EXTENDING FREITAG'S FIBONACCI–LIKE MAGIC SQUARE TO OTHER DIMENSIONS

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Abstract. The Fibonacci–like 4×4 magic square of Herta Freitag is analyzed from the standpoint of orthogonal diagonal latin squares and generalized to similar squares of higher dimension. Construction of such arrays are investigated and several examples are presented. The special case of 3×3 is constructed by other means leaving as the only unknown constructions of Fibonacci–like magic squares the 6×6 case.

1. INTRODUCTION

In [5], Brown showed that for $n \geq 2$ there do not exist any $n \times n$ magic squares with distinct entries chosen from a set of Fibonacci numbers. In [12, Array 5], Freitag discovered a 4×4 magic square and an algorithm for constructing an infinite family of such magic squares, $[F_a]_4$, having magic constant F_{a+8},

$$[F_a]_4 = \begin{bmatrix} F_{a+2} & F_{a+6} & F_{a+1}+F_{a+6} & F_{a+4} \\ F_{a+3}+F_{a+6} & F_{a+3} & F_{a+1}+F_{a+5} & F_a+F_{a+4} \\ F_{a+2}+F_{a+5} & F_a+F_{a+6} & F_{a+5} & 2F_{a+1} \\ F_{a+1}+F_{a+4} & F_{a+1}+F_{a+3} & F_a+F_{a+2} & F_{a+7} \end{bmatrix}, \tag{1.1}$$

and provided the example

$$[F_5]_4 = \begin{bmatrix} 13 & 89 & 97 & 34 \\ 110 & 21 & 63 & 39 \\ 68 & 94 & 55 & 16 \\ 42 & 29 & 18 & 144 \end{bmatrix}.$$

We note that the entry $2F_{a+1} = F_{a-1} + F_{a+2}$ so if $a > 2$ Zeckendorf's Theorem guarantees that the entries in (1.1) are unique. The entries in (1.1) are the sum of at most two Fibonacci numbers. Such squares will be referred to as Fibonacci–like magic squares.

Except for the case of $n = 3$, our construction of Fibonacci–like magic squares uses matrices known as orthogonal or graeco–latin squares. Using latin squares to construct magic squares is not original with this paper. Indeed, the first reference using this technique was presented by Euler [10] to the St. Petersburg Academy in 1776. However, whereas magic squares are categorized as recreational mathematics, latin squares have significant applications. For example, they are used in coding theory, statistical design, combinatorial group theory, biology, marketing, etc., and so, in general, material on latin squares can be obtained from a variety of sources. See, for example [8, 16, 24] and the extensive bibliography in [8]. It will be informative if we present various definitions and properties of latin squares, most of which are found in [8, 9].

Definition 1.1. A **latin square**, $L = [a_{ij}]$ of order or dimension n is an $n \times n$ matrix with entries from a set S of n elements, where none of the entries occur more than once in the same row or column.

Definition 1.2. A latin square L is **diagonal** if the entries on the main diagonal are distinct and the entries on the counter diagonal are distinct. L is called **pandiagonal** if L is diagonal and the entries on each of the broken diagonals are distinct.

Definition 1.3. Two order-n latin squares, $L_1 = [a_{ij}]$ with entries from a set S and $L_2 = [b_{ij}]$ with entries from a set T, are **orthogonal**, if the set of ordered pairs $\{(a_{ij}, b_{ij})\} = S \times T$, are distinct.

Definition 1.4. A latin square that is orthogonal to its transpose is called **self-orthogonal**.

Definition 1.5. A **magic square** is an $n \times n$ array with distinct whole number entries whose sum, in any row, column, main diagonal, and counter diagonal is a constant. The constant is called the **magic number** or the **magic constant**. If the n^2 entries are $1, \dots, n^2$ (or $0, \dots, n^2 - 1$) then the magic square is called an **ordinary magic square**.

Definition 1.6. A magic square is called **Fibonacci–like** if each entry is at most the sum of two Fibonacci numbers.

