PATTERNS IN DIFFERENCES BETWEEN ROWS IN k-ZECKENDORF ARRAYS

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ABSTRACT. For a fixed integer $k \geq 2$, we study the k-Zeckendorf array, $\mathcal{X}_k$, based upon the kth order recurrence $u_n = u_{n-1} + u_{n-k}$. We prove that the pattern of differences between successive rows is a k-letter infinite word generalizing the infinite Fibonacci word.

1. Introduction and Background

Definition 1. The *k-Zeckendorf array* [3], $\mathcal{X}_k = \{x_{r,c} \mid r, c \geq 0\}$, is a doubly subscripted array of positive integers. The first row begins with $x_{0,c} = c + 1$ for $0 \leq c < k$. For $i \geq k$,

$$x_{0,i} = x_{0,i-1} + x_{0,i-k}. \tag{1.1}$$

Subsequent rows are specified inductively as follows. For $r > 0$, $x_{r,0}$ is the smallest integer not in previous rows. Let the k-Zeckendorf representation (see Definition 2 below) of $x_{r,0}$ be $\sum_{i=0}^{m} d_i x_{0,i}$. Then for $c > 0$,

$$x_{r,c} = \sum_{i=0}^{m} d_i x_{0,i+c}. \tag{1.2}$$

Definition 2. The *k-Zeckendorf representation of* n is $\sum_{i=0}^{m} d_i x_{0,i}$, where for all i, $d_i \in \{0, 1\}$ and every sequence $\{d_i, d_{i+1}, \ldots, d_{i+k-1}\}$ contains at most one 1. The upper limit m in the sum is the largest integer such that $x_{0,m} \leq n$.

The well-known Zeckendorf theorem is for $k = 2$, and the sequence $\{x_{0,c}\}$ is the Fibonacci sequence $\{F_{c+2}\}$ (see [5, 8]). This generalizes easily to the k-Zeckendorf representation, which is also unique.

Definition 3. If the k-Zeckendorf representation of n is $\sum_{i=0}^{m} d_i x_{0,i}$, the *$k$-shift of n* is $S(n) = \sum_{i=0}^{m} d_i x_{0,i+1}$.

Taking $n = x_{r,0}$, we can write equation (1.2) as

$$x_{r,c} = S^c(x_{r,0}) = S \cdots S(x_{r,0}). \tag{1.3}$$

The arrays $\mathcal{X}_k$ have several well-known properties [2, 3, 4]:

(1) Every row of $\mathcal{X}_k$ satisfies the recurrence $x_{r,c} = x_{r,c-1} + x_{r,c-k}$.
(2) $\mathcal{X}_k$ contains every positive integer exactly once.
(3) $\mathcal{X}_k$ is an interspersion [3]. If $x_{r,c} < x_{r',c'} < x_{r,c+1}$, then $x_{r,c+1} < x_{r',c'+1} < x_{r,c+2}$.

Portions of $\mathcal{X}_2$, $\mathcal{X}_3$, and $\mathcal{X}_4$ are displayed below. $\mathcal{X}_2$ is also known as the Wythoff array [3] and is given as OEIS # A035513 in [7]. We use *precursion*:[1] $x_{r,n-k} = x_{r,n} - x_{r,n-1}$, to prepend k columns to each $\mathcal{X}_k$. We will establish later in Theorem 2 that $x_{r,-k} = r$ for $r \geq 0$ in all $\mathcal{X}_k$, as shown by column $c = -k$ in Table 1, Table 2, and Table 3.

[1]Clark Kimberling suggested the terminology at the 2010 Fibonacci Association conference in Mexico.

TABLE 1. k-Zeckendorf array at $k = 2$.

$\mathcal{X}_2$	$c:$ -2	-1	0	1	2	3	4	5	6	7
$r:0$	0	1	1	2	3	5	8	13	21	34
1	1	3	4	7	11	18	29	47	76	123
2	2	4	6	10	16	26	42	68	110	178
3	3	6	9	15	24	39	63	102	165	267
4	4	8	12	20	32	52	84	136	220	356
5	5	9	14	23	37	60	97	157	254	411
6	6	11	17	28	45	73	118	191	309	500

TABLE 2. k-Zeckendorf array at $k = 3$.

