GCD PROPERTIES IN HOSOYA'S TRIANGLE

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ABSTRACT. We generalize several properties from Pascal's triangle to Hosoya's triangle. In particular, we prove the GCD property for the Star of David and other polygons. We also give a criterion to determine whether a sequence of points in a polygon or in a rhombus have GCD equal to one.

1. Introduction

Several authors have studied the properties of a hexagon surrounding a given point in the Pascal triangle. That hexagon gives rise to two triangles each surrounding the given point. These two triangles form what is called *Star of David*. The first study of the Star of David appeared in 1971 by Hoggatt, Jr. and W. Hansell [4]. Several other articles have been published since then, either proving open questions or generalizations of this concept. (See for example [1, 3, 6, 8].)

Hoggatt, Jr. and W. Hansell [4] proved that the points in a Star of David satisfy the multiplication property. That is, $a_1a_2a_3 = b_1b_2b_3$ (see Figure 2 part (a)). In 1972 Gould [1] conjectured that $\gcd(a_1, a_2, a_3) = \gcd(b_1, b_2, b_3)$ in the Star of David. In 1972 Hillman and Hoggatt [3] proved Gould's conjecture. In 1994 Korntved [6] proved that this property extends to other configurations. He constructed a configuration with three hexagons where each hexagon intersects the other two in a point. This configuration satisfies that the GCD of the alternating points is equal to the GCD of the others. For example, if Figure 3 holds in the Pascal triangle, then $\gcd(a_1, a_2, a_3, a_4, a_5, a_6) = \gcd(b_1, b_2, b_3, b_4, b_5, b_6)$.

In 1976 Hosoya [5, 7] introduced a new array of numbers that he called Fibonacci triangle. Later the name was changed to avoid confusion with another arrangement that is also called Fibonacci triangle. So, now Hosoya's arrangement is called *Hosoya's triangle.* Each entry of this triangle is a product of two Fibonacci numbers (see Table 1).

Koshy [7] has a complete chapter dedicated to the study of Hosoya's triangle and some of its properties. Some identities and additive properties of several configurations such as triangles and rhomboids can be found in [7, Chapter 15].

It is natural to ask the question, what properties that hold for Pascal's triangle generalize to Hosoya's triangle? In 2011 Griffiths [2], using ordinary generating functions, found an expression for the sum of the elements lying on the nth diagonal of Hosoya's triangle.

We believe that Hosoya's triangle may have as many properties as Pascal's triangle. Unfortunately, this triangle has not been studied enough. In this paper we prove that some properties that are true in Pascal's triangle generalize to Hosoya's triangle. In particular, we prove all properties mentioned above. Those properties behave better than in Pascal's triangle. For instance, we prove that the GCD's mentioned above are equal to one.

We prove the multiplication property and GCD property for the Star of David. We also prove the GCD property for the configuration with three hexagons constructed by Korntved (see Figure 3). It is interesting to mention that the GCD of those points is one. The main theorem proves that the GCD of an array of points in a rhombus is either one or two, if no two points are in the same diagonal.

2. Preliminaries

In this section we introduce some notations and definitions and give results that are going to be used throughout the paper. Some of them are well-known, but we prefer to restate them here as it will avoid ambiguities.

The *Hosoya's sequence* $\{H(r,k)\}_{r,k\geq 1}$ is defined recursively by

$$H(r,k) = H(r-1,k) + H(r-2,k) \text{ and } H(r,k) = H(r-1,k-1) + H(r-2,k-2)$$

$$\text{where } r > 2 \text{ and } 1 \leq k \leq r \text{ with } H(1,1) = H(2,1) = H(2,2) = H(3,2) = 1.$$

Note that $\{H(r,k)\}_{r,k\geq 1}$ begins with $r = 1$ and $k = 1$, instead of $r = 0$ and $k = 0$ as in [7]. In [7] it is proved that $H(r,k) = F_k F_{r-k+1}$. This sequence gives rise to *Hosoya's triangle* [5, 7, 11], where the entry in position k (taken from left to right) of the rth row is equal to $H(r,k) = F_k F_{r-k+1}$ (see Table 1).

									1									
								1		1								
							2		1		2							
						3		2		2		3						
					5		3		4		3		5					
				8		5		6		6		5		8				
			13		8		10		9		10		8		13			
		21		13		16		15		15		16		13		21		
	34		21		26		24		25		24		26		21		34	
55		34		42		39		40		40		39		42		34		55

Table 1. Hosoya's Triangle.