On a historical note, Euler [11] used the first n Latin letters and the first n Graeco–Roman letters for S and T and hence the names latin and graeco–latin squares. As an aside we invite the reader to visit The Euler Archive, an e-library, at `www.math.dartmouth.edu/∼euler` to peruse the complete works (866 articles!) that are available.

In Section 3, we generalize (1.1). To facilitate understanding, we point out the following easily proven properties of latin squares. Assume L, L_1 and L_2 are latin squares of order n on $\{0, \dots, n-1\}$. If $L = [a_{ij}]$ and $\sigma \in S_n$, the permutation group on n letters, then $\sigma(L) = [\sigma(a_{ij})]$ is a latin square. If L_1, L_2 are orthogonal (diagonal) latin squares and $\sigma, \gamma \in S_n$, then $\sigma(L_1), \gamma(L_2)$ are orthogonal (diagonal) latin squares. Given a diagonal latin square $L = [a_{ij}]$, let $\sigma(a_{ii}) = i$. Then $\sigma(L)$ has the additional property that $0, 1, 2 \dots, n-1$ are sequential on the main diagonal.

We will show that given two orthogonal diagonal latin squares it will always be possible to construct a Fibonacci–like magic square. However, for what values of n is it possible to construct two diagonal orthogonal latin squares? In the case of n odd not divisible by 3 it is easy to construct self-orthogonal diagonal latin squares by the following theorem.

Theorem 1.7. [8, p 109] *If n is odd not divisible by 3 then the $n \times n$ matrix $L = [a_{ij}]$ where $0 \le i, j \le n-1$ and*

$$a_{ij} = 2i + j \pmod n$$

is a self-orthogonal, pan-diagonal, latin square.

For $n = 1$, obviously the array $[0]$ is a self-orthogonal diagonal latin square. No magic square of order 2 exists and the two latin squares of order 2 are not orthogonal. None of the twelve latin squares for $n = 3$ are diagonal [24] and no orthogonal latin squares exist for $n = 6$ [19]. For a historical account of orthogonal latin squares of order $4n + 2$ see [2, 3, 11, 18, 19]. For all other values of n, orthogonal diagonal latin squares exist and the solutions/techniques used can be found in [1, 8, 9, 14, 20, 21, 22, 23]. We summarize these results in the following theorem.

Theorem 1.8. *If $n \notin \{2,3,6\}$ then self-orthogonal pairs of diagonal latin squares of order n exist. If $n \in \{2,3,6\}$ then orthogonal pairs of diagonal latin squares do not exist.*

Our extension of Frietag's Fibonacci–like magic squares to higher dimensions is a specific application of the basic method by which one constructs a general magic square by means of orthogonal latin squares. See, for example [8, 10] or less formally [16].

Construction 1.9. *Let $L_1 = [b_{ij}], L_2 = [c_{ij}]$ be orthogonal diagonal latin squares of order n on $\{0,\dots,n-1\}$. Let $S = \{s_i | i = 0,\dots,n-1\}$ and $T = \{t_i | i = 0,\dots,n-1\}$ be sets each containing n distinct values. If $\{s+t | s \in S, t \in T\}$ contains n^2 distinct values then*

$$[s_{b_{ij}}] + [t_{c_{ij}}]$$

is a magic square with magic constant $MN = \sum_{i=0}^{n-1}(s_i + t_i)$.

The proof of this follows directly from Definitions 1.1, 1.2, 1.3 and the hypothesis that $\{s+t | s \in S, t \in T\}$ contains n^2 distinct values.

By this method, an ordinary $n \times n$ magic square $[O]_n$ with entries $0,\dots,n^2-1$ can be constructed using orthogonal diagonal latin squares $[b_{ij}],[c_{ij}]$ of order n where

$$O_{ij} = b_{ij} + n \cdot c_{ij}, \; 0 \le i,j \le n-1.$$

For example,

$$\begin{bmatrix} 0 & 3 & 1 & 2 \\ 2 & 1 & 3 & 0 \\ 3 & 0 & 2 & 1 \\ 1 & 2 & 0 & 3 \end{bmatrix} + 4 \begin{bmatrix} 0 & 2 & 3 & 1 \\ 3 & 1 & 0 & 2 \\ 1 & 3 & 2 & 0 \\ 2 & 0 & 1 & 3 \end{bmatrix} = \begin{bmatrix} 0 & 11 & 13 & 6 \\ 14 & 5 & 3 & 8 \\ 7 & 12 & 10 & 1 \\ 9 & 2 & 4 & 15 \end{bmatrix}$$

is an ordinary magic square of order 4, with magic number $MN = 30$.