$\mathcal{X}_3$	$c:$ -3	-2	-1	0	1	2	3	4	5	6	7
$r:0$	0	1	1	1	2	3	4	6	9	13	19
1	1	3	4	5	8	12	17	25	37	54	79
2	2	4	5	7	11	16	23	34	50	73	107
3	3	5	7	10	15	22	32	47	69	101	148
4	4	7	10	14	21	31	45	66	97	142	208
5	5	9	13	18	27	40	58	85	125	183	268
6	6	10	14	20	30	44	64	94	138	202	296

TABLE 3. k-Zeckendorf array at $k = 4$.

$\mathcal{X}_4$	$c:$ -4	-3	-2	-1	0	1	2	3	4	5	6	7
$r:0$	0	1	1	1	1	2	3	4	5	7	10	14
1	1	3	4	5	6	9	13	18	24	33	46	64
2	2	4	5	6	8	12	17	23	31	43	60	83
3	3	5	6	8	11	16	22	30	41	57	79	109
4	4	6	8	11	15	21	29	40	55	76	105	145
5	5	8	11	15	20	28	39	54	74	102	141	195
6	6	10	14	19	25	35	49	68	93	128	177	245

Sequences in $\mathcal{X}_k$ for rows 0 and columns 0 that are recorded in [7] are referenced here.

TABLE 4. Integer sequences.

	$\mathcal{X}_2$	$\mathcal{X}_3$	$\mathcal{X}_4$
Row 0	A000045	A000930	A003269
Column 0	A003622	A020942	

2. Preliminaries

We focus our attention now on column zero of $\mathcal{X}_k$, $\{x_{r,0} \mid r \geq 0\}$. The elements of column zero of $\mathcal{X}_k$ are those numbers whose k-Zeckendorf representation ends with the least significant portion (abbreviated LSP, with its complement MSP as the most significant portion) given by

$$(d_{k-1}, d_{k-2}, \ldots, d_1, d_0) = (0, 0, \ldots, 0, 1). \tag{2.1}$$

In this notation we mimic the usual binary number representation with bit strings. The rightmost bit is the coefficient of $x_{0,0}$.

Lemma 1. $x_{r,0} = S^k(r) + 1 = S \cdots S(r) + 1.$

Proof. This follows from the definition of $\mathcal{X}_k$ and equation (2.1). □

The shift function preserves order, so $x_{r,0} < x_{r+1,0}$ for all $r \geq 0$.

In Theorem 1 below, we examine the pattern of the differences $x_{r+1,0} - x_{r,0}$ and then the differences between successive rows of $\mathcal{X}_k$. This pattern is captured in the sequence $\mathcal{W}_k$ of words, which for $k = 2$ is the simplest Sturmian word (the Fibonacci word, see [1]), with its higher order generalizations.

Definition 4. $\mathcal{W}_k = \{w_i\}_{i \geq 0}$ is an infinite sequence of words over the k-letter alphabet $\Sigma = \{a_i \mid 0 \leq i < k\}$. w_i is a word of length $x_{0,i}$, as follows:

$$\begin{aligned} w_0 &= a_0, \\ w_1 &= a_0 a_1, \\ w_2 &= a_0 a_1 a_2, \\ &\vdots \quad \vdots \\ w_{k-1} &= a_0 a_1 a_2 \cdots a_{k-1}, \\ w_i &= w_{i-1} w_{i-k} \text{ for } i \geq k. \end{aligned}$$

We define another infinite sequence $\mathcal{W}'_k$ of words, which is related to $\mathcal{W}_k$. Each derivation has useful word properties, which will be highlighted by Table 5 in Lemma 2.

Definition 5. $\mathcal{W}'_k = \{w'_i\}_{i \geq 1}$ is an infinite sequence of words over the same k-letter alphabet $\Sigma = \{a_i \mid 0 \leq i < k\}$, determined by the iterative algorithm:

$$w'_i = \begin{cases} a_i & \text{for } 1 \leq i < k, \\ a_0 & \text{at } i = k, \end{cases}$$

$$w'_i = w'_{i-1}\, w'_{i-k} \text{ for } i > k.$$

w'_i is a word of length $x_{0,i-k}$, where the word length is one for $0 < i < k$ according to precursion in row 0 of $\mathcal{X}_k$. The shifted relationships between w_i and w'_i and between their respective lengths $|w_i|$ and $|w'_i|$ are given by

$$w_i = w'_{i+k} \quad \text{and} \quad |w_i| = |w'_{i+k}| = x_{0,i} \text{ for } i \geq 0.$$

The Fibonacci case ($k = 2$) has $|w_i| = x_{0,i} = F_{i+2}$ for $i \geq 0$, and $|w'_i| = x_{0,i-2} = F_i$ for $i \geq 1$.

