 1 \pmod 2$ *then*

$$\sum_{1\leq j<k\leq m} (-1)^{i_j+1}F_{i_k-i_j} \geq F_{d_2} + \sum_{k=3}^{m} F_{d_k-1} + (m-2). \tag{4.6}$$

(b) If $i_1 \equiv 0 \pmod 2$ *then*

$$n + \sum_{1\leq j<k\leq m} (-1)^{i_j+1}F_{i_k-i_j} \geq F_{i_1} + \sum_{j=2}^{m} F_{i_j-2} + (m-1). \tag{4.7}$$

(c) These bounds are best possible.

Proof. (a) Assume $i_1 \equiv 1 \pmod 2$. Then

$$\sum_{1\leq j<k\leq m} (-1)^{i_j+1}F_{i_k-i_j} \geq \sum_{j=2}^{m} F_{i_j-i_1} + \sum_{k=3}^{m}\sum_{j=2}^{k-1}(-1)^{i_j+1}F_{i_k-i_j}. \tag{4.8}$$

There are two cases to consider according to the parity of i_2.

Case - i_2 even. Since $i_1 \equiv 1 \pmod 2$ and $i_2 \equiv 0 \pmod 2$, by (1.2), we have

$$i_2 - i_1 \geq 3. \tag{4.9}$$

To estimate (4.8), we will need the well-known Lucas identity [3, p. 71]

$$\sum_{\substack{i \geq 0 \\ n-2i \geq 2}} F_{n-2i} = F_{n+1} - 1. \tag{4.10}$$

By (4.8), we therefore have

$$\begin{aligned}
\sum_{1 \leq j < k \leq m} (-1)^{i_j+1} F_{i_k - i_j} &\geq \sum_{j=2}^{m} F_{i_j - i_1} - \sum_{j=3}^{m} \sum_{k=2}^{j-1} F_{i_j - i_k}, \\
&\geq \sum_{j=2}^{m} F_{i_j - i_1} - \sum_{j=3}^{m} \left(F_{i_j - i_2 + 1} - 1 \right), \text{ by (4.10)}, \\
&\geq F_{i_2 - i_1} + \sum_{j=3}^{m} \left(F_{i_j - i_1} - F_{i_j - i_2 + 1} \right) + (m-2), \\
&\geq F_{i_2 - i_1} + \sum_{j=3}^{m} \left(F_{i_j - i_1 - 1} \right) + (m-2), \text{by (4.9)}, \\
&= F_{d_2} + \sum_{j=3}^{m} \left(F_{d_j - 1} \right) + (m-2), \text{by (4.1)}.
\end{aligned}$$

The proof for odd i_2 is similar and omitted.

The proof of Theorem 4.4(b) is similar to the proof of Theorem 4.4(a) and is omitted.

Proof of (c). For integer $n \geq 4$, define $I_n = \{3, 6, 8, \ldots, 2n\}$. It is straightforward to verify (using the identities in the above proof) that $m = m(I_n) = n - 1$ and

$$\sum_{1 \leq j \leq k \leq (n-1)} (-1)^{i_j+1} F_{i_k - i_j} = F_{2n-3} + (n-1) - 2 = F_{d_2} + \sum_{j=3}^{m} F_{d_k - 1} + m - 2.$$

This shows the bounds best possible for (4.6).

Similarly, by defining the sets $I_n = \{2, 4, \ldots, 2n\}, n \geq 2$, we can verify that $m = m(I_n) = n$ and

$$n + \sum_{1 \leq j \leq k \leq n} (-1)^{i_j+1} F_{i_k - i_j} = F_{2n-1} + n - 1 = F_{i_1} + \sum_{j=2}^{m} F_{i_j - 2} + m - 1,$$

proving the bounds best for (4.7). □

5. Applications

In this section we show how the theorems of the previous sections can reduce proof of the impossibility of a specific value being in the range of κ to a finite quick computational verification. We have chosen to illustrate by showing the impossibility that $\kappa(n) = 3$. Even though this result has already been proven [1] it is the computationally easiest case. This example also has sufficient richness to fully illustrate all machinery developed in this paper. Additionally many of the intermediate results needed have intrinsic interest in their own right.

We shall present the proof by a series of *computational facts.* Throughout, we assume the notations of (1.1).

Computational Fact 5.1. *If* $i_1 \equiv 1 \pmod 2$ *and* $m \geq 4$ *then* $\kappa(n) \neq 3$.

Proof. By (1.2) and (4.1), $d_2 \geq 2, d_3 \geq 4$, and $d_4 \geq 6$. Therefore, by (4.6), Theorems 1.1 and 1.2, $\kappa(n) \geq F_2 + F_{4-1} + F_{6-1} + (m-2) - (m-1) \geq 7$. □

Computational Fact 5.2. *If* $i_1 \equiv 0 \pmod 2$ *and* $m \geq 3$ *then* $\kappa(n) \neq 3$.

Proof. By (1.2),(4.7), Theorems 1.2 and 1.1, $\kappa(n) \geq F_2 + F_2 + F_4 + (m-1) - m \geq 4$. □

Computational Fact 5.3. *If* $m = 1$ *then*

$$\kappa(n) = \begin{cases} 0, & \text{if } i_1 \equiv 1 \pmod 2, \\ n-1, & \text{if } i_1 \equiv 0 \pmod 2. \end{cases}$$

Hence for all n, $\kappa(n) \neq 3$.

Proof. Note that since $m = 1$, the sum of Fibonacci numbers on the right hand side of Theorem 1.1(a), (b) is 0. Similarly $C(I) = 0$, $D(I) = 0$, $A(I) = (-1)^{i_1+1}\frac{1}{\sqrt{5}}$, and $B(I) = \frac{\beta^{2i_1}}{\sqrt{5}}$.