2. FREITAG'S FIBONACCI MAGIC SQUARE REVISITED

Freitag's procedure involved constructing three magic squares. This first array [12, Array 3]

$$\begin{bmatrix} F_{a_0}+F_{a_4} & F_{a_3}+F_{a_6} & F_{a_1}+F_{a_7} & F_{a_2}+F_{a_5} \\ F_{a_2}+F_{a_7} & F_{a_1}+F_{a_5} & F_{a_3}+F_{a_4} & F_{a_0}+F_{a_6} \\ F_{a_3}+F_{a_5} & F_{a_0}+F_{a_7} & F_{a_2}+F_{a_6} & F_{a_1}+F_{a_4} \\ F_{a_1}+F_{a_6} & F_{a_2}+F_{a_4} & F_{a_0}+F_{a_5} & F_{a_3}+F_{a_7} \end{bmatrix}, \tag{2.1}$$

where $F_{a_0},\dots,F_{a_7}$ are arbitrary Fibonacci numbers, generates an infinite family of Fibonacci–like magic squares. If $F_{a_0},\dots,F_{a_3}$ are distinct and $F_{a_4},\dots,F_{a_7}$ are distinct then this magic square can be written as

$$\begin{bmatrix} F_{a_0} & F_{a_3} & F_{a_1} & F_{a_2} \\ F_{a_2} & F_{a_1} & F_{a_3} & F_{a_0} \\ F_{a_3} & F_{a_0} & F_{a_2} & F_{a_1} \\ F_{a_1} & F_{a_2} & F_{a_0} & F_{a_3} \end{bmatrix} + \begin{bmatrix} F_{a_4} & F_{a_6} & F_{a_7} & F_{a_5} \\ F_{a_7} & F_{a_5} & F_{a_4} & F_{a_6} \\ F_{a_5} & F_{a_7} & F_{a_6} & F_{a_4} \\ F_{a_6} & F_{a_4} & F_{a_5} & F_{a_7} \end{bmatrix} \tag{2.2}$$

which by Definitions 1.2 and 1.3 is the sum of two orthogonal diagonal latin squares. However, $\sum_{i=0}^{7} F_{a_i}$ is not in general a Fibonacci number. By using a variation on the identity

$\sum_{i=1}^{n} F_{2i-1} = F_{2n}$ [17, Theorem 5.2, p.71] and imposing restrictions on the F_{a_i}, her second array [12, Array 4], which can be written as

$$\begin{bmatrix} F_a & F_{a+5} & F_{a+1} & F_{a+3} \\ F_{a+3} & F_{a+1} & F_{a+5} & F_a \\ F_{a+5} & F_a & F_{a+3} & F_{a+1} \\ F_{a+1} & F_{a+3} & F_a & F_{a+5} \end{bmatrix} + \begin{bmatrix} F_{a+7} & F_{a+11} & F_{a+13} & F_{a+9} \\ F_{a+13} & F_{a+9} & F_{a+7} & F_{a+11} \\ F_{a+9} & F_{a+13} & F_{a+11} & F_{a+7} \\ F_{a+11} & F_{a+7} & F_{a+9} & F_{a+13} \end{bmatrix}, \tag{2.3}$$

corrects this so that the magic number $MN = F_{a+4(4)-2} = F_{a+14}$. By imposing different restrictions on the F_{a_i}, her last array (1.1) has magic number $MN = F_{a+8}$ and displays the main diagonal as single Fibonacci numbers.

Using the observations above, in the following section, we extend (1.1), (2.1), and (2.3) to higher dimensional magic squares for sequences arising from second order recurrence relations.