For $i \geq k$, w_{i-1} is a prefix of w_i and $w_{i-k} = w'_i$ is a suffix of w_i, yielding the infinite word

$$w = \lim_{i\to\infty} w_i = \lim_{i\to\infty} w'_i = \{w(n)\}_{n\geq 0} = w(0)w(1)w(2)\ldots, \tag{2.2}$$

where $w(n) \in \Sigma$. At $k = 2$, the infinite Fibonacci word w begins $1011010110110\ldots$ with $a_0 = 1, a_1 = 0$.

Lemma 2. *The least significant portion LSP of $k-1$ coefficients of the k-Zeckendorf representation of a sequence of nonnegative integers follows the algorithm used to construct $\mathcal{W}_k$.*

Proof. We express the k-Zeckendorf representation using at least $k-1$ bits. The LSP of length $k-1$ may take on k different patterns (composed of either all zeros or zeros with a single one). From these k patterns, we create our k-letter alphabet $\Sigma = \{a_i \mid 0 \leq i < k\}$. The k-Zeckendorf representation of 0 is 0, and we assign a_0 to instances when the LSP is 0^{k-1} ($k-1$ 0's). For

$1 \le i \le k-1$, we have by Definition 1 that $i = x_{0,i-1}$. The k-Zeckendorf representation of i consists of a single term $x_{0,i-1}$ with LSP of $0^{k-1-i}10^{i-1}$, to which we assign the letter a_i.

Table 5 with $k = 3$ illustrates the principle of this proof. Association between letters a, b, c in Table 5 and letters a_i in $\mathcal{W}_k$ and $\mathcal{W}'_k$ is made by substitutions $a = a_1$, $b = a_2$, and $c = a_0$, with indices of a_i listed in the last column of Table 5.

TABLE 5. k-Zeckendorf representations at $k = 3$.

n	MSP of k-Zeck.	LSP of k-Zeck.	letters	i
0		0 0	c	0
1		0 1	a	1
2		1 0	b	2
3	1	0 0	c	0
4	1 0	0 0	c	0
5	1 0	0 1	a	1
6	1 0 0	0 0	c	0
7	1 0 0	0 1	a	1
8	1 0 0	1 0	b	2
9	1 0 0 0	0 0	c	0
10	1 0 0 0	0 1	a	1
11	1 0 0 0	1 0	b	2
12	1 0 0 1	0 0	c	0

The horizontal lines in Table 5 are placed over the values n when $n = x_{0,c}$, the entries of $\mathcal{X}_3$ in row 0 at $c \ge 0$ beginning with $1, 2, 3, 4, 6, 9, 13$. The sequence of words in the letters column from the top of the table down to each horizontal line is the following:

$$c, ca, cab, cabc, cabcca, cabccacab, cabccacabcabc, \ldots.$$

Thus from Definition 4, each successive word sequence w_i from $i \ge 0$ gives the letter sequence from the table top to the next horizontal line, and each word length is $|w_i| = x_{0,i}$ as required.

The sequence of words in the letters column between the consecutive horizontal lines in Table 5 are the words w'_i for $i \ge 1$ listed in sequence as

$$a, b, c, ca, cab, cabc, cabcca, cabccacab, cabccacabcabc, \ldots.$$

At $k = 3$, row 0 of $\mathcal{X}_3$ for $x_{0,c-k}$ at $c \ge 1$ beginning with $1, 1, 1, 2, 3, 4, 6, 9, 13$ gives $|w'_c|$ as the number of terms between the horizontal lines in the table.