An nth *diagonal* in Hosoya's triangle is the collection of all Fibonacci numbers multiplied by F_n. We distinguish between *slash diagonals* and *backslash diagonals*, with the obvious meaning. We write $\mathrm{S}(F_n)$ and $\mathrm{B}(F_m)$ to mean the slash diagonal and backslash diagonal, respectively, (see Figure 1). We formally define these two diagonals as

$$\mathrm{S}(F_n) = \{H(n+i-1,n)\}_{i=1}^{\infty} = \{F_i F_n | i \in \mathbb{N}\},$$

and

$$\mathrm{B}(F_m) = \{H(m+i-1,i)\}_{i=1}^{\infty} = \{F_m F_i | i \in \mathbb{N}\}.$$

We can associate a pair of Fibonacci numbers to every element of Hosoya's triangle. If a is a point in Hosoya's triangle, then there are two Fibonacci numbers F_m and F_n such that $a \in \mathrm{B}(F_m) \cap \mathrm{S}(F_n)$. Thus, $a = F_m F_n$. Therefore, the element a corresponds to the pair (F_m, F_n). We denoted this correspondence by "$\longleftrightarrow$". That is, if $a \in \mathrm{B}(F_m) \cap \mathrm{S}(F_n)$, then $a \longleftrightarrow (F_m, F_n)$. It is clear that this correspondence is a bijection between points of Hosoya's triangle and pairs of Fibonacci numbers.

The *Star of David* is a configuration of 6 points in Hosoya's triangle formed by two triangles with vertices a_1, a_2, a_3 and b_1, b_2, b_3 (see Figure 2 parts (a) and (b)).

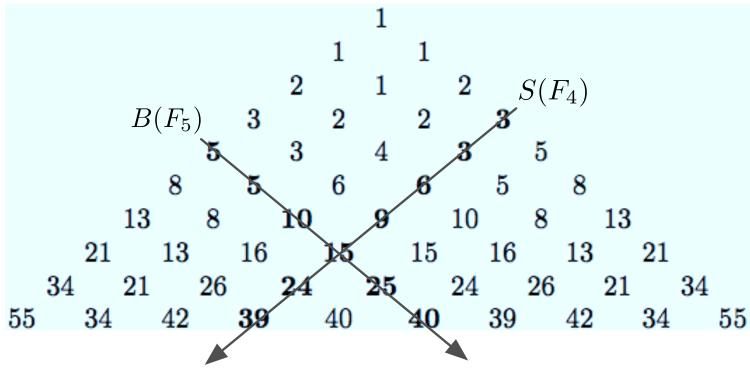
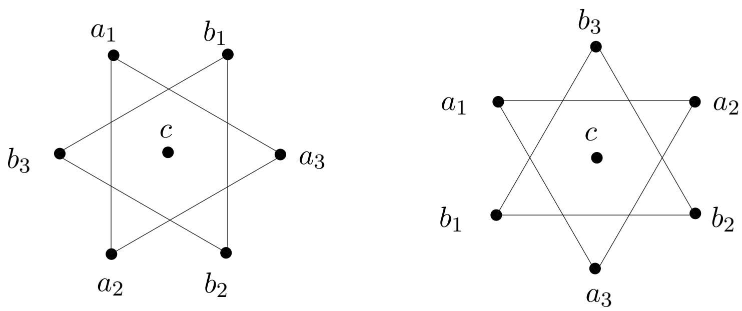


FIGURE 1. The slash diagonal $\mathrm{B}(F_5)$ and backslash diagonal $\mathrm{S}(F_4)$.



(a) (b)

FIGURE 2. Star of David.

The *multiplication property* of the *GCD* says that if a and b are co-primes, then $\gcd(ab, c) = \gcd(a, c)\gcd(b, c)$. We use this property to prove Proposition 2.2, which will play a special role throughout the paper. Proposition 2.2 is a natural relation. Therefore, we believe that it is known, but unfortunately we have not found this property in the mathematics literature.

Let p be a prime number and $A \in \mathbb{Z}^k$ for some $k \in \mathbb{N}$. We denote by $n_p(A)$ the number of entries in the k-tuple A that are divisible by p. For instance, $n_3(2, 3, 3, 6, 5, 12) = 4$.

Lemma 2.1 ([7, Theorem 16.3])**.** $\gcd(F_m, F_n) = F_{\gcd(m,n)}$.