3. HIGHER DIMENSIONAL FREITAG–TYPE MAGIC SQUARES

In this section, we extend the results of Freitag to higher dimensions. Since the extensions are by Construction 1.9, Theorem 1.8 applies and so excludes the dimensions 2,3,6 for these constructions. We will need the following two technical results.

Proposition 3.1. *Let S and T be two sets of Fibonacci numbers with $|S| = m, |T| = n$ and $S \cap T = \{\ \}$. Then the set $\{s + t | s \in S, t \in T\}$ contains mn distinct sums.*

Proof. Suppose s, $s' \in S$, t, $t' \in T$, and $s + t = s' + t'$. First note that since $S \cap T = \{\ \}$, then $\{s, t\} = \{s', t'\}$ implies $s = s'$ and $t = t'$. Suppose both $s + t$ and $s' + t'$ are the Zeckendorf representation [4, 13] of the sum of two nonconsecutive Fibonacci numbers. Then by the Zeckendorf Theorem it follows $\{s, t\} = \{s', t'\}$ and so $s = s'$ and $t = t'$. Now suppose at least one of $s + t$ or $s' + t'$ is not a Zeckendorf representation of the sum of two nonconsecutive Fibonacci numbers. Without loss of generality, we can assume $s + t$ is not written as the sum of two nonconsecutive Fibonacci numbers. Then by the Zeckendorf Theorem it follows that either $s = t$ or s, t are consecutive Fibonacci numbers. Since $S \cap T = \{\ \}$ $s \neq t$, and so s, t are consecutive Fibonacci numbers. It follows that s', t' are the same consecutive Fibonacci numbers and thus $\{s, t\} = \{s', t'\}$ and so $s = s', t = t'$. Hence, $\{s + t | s \in S, t \in T\}$ contains mn distinct sums. □

Proposition 3.2. *Let $a > 2$, $S = \{F_a\} \cup \{F_{a+2i-1} | i = 1, \ldots, n-1\}$ and $T = \{F_{a+1}\} \cup \{F_{a+2i} | i = 1, \ldots, n-1\}$. Then the set $\{s + t | s \in S, t \in T\}$ contains n^2 distinct sums.*

Proof. Since $a > 2$, F_{a-1}, F_a, and F_{a+1} are distinct Fibonacci numbers. Let $S' = S - \{F_{a+1}\}$. Then by Proposition 3.1 the set $\{s' + t | s \in S', t \in T\}$ contains $n^2 - n$ distinct sums. The sums $s' + t$ written with the minimal number of summands of Fibonacci numbers are given by (3.1) and (3.2). We have the $2n - 3$ sums of consecutive Fibonacci numbers

$$\begin{cases} F_a + F_{a+1} = F_{a+2} \\ F_{a+2i-1} + F_{a+2i} = F_{a+2i+1}, i = 2, \ldots, n-1 \\ F_{a+2(i+1)-1} + F_{a+2i} = F_{a+2i+2}, i = 1, \ldots, n-2, \end{cases} \tag{3.1}$$

and the $n^2 - 3n + 3$ minimal sums of two non-consecutive Fibonacci numbers

$$\begin{cases} F_a + F_{a+2i},\ i = 1, \ldots, n-1 \\ F_{a+2i-1} + F_{a+1},\ i = 2, \ldots, n-1 \\ F_{a+2j-1} + F_{a+2i},\ j = 2, \ldots, n-1, i = 1, \ldots, n-1, i \neq j, i \neq j-1. \end{cases} \tag{3.2}$$

The remaining sums of $\{s+t|s \in S, t \in T\}$ are the sums $\{F_{a+1}+t|t \in T\}$ and are

$$\begin{cases} F_{a+1}+F_{a+1}=F_{a-1}+F_{a+2} \\ F_{a+1}+F_{a+2}=F_{a+3} \\ F_{a+1}+F_{a+2i}, i=2,\dots,n-1 \end{cases}. \tag{3.3}$$

Since $a>2$, none of the sums of (3.3) are listed in (3.1) or (3.2) and so $\{s+t|s \in S, t \in T\}$ has n^2 distinct sums. □