We now return to the general case, $k \le 2$, to form a table such as Table 5. We initialize the first k lines ($k = 3$ in the illustration) straightforwardly: for $0 \le i < k$, $x_{0,i} = i+1$, so that the k-Zeckendorf representation of i corresponds to the letter a_i. Then proceed inductively. Having constructed the first $x_{0,n-1}$ lines, the next $x_{0,n-k}$ lines are then built. If $x_{0,n-1} \le m < x_{0,n}$, the k-Zeckendorf representation of m is the k-Zeckendorf representation of $m - x_{0,n-k}$ augmented by $x_{0,n-1}$. As bit strings, take the initial $x_{0,n-k}$ lines of the table, pad their k-Zeckendorf representations with zeros on the left to make them all bit strings of length $n - k$. Then prepend each of these strings with 10^{k-1}, giving the new $x_{0,n-k}$ lines, extending the table to $x_{0,n}$ lines.

The list of LSP's follows the same algorithm used to construct $\mathcal{W}_k$. $\square$

In the above, in the words in $\{a, b, c\}$, a is always followed by b or c, b is always followed by c, and c is always followed by a or c. For general k, $\Sigma = \{a_i \mid 0 \le i < k\}$, a_i will always be followed by either a_{i+1} or a_0, with the latter occurring when there is a *carry*, a replacement of $x_i + x_{i+k-1}$ with x_{i+k}. a_{k-1} is always followed by a_0.

The subscript i of a_i is simply the value of the length $k-1$ LSP of the Zeckendorf representation of i. In other words, if the k-Zeckendorf representation of r is $\sum_{h=0}^{m} d_h\, x_{0,h}$, then $i = \sum_{h=0}^{k-2} d_h x_{0,h}$.

3. Main Theorems

Theorem 1. *The difference between two adjacent rows of $\mathcal{X}_k$ is a shift of row 0, where the difference $\delta_r(c) = x_{r+1,c} - x_{r,c} = x_{0,c+j(r)}$ with $1 \leq j(r) \leq k$. Specifically the shift index $j(r)$ is*

$$j(r) = \begin{cases} i & \text{if} \quad w(r) = a_i \quad \text{for} \quad 0 < i < k, \\ k & \text{if} \quad w(r) = a_0 \end{cases} \tag{3.1}$$

where $w(r)$ is the letter located at position r of the infinite word w of equation (2.2).

Proof. By Lemma 1, $x_{r,0} = S^k(r) + 1$, so $x_{r+1,0} - x_{r,0} = S^k(r+1) - S^k(r)$.

Suppose $w(r) = a_i$. Then $w(r+1) = a_{i+1}$ or $w(r+1) = a_0$, with the latter case occurring when there is one or more *carries*.

Case 1: $w(r) = a_i$ with $0 < i < k$. Express the k-Zeckendorf representation of r as $\sum d_n x_{0,n}$. $i-1$ is the smallest subscript such that $d_{i-1} = 1$, and $i-1+k$ is the smallest subscript such that $d'_{i-1+k} = 1$ in the k-Zeckendorf representation of $S^k(r)$ as $\sum d'_n x_{0,n}$.

Case 1.1: Suppose $w(r+1) = a_{i+1}$ and the k-Zeckendorf representation of $S^k(r+1)$ is $\sum d''_n x_{0,n}$. We have $d''_n = d'_n$ for $n > i+k$, so $S^k(r+1) - S^k(r) = x_{0,i+k} - x_{0,i-1+k} = x_{0,i}$.

Case 1.2: If $w(r+1) = a_0$, we may execute a sequence of *borrows* reversing the above mentioned sequence of *carries*, to write $S^k(r+1) = \sum d''_n x_{0,n}$ (which, after borrowing is *not* k-Zeckendorf). We have $d''_n = d'_n$ for $n > i+1$, so again $S^k(r+1) - S^k(r) = x_{0,i+k} - x_{0,i-1+k} = x_{0,i}$.

Case 2: $w(r) = a_0$. The two sub-cases, $w(r+1) = a_1$ and $w(r+1) = a_0$ follow a very similar argument to that given in Case 1. In both these sub-cases, $S^k(r+1) - S^k(r) = x_{0,k} - 0 = x_{0,k}$.