To apply Theorems 1.1 and 1.2 note that if $i_1 \geq 3$ is odd then Corollary 4.2 implies that $\frac{1}{\sqrt{5}} = 0.4472 < c < \frac{1+\beta^{2i_1}}{\sqrt{5}} \leq 0.4721$ implying $\kappa(n) = 0$, while if $i_1 \geq 2$ is even then Corollary 4.2 implies that $-0.3820 \leq \frac{-1+b^{2i_1}}{\sqrt{5}} < c < -\frac{1}{\sqrt{5}} = -0.4472$, implying $\kappa(n) = n-1$. □

Computational Fact 5.4. *If* $m = 2$ *then* $\kappa(n) \neq 3$ *for all* n.

Comment. It is an instructive exercise, similar to our treatment of the $m = 1$ case, to use Theorems 1.1 and 1.2 to derive a complete set of exact values for $\kappa(n)$. For example, we can show that

$$\kappa(n) = \begin{cases} F_{d_2} + 1, & \text{for } d_2 \in \{2,4\}, \text{ and any odd } i_1, \\ F_{d_2}, & \text{for } d_2 \geq 6, d_2 \equiv 0 \pmod 2, \text{ and any odd } i_1, \\ F_{d_2} - 1, & \text{for } d_2 \text{ odd}, d_2 \in \{3,5,7\}, \text{ and any odd } i_1. \end{cases}$$

However for $d_2 \geq 9$, $\kappa(n)$ equals F_{d_2}, for odd $i_1 \leq u(d_2)$, but equals $F_{d_2} - 1$, for odd $i_1 > u(d_2)$ where $u(d_2)$ is a non-decreasing function. For example, $u(9) = 3, u(11) = 3, u(13) = 5, u(15) = 5, u(17) = 7, \ldots$. The numerical evidence suggests the conjecture that $u(4m+1) = u(4m+3) = 2m-1$. Thus, even in the relatively simple case of $m = 2, i_1$ odd, a complete description of values of $\kappa(n)$ is complex. As m grows, the number of cases to consider increases significantly. Therefore, we must introduce a different proof approach.

Proof. First suppose $i_1 \equiv 1 \pmod 2$. Then by Theorems 1.1 and 1.2, $\kappa(n) \geq F_{d_2} + \lfloor c \rfloor \geq F_{d_2} - 1$, implying that if $d_2 \geq 5$, $\kappa(n) \geq 4$. Therefore, to complete the proof of the impossibility of $\kappa(n) = 3$ for the i_1 odd case, we only need to numerically verify the impossibility for $d_2 \in \{2,3,4\}$.

Similarly if $i_1 \equiv 0 \pmod 2$, then by Theorem 1.1 $\kappa(n) \geq n - F_{d_2} + \lfloor c \rfloor$. But $n = F_{i_1} + F_{i_1+d_2} \geq F_2 + F_{2+d_2}$, and by Theorem 1.2, $c \geq -2$. It follows that $\kappa(n) \geq F_{d_2+1} - 1$. The proof of the impossibility of $\kappa(n) = 3$ is completed by numerically verifying the impossibility for $d_2 \in \{2,3\}$. □

To complete the proof that for all n, $\kappa(n) \neq 3$, in light of Computational Facts 5.1 and 5.2, we are left to deal with the case $m = 3$ and $i_1 \equiv 1 \pmod 2$. We use the proof approach presented for the $m = 2$ case.

Computational Fact 5.5. *If $i_1 \equiv 1 \pmod 2$ and $m = 3$, then for all positive integer n, $\kappa(n) \neq 3$.*

Proof. By (1.2), (4.1) and Theorems 1.1 and 1.2, $\kappa(n) \geq F_{d_2} + F_{d_3} - F_{d_3-d_2} - 2$, with $d_i \geq 2, i = 2, 3$ and $d_3 - d_2 \geq 2$. Clearly $F_{d_3-d_2} \leq F_{d_3-2}$, implying $\kappa(n) \geq F_{d_2} + F_{d_3-1} - 2$. So for $d_3 \geq 6$, $\kappa(n) \geq 4$. The proof is completed by checking all integer lattice points (d_2, d_3) with $2 \leq d_2 < d_2 + 2 \leq d_3 < 6$. This completes the proof. □

6. Conclusion

Kimberling's κ function has many interesting properties. Because of its discontinuities, classical continuous methods do not suffice to analyze it. Nevertheless, $\kappa(n)$ has well-defined algebraic and analytic properties allowing us to obtain bounds.

Connected with $\kappa(n)$ there are many interesting conjectures and unsolved problems which appear solvable by the methods introduced in this paper. We list a few unsolved problems.

1. (Kimberling) Describe the complement of the range of κ.
2. Describe those n such that $\kappa(m) = n$ has (in)finitely many solutions.
3. Prove that the set of values (not) in the range of κ has positive density and compute this density.
4. We have seen that proofs of impossibility may require analyzing many cases depending on parity. Find and prove theorems which reduce the number of cases necessary to consider.

The methods of this paper as well as the four problems just enumerated should be of interest to those who work with Beatty sequences in general and with Wythoff sequences in particular.

References

[1] M. Behrend, http://www.cantab.net/users/michael.behrend/problems/never3/solution.html.
[2] C. Kimberling, *Problem 14*, http://faculty.Evansville.edu/ck6/integer/unsolved.html.
[3] T. Koshy, *Fibonacci and Lucas Numbers and Applications*, John Wiley and Sons, N.J., 2001.

MSC2010: 11B37, 11B39

Department of Mathematics, Towson University, Towson, Maryland 21252
E-mail address: RHendel@Towson.edu