The higher dimensional extension of (2.2) is now a direct consequence of Proposition 3.1 and Construction 1.9. Let n be given with the restrictions of Theorem 1.8 and F_{a_0}, …, $F_{a_{n-1}}$, F_{a_n}, …, $F_{a_{2n-1}}$ be a sequence of $2n$ distinct Fibonacci numbers. Let $S=\{s_i|s_i=F_{a_i}, i=0,\dots,n-1\}$, $T=\{t_i|t_i=F_{a_{n+i}}, i=0,\dots,n-1\}$ and $L_1=[b_{ij}], L_2=[c_{ij}]$ be orthogonal diagonal latin squares of order n on the set $\{0,\dots,n-1\}$. Let $[F_a]_n$ be the matrix from Construction 1.9, i.e.,

$$[F_a]_n=[F_{a_{b_{ij}}}]+[F_{a_{n+c_{ij}}}]. \tag{3.4}$$

Then by Proposition 3.1 the sums $\{s+t|s \in S, t \in T\}$ are distinct and so $[F_a]_n$ is a magic square with magic constant $MN=\sum_{i=0}^{2n-1} F_{a_i}$ and with entries that are the sum of at most two Fibonacci numbers.

Similar to Frietag's construction, for each n, (3.4) is a prescription for generating an infinite family of Fibonacci–like magic squares.

Example 3.3. *For $n=5$, use Theorem 1.7 to generate the pandiagonal orthogonal latin squares*

$$L=\begin{bmatrix} 0&1&2&3&4 \\ 2&3&4&0&1 \\ 4&0&1&2&3 \\ 1&2&3&4&0 \\ 3&4&0&1&2 \end{bmatrix}, L^T=\begin{bmatrix} 0&2&4&1&3 \\ 1&3&0&2&4 \\ 2&4&1&3&0 \\ 3&0&2&4&1 \\ 4&1&3&0&2 \end{bmatrix}.$$

The matrix $[F_a]_5$ in this case is

$$[F_a]_5=\begin{bmatrix} F_{a_0}&F_{a_1}&F_{a_2}&F_{a_3}&F_{a_4} \\ F_{a_2}&F_{a_3}&F_{a_4}&F_{a_0}&F_{a_1} \\ F_{a_4}&F_{a_0}&F_{a_1}&F_{a_2}&F_{a_3} \\ F_{a_1}&F_{a_2}&F_{a_3}&F_{a_4}&F_{a_0} \\ F_{a_3}&F_{a_4}&F_{a_0}&F_{a_1}&F_{a_2} \end{bmatrix}+\begin{bmatrix} F_{a_5}&F_{a_7}&F_{a_9}&F_{a_6}&F_{a_8} \\ F_{a_6}&F_{a_8}&F_{a_5}&F_{a_7}&F_{a_9} \\ F_{a_7}&F_{a_9}&F_{a_6}&F_{a_8}&F_{a_5} \\ F_{a_8}&F_{a_5}&F_{a_7}&F_{a_9}&F_{a_6} \\ F_{a_9}&F_{a_6}&F_{a_8}&F_{a_5}&F_{a_7} \end{bmatrix}, \tag{3.5}$$

and is a panmagic square with magic constant $MN=\sum_{i=0}^{9} F_{a_i}$.

Example 3.4. *If we let $F_{a_i}=F_{i+2}$ for $i=0,\dots,9$ in the preceding example then*

$$[F_a]_5=\begin{bmatrix} 14&36&92&26&63 \\ 24&60&21&35&91 \\ 42&90&23&58&18 \\ 57&16&39&97&22 \\ 94&29&56&15&37 \end{bmatrix},$$

is panmagic with magic constant 231, which is not a Fibonacci number.

We now place restrictions on the Fibonacci numbers $F_{a_0}, \ldots, F_{a_{n-1}}, F_{a_n}, \ldots, F_{a_{2n-1}}$ to generalize (2.3) and (1.1). We use the extension of [17, Theorem 5.2, p.71]:

$$F_a + \sum_{i=1}^{k} F_{a+2i-1} = F_{a+2k}, \tag{3.6}$$

which is easily proven by induction.