The above argument carries through identically for the general case

$$\begin{aligned} \delta_r(c) \;=\; x_{r+1,c} - x_{r,c} &= S^{c+k}(r+1) - S^{c+k}(r) \\ &= S^c(S^k(r+1) - S^k(r)) \\ &= S^c(x_{0,j(r)}) \\ &= x_{0,j(r)+c} \end{aligned} \tag{3.2}$$

where $j(r)$ is given in equation (3.1). $\square$

For illustration, we show that the first 13 differences in column 0 of $\mathcal{X}_3$ have the same pattern as the 3-letter word of $\mathcal{W}_3$. The first 13 terms of the sequences $\{\delta_r(0)\}$, $\{j(r)\}$ and $\{w(r)\}$ for $0 \leq r \leq 12$ are shown below. The substitution of $\{c, a, b\}$ for $\{a_0, a_1, a_2\}$ is made in the word $\{w(r)\}_{r \geq 0}$.

TABLE 6. Terms in Theorem 1 for k-Zeckendorf array at $k = 3$.

$\delta_{0,\ldots,12}(0)$	$j(0,\ldots,12)$	$w(0,\ldots,12)$
4,2,3,4,4,2,4,2,3,4,2,3,4	3,1,2,3,3,1,3,1,2,3,1,2,3	*cabccacabcabc*

Theorem 2. *For all r, $x_{r,-k} = r$.*

Proof. $x_{r+1,-k} - x_{r,-k} = x_{0,-k+j(r)} = x_{0,n}$. Because $1 \leq j(r) \leq k$, we have $1 - k \leq n \leq 0$. It follows from Definition 1 and a simple precursion argument that $x_{0,n}$ is always 1, and that $x_{0,-k} = 0$. □

Theorem 3. *For all $r \geq 0$, $w(x_{r,c}) = a_{c+1}$ if $0 \leq c \leq k-2$, and $w(x_{r,c}) = a_0$ if $c > k-2$.*

Proof. This follows from the definition of $\mathcal{X}_k$, Definition 3, equation (1.3), Lemma 1, equation (2.1), and Lemma 2.

By Lemma 2, the LSP of $0^{k-1-i}10^{i-1}$ assigns $w(n) = a_i$ if $1 \leq i \leq k-1$ and $w(n) = a_0$ if $i = 0$ or $i \geq k$. Likewise by Lemma 1, column 0 has LSP of $0^{k-1-i}1$; and by Definition 3 (equation (1.3)), column c has LSP of $0^{k-1-c}10^{c-1}$. □

Some examples of Theorem 3 include:

(1) $w(x_{r,0}) = a_1$ for $r \geq 0, k \geq 2$,
(2) $w(x_{r,1}) = a_2$ for $r \geq 0, k \geq 3$,
(3) $w(x_{r,k-1}) = a_0$ for $r \geq 0$.

The following results deal with how the numbers in column c of $\mathcal{X}_k$ punctuate (i.e., break into factors) the infinite word w. By Theorem 1, we know there are k different intervals $[x_{r,c}, x_{r+1,c})$, and the lengths of these intervals are k successive numbers of the sequence $\{x_{0,i}\}$.

Definition 6. Let the sequence

$$v_{r,c} = \langle w(i) \mid x_{r,c} \leq i < x_{r+1,c} \rangle$$

be a factor of the infinite word w. Its length is

$$|v_{r,c}| = \delta_r(c) = x_{r+1,c} - x_{r,c} = x_{0,c+j(r)}.$$

Definition 7. Let $\sigma : \Sigma^* \to \Sigma^*$ be a morphism defined by

$$\begin{aligned} \sigma(a_0) &= a_0 a_1, \\ \sigma(a_i) &= a_{i+1} \quad \text{for } 1 \leq i \leq k-1. \end{aligned}$$

As specified in Definition 5 and Theorem 1, we take $a_k = a_0$.

Lemma 3. *The infinite word w is the fixed point of σ.*

Proof. A straightforward inductive argument shows $\sigma(w_i) = w_{i+1}$ for all $i \geq 0$. □

Definition 8. For $0 \leq \lambda < k$, define words $\psi_\lambda(c)$ as follows:

$$\begin{aligned} \psi_0(0) &= a_1\ a_0, \\ \psi_\lambda(0) &= a_1 \cdots a_{\lambda+1}\ a_0 \quad \text{for } 1 \leq \lambda \leq k-2, \\ \psi_{k-1}(0) &= a_1 \cdots a_{k-1}\ a_k\ a_0. \end{aligned}$$

For $c > 0$, let $\psi_\lambda(c) = \sigma(\psi_\lambda(c-1))$.