First, we extend (2.3) so that the entries of $[F_a]_n$ are at most the sum of two Fibonacci numbers and the magic number is a prescribed Fibonacci number.

Let $a > 1$, $F_{a_0} = F_a$, and $F_{a_i} = F_{a+2i-1}, i = 1\ldots, 2n-1$. Since $a > 1$, the set $F = \{F_a\} \cup \{F_{a+2i-1} | i = 1, \ldots, 2n-1\}$ has $2n$ distinct Fibonacci numbers. Let $[F_a]_n$ be the magic square of (3.4). Then the magic constant of $[F_a]_n$ is now

$$MN = \sum_{i=0}^{2n-1} F_{a_i} = F_a + \sum_{i=1}^{2n-1} F_{a+2i-1} = F_{a+2(2n-1)} = F_{a+4n-2}.$$

Example 3.5. *In* (3.5), *let* $F_{a_0} = F_a$ *and* $F_{a_i} = F_{a+2i-1}$ *for* $i = 1, \ldots n-1$. *Then*

$$[F_a]_5 = \begin{bmatrix} F_a & F_{a+1} & F_{a+3} & F_{a+5} & F_{a+7} \\ F_{a+3} & F_{a+5} & F_{a+7} & F_a & F_{a+1} \\ F_{a+7} & F_a & F_{a+1} & F_{a+3} & F_{a+5} \\ F_{a+1} & F_{a+3} & F_{a+5} & F_{a+7} & F_a \\ F_{a+5} & F_{a+7} & F_a & F_{a+1} & F_{a+3} \end{bmatrix} + \begin{bmatrix} F_{a+9} & F_{a+13} & F_{a+17} & F_{a+11} & F_{a+15} \\ F_{a+11} & F_{a+15} & F_{a+9} & F_{a+13} & F_{a+17} \\ F_{a+13} & F_{a+17} & F_{a+11} & F_{a+15} & F_{a+9} \\ F_{a+15} & F_{a+9} & F_{a+13} & F_{a+17} & F_{a+11} \\ F_{a+17} & F_{a+11} & F_{a+15} & F_{a+9} & F_{a+13} \end{bmatrix},$$

is a panmagic square with magic constant $F_{a+4(5)-2} = F_{a+18}$.

Example 3.6. *If* $a = 2$ *in the previous example, then*

$$[F_2]_5 = \begin{bmatrix} 90 & 612 & 4186 & 246 & 1631 \\ 238 & 1610 & 123 & 611 & 4183 \\ 644 & 4182 & 235 & 1602 & 102 \\ 1599 & 94 & 623 & 4215 & 234 \\ 4194 & 267 & 1598 & 91 & 615 \end{bmatrix},$$

is a panmagic square with magic number $F_{20} = 6765$.

We now generalize (1.1) to higher dimensions. Let $a > 2$ and let $L_1 = [b_{ij}]$ and $L_2 = [c_{ij}]$ be orthogonal, diagonal latin squares on $\{0, \ldots, n-1\}$. We can assume that $0, 1, 2, \ldots, n-1$ are sequential on the main diagonals of L_1 and L_2, i.e., $b_{ii} = c_{ii} = i$ for $i = 0, \ldots, n-1$. Let $S = \{s_i | s_0 = F_a, s_i = F_{a+2i-1}, i = 1, \ldots n-1\}$ and $T = \{t_i | t_0 = F_{a+1}, t_i = F_{a+2i}, i = 1, \ldots n-1\}$. S contains n distinct values, T contains n distinct values, and by Proposition 3.2 the set $\{s + t | s \in S, t \in T\}$ contains n^2 distinct sums. Let $[F_a]_n$ be the magic square derived from Construction 1.9. The diagonal entries of $[F_a]_n$ are $s_{b_{00}} + t_{c_{00}} = s_0 + t_0 = F_a + F_{a+1} = F_{a+2}$ and for $i = 1, \ldots, n-1$, $s_{b_{ii}} + t_{c_{ii}} = s_i + t_i = F_{a+2i-1} + F_{a+2i} = F_{a+2i+1}$.

The magic number is given by the sum along the main diagonal which by (3.6) is

$$\begin{aligned}\sum_{i=0}^{n-1}(s_i+t_i) &= (F_a+F_{a+1})+\sum_{i=1}^{n-1}F_{a+2i+1}\\ &= (F_a+F_{a+1})+\sum_{i=2}^{n}F_{a+2i-1}\\ &= F_a+\sum_{i=1}^{n}F_{a+2i-1}=F_{a+2n}.\end{aligned}$$

Hence, $[F_a]_n$ is magic with main diagonal entries Fibonacci numbers and magic constant $MN = F_{a+2n}$.

We note in passing that it follows from Proposition 3.2 that all the Fibonacci numbers $F_{a+2}, F_{a+3}, \ldots, F_{a+2n-1}$ occur as entries in $[F_a]_n$.

It will be informative to present a complete example as to how this construction is done.

Example 3.7. *Using Theorem 1.7 with $n = 5$ begin with the self orthogonal array*

$$L=\begin{bmatrix}0&1&2&3&4\\2&3&4&0&1\\4&0&1&2&3\\1&2&3&4&0\\3&4&0&1&2\end{bmatrix}.$$

Let $\sigma(0) = 0, \sigma(3) = 1, \sigma(1) = 2, \sigma(4) = 3, \sigma(2) = 4$. Then

$$\sigma(L)=\begin{bmatrix}0&2&4&1&3\\4&1&3&0&2\\3&0&2&4&1\\2&4&1&3&0\\1&3&0&2&4\end{bmatrix},(\sigma(L))^T=\begin{bmatrix}0&4&3&2&1\\2&1&0&4&3\\4&3&2&1&0\\1&0&4&3&2\\3&2&1&0&4\end{bmatrix}.$$

Our L_1, L_2 are $L_1 = \sigma(L), L_2 = (\sigma L)^T$. Let $s_0 = F_a, s_1 = F_{a+1}, s_2 = F_{a+3}, s_3 = F_{a+5}, s_4 = F_{a+7}$ and $t_0 = F_{a+1}, t_1 = F_{a+2}, t_2 = F_{a+4}, t_3 = F_{a+6}, t_4 = F_{a+8}$. Construct $[F_a]_n$ using Construction 1.9. The (pan)magic square $[F_a]_5$ is now

$$\begin{bmatrix}F_a&F_{a+3}&F_{a+7}&F_{a+1}&F_{a+5}\\F_{a+7}&F_{a+1}&F_{a+5}&F_a&F_{a+3}\\F_{a+5}&F_a&F_{a+3}&F_{a+7}&F_{a+1}\\F_{a+3}&F_{a+7}&F_{a+1}&F_{a+5}&F_a\\F_{a+1}&F_{a+5}&F_a&F_{a+3}&F_{a+7}\end{bmatrix}+\begin{bmatrix}F_{a+1}&F_{a+8}&F_{a+6}&F_{a+4}&F_{a+2}\\F_{a+4}&F_{a+2}&F_{a+1}&F_{a+8}&F_{a+6}\\F_{a+8}&F_{a+6}&F_{a+4}&F_{a+2}&F_{a+1}\\F_{a+2}&F_{a+1}&F_{a+8}&F_{a+6}&F_{a+4}\\F_{a+6}&F_{a+4}&F_{a+2}&F_{a+1}&F_{a+8}\end{bmatrix}$$

and equals

$$\begin{bmatrix}F_{a+2}&F_{a+3}+F_{a+8}&F_{a+8}&F_{a+1}+F_{a+4}&F_{a+2}+F_{a+5}\\F_{a+4}+F_{a+7}&F_{a+3}&F_{a+1}+F_{a+5}&F_a+F_{a+8}&F_{a+3}+F_{a+6}\\F_{a+5}+F_{a+8}&F_a+F_{a+6}&F_{a+5}&F_{a+2}+F_{a+7}&F_{a-1}+F_{a+2}\\F_{a+4}&F_{a+1}+F_{a+7}&F_{a+1}+F_{a+8}&F_{a+7}&F_a+F_{a+4}\\F_{a+1}+F_{a+6}&F_{a+6}&F_a+F_{a+2}&F_{a+1}+F_{a+3}&F_{a+9}\end{bmatrix}, \tag{3.7}$$

with magic constant F_{a+10}.