Lemma 4. $|\psi_\lambda(c)| = x_{0,c+\lambda+1}$ *for $c \geq 0$.*

Proof. For $0 \leq \lambda \leq k-1$, the word $\psi_\lambda(0)$ has length $x_{0,\lambda+1}$ and is a permutation of the word $w_{\lambda+1}$. The result then follows from the observation that $\sigma(w_i) = w_{i+1}$ for all $i \geq 0$ (see Lemma 3) and that σ is a morphism. □

Theorem 4. *For $c \geq 0$, the $\psi_\lambda(c)$ are the only strings among the words $v_{r,c}$.*

Proof. From Definition 8, each string $\psi_\lambda(0)$ begins with the letter a_1, and it contains no other instance of a_1. In the infinite word w, every factor $\psi_\lambda(0)$ is followed by the letter a_1, beginning the next factor. The infinite string w, therefore, satisfies the following equations:

$$\begin{aligned} w &= w_0, \langle \psi_{j(r)+1}(0) \mid r = 0,1,2,\ldots \rangle, \\ \sigma(w) &= w_1, \langle \psi_{j(r)+1}(1) \mid r = 0,1,2,\ldots \rangle, \\ \vdots \;\; & \;\; \vdots \\ \sigma^c(w) &= w_c, \langle \psi_{j(r)+1}(c) \mid r = 0,1,2,\ldots \rangle. \end{aligned} \tag{3.3}$$

Because w is the fixed point of σ, the result follows. ☐

We illustrate the first few iterations from (3.3) using $k = 3$. For clarity of presentation, we substitute for letters a_i in Theorem 4 and Definition 8 according to $a = a_1$, $b = a_2$, and $c = a_0$.

$$\begin{aligned} w &= c, abcc, ac, abc, abcc, abcc, ac, abcc, ac, abc, abcc, ac, abc, abcc, abcc, ac, abc, \ldots \\ \sigma(w) &= ca, bccaca, bca, bcca, bccaca, bccaca, bca, bccaca, bca, bcca, bccaca, bca, bcca, \ldots \\ \sigma^2(w) &= cab, ccacabcab, ccab, ccacab, ccacab, cab, ccacab, cab, ccab, ccacab, cab, ccab, \ldots. \end{aligned}$$

Finally for $k = 3$, we display the k distinct strings $\psi_\lambda(c)$ in the first few columns $c \geq 0$.

TABLE 7. Words w_c and strings $\psi_\lambda(c)$ for k-Zeckendorf array at $k = 3$.

$\psi_\lambda(c)$	c : 0	1	2	3	4
w_c	c	ca	cab	$cabc$	$cabcca$
λ : 0	ac	bca	$ccab$	$cacabc$	$cabcabcca$
1	abc	$bcca$	$ccacab$	$cacabcabc$	$cabcabccabcca$
2	$abcc$	$bccaca$	$ccacabcab$	$cacabcabccabc$	$cabcabccabccacabcca$

We note that the strings, $\psi_\lambda(c)$ and $\psi_{\lambda-1}(c+1)$, have the same length and are a cyclic permutation of each other. We thus define a cycle operator ρ such that $\rho(a_i x) = x a_i$ for $a_i \in \Sigma$ and $a_i x \in \Sigma^*$, and we present the following proposition without proof.

Proposition 1. $\psi_\lambda(c) = \rho \cdots \rho(w_{c+\lambda+1}) = \rho^{|w_c|}(w_{c+\lambda+1})$.

Taking $k = 3$ and $c = 1$, Proposition 1 gives $\psi_\lambda(1) = \rho^{|w_1|}(w_{2+\lambda})$ with $\psi_0(1) = \rho^2(cab) = bca$, $\psi_1(1) = \rho^2(cabc) = bcca$, and $\psi_2(1) = \rho^2(cabcca) = bccaca$. As noted for diagonals, at $\lambda + c = 3$ we get $\psi_\lambda(c) = \rho^{|w_c|}(w_4)$ with $\psi_2(1) = \rho^2(cabcca) = bccaca$, $\psi_1(2) = \rho^3(cabcca) = ccacab$, and $\psi_0(3) = \rho^4(cabcca) = cacabc$; having equal lengths $|\psi_\lambda(c)| = |w_4| = x_{0,4} = 6$ as required by Proposition 1, Theorem 1, and Lemma 4.

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