Example 3.8. *If $a = 3$ then (3.7) is*

$$[F_3]_5 = \begin{bmatrix} 5 & 97 & 89 & 16 & 26 \\ 68 & 8 & 24 & 91 & 42 \\ 110 & 36 & 21 & 60 & 6 \\ 13 & 58 & 92 & 55 & 15 \\ 37 & 34 & 7 & 11 & 144 \end{bmatrix}$$

which is panmagic with magic constant $F_{13} = 233$.

4. CONCLUDING REMARKS

In conclusion, Section 3 provides a way to construct an infinite family of Fibonacci–like magic squares for each $n \geq 4$, except $n = 6$. The obvious Fibonacci-like magic square for $n = 1$ is $[F_a]_1 = [F_a]$ where F_a is any Fibonacci number. No order 2 magic squares exist. Using Chernick's [6] special form for 3×3 magic squares it can be shown that any 3×3 Fibonacci–like magic square has a magic constant of one of five different types. None of these magic constants are Fibonacci numbers. Thus no 3×3 Fibonacci–like magic square can have a Fibonacci number as a magic number. One such type is

$$\begin{bmatrix} F_{a+5} & F_a & F_{a+1} + F_{a+4} \\ F_{a+3} & F_{a+4} & F_{a+2} + F_{a+4} \\ F_a + F_{a+3} & F_{a+1} + F_{a+5} & F_{a+2} \end{bmatrix}$$

which has magic constant $F_{a+2} + F_{a+6}$ and so 3×3 Fibonacci–like magic squares can be constructed. It is our belief that 6×6 Fibonacci–like magic squares can be constructed, however at this time we have yet to find any specific 6×6 Fibonacci-like magic squares. For example, since a 6×6 magic square cannot be obtained by means of orthogonal latin squares the example [8, p. 212]

$$\begin{bmatrix} 35 & 1 & 6 & 26 & 19 & 24 \\ 3 & 32 & 7 & 21 & 23 & 25 \\ 31 & 9 & 2 & 22 & 27 & 20 \\ 8 & 28 & 33 & 17 & 10 & 15 \\ 30 & 5 & 34 & 12 & 14 & 16 \\ 4 & 36 & 29 & 13 & 18 & 11 \end{bmatrix}$$

can be written uniquely by Zeckendorf's theorem as

$$\begin{bmatrix} F_9+F_2 & F_2 & F_5+F_2 & F_8+F_5 & F_7+F_5+F_2 & F_8+F_4 \\ F_4 & F_8+F_6+F_4 & F_5+F_3 & F_8 & F_8+F_3 & F_8+F_4+F_2 \\ F_8+F_6+F_3 & F_6+F_2 & F_3 & F_8+F_2 & F_8+F_5+F_2 & F_7+F_5+F_3 \\ F_6 & F_8+F_5+F_3 & F_8+F_6+F_4+F_2 & F_7+F_4+F_2 & F_6+F_3 & F_7+F_3 \\ F_8+F_6+F_2 & F_5 & F_9 & F_6+F_4+F_2 & F_7+F_2 & F_7+F_4 \\ F_4+F_2 & F_9+F_3 & F_8+F_6 & F_7 & F_7+F_5 & F_6+F_4 \end{bmatrix},$$

which, though magic, is not Fibonacci–like.

We have used Construction 1.9 to extend (2.1), (2.3), and (1.1) to higher dimensions for general Fibonacci numbers $u_1 = a, u_2 = b$, $u_{n+2} = Au_{n+1} + Bu_n$ for constants a, b, A, B. In particular we have constructed the higher dimensional analogues for the Jacobsthal numbers and generalized the Pell-like magic square, $[P_a]$, that was presented in [7]. However for the Pell, Jacobsthal, and general second order recurrence relations, problems about uniqueness of sums occur and the forms of the extensions depend upon A and B. These results will be presented at a later date.

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