Proposition 2.2. *Let a, b, c and d be positive integers.*

(1) If $\gcd(a, b) = 1$ and $\gcd(c, d) = 1$, then $\gcd(ab, cd) = \gcd(a, c)\gcd(a, d)\gcd(b, c)\gcd(b, d)$.
(2) If $\gcd(a, c) = \gcd(b, d) = 1$, then $\gcd(ab, cd) = \gcd(a, d)\gcd(b, c)$.

Proof. The proof of part (1) follows from the multiplication property of the GCD.

Proof of part (2). Let $d_1 = \gcd(a, d), d_2 = \gcd(b, c)$ and $x = \gcd(ab, cd)$. It is easy to see that $d_1d_2 \mid ab$ and $d_1d_2 \mid cd$. Thus,

$$d_1d_2 \mid x. \tag{2.1}$$

We now prove that $x \mid d_1d_2$. If $x = 1$, then it is clear that $x \mid d_1d_2$. We assume that $x \geq 2$. Let $x = p_1^{r_1}p_2^{r_2}\dots p_n^{r_n}$ be the prime decomposition of x. We consider a prime power p^r that divides x with $r \geq 1$. Thus, $p^r \mid ab$ and $p^r \mid cd$. Since $\gcd(b,d) = \gcd(a,c) = 1$, it is easy to see that either $p^r \mid a$ and $p^r \mid d$ or $p^r \mid b$ and $p^r \mid c$. Thus, $p^r \mid d_1$ or $p^r \mid d_2$. Therefore, $p^r \mid d_1 d_2$.

The previous argument shows that $p_i^{r_i} \mid d_1 d_2$ for every $i \in \{1, \dots, n\}$. Since $p_1^{r_1}, \dots, p_n^{r_n}$ are relatively prime, $(p_1^{r_1} p_2^{r_2} \dots p_n^{r_n}) \mid d_1 d_2$. That is, $x \mid d_1 d_2$. This and (2.1) imply that $x = d_1 d_2$. This proves the lemma. □

3. Properties that Generalize from Pascal's Triangle to Hosoya's Triangle

Hillman and Hoggatt [3] proved that if the configuration of points in Figure 2 part (a) holds in Pascal's triangle, then $\gcd(a_1, a_2, a_3) = \gcd(b_1, b_2, b_3)$. It is called the *GCD property* of the Star of David. However, this property is not true for the configuration of points in Figure 2 part (b). As a counterexample we could choose

$$a_1 = \binom{3}{1}, \quad a_2 = \binom{3}{2}, \quad a_3 = \binom{6}{3}, \quad b_1 = \binom{5}{2}, \quad b_2 = \binom{5}{3}, \quad b_3 = \binom{2}{1}.$$

In this section we prove that $\gcd(a_1, a_2, a_3) = \gcd(b_1, b_2, b_3) = 1$ for the configuration of points in Figure 2 parts (a) and (b) in Hosoya's triangle. This generalizes the GCD property of the Star of David. In Theorem 3.2 we prove that the product of $\gcd(a_1, b_2)$ and $\gcd(b_1, a_2)$ is equal to the interior point c of the hexagon. Theorem 3.4 generalizes this property for every polygon.

In [6] it is proved that if the configuration of points in Figure 3 in Pascal's triangle, then $\gcd(a_1, a_2, a_3, a_4, a_5, a_6) = \gcd(b_1, b_2, b_3, b_4, b_5, b_6)$. In Proposition 3.3 we prove that this property is also true for Hosoya's triangle.

Lemma 3.1. *Let m, n, s and t be positive integers. If $|m-n| \leq 2$ and $|s-t| \leq 2$, then*
(1) $\gcd(F_m, F_n) = 1$,
(2) $\gcd(F_m F_s, F_n F_t) = \gcd(F_m, F_t) \gcd(F_s, F_n) = F_{\gcd(m,t)} F_{\gcd(n,s)}$.

Proof. We prove part (1). The Lemma 2.1 and $\gcd(m,n) \leq 2$, imply that $\gcd(F_m, F_n) = F_{\gcd(m,n)} = F_1$ or F_2. This proves part (1).

We now prove part (2). From part (1) we know that $\gcd(F_t, F_s) = \gcd(F_m, F_n) = 1$. These and Proposition 2.2 part (2) imply that $\gcd(F_m F_s,\ F_n F_t) = \gcd(F_m, F_t) \gcd(F_s,\ F_n) = F_{\gcd(m,t)} F_{\gcd(s,n)}$. □

Theorem 3.2. *If a_1, a_2, a_3 and b_1, b_2, b_3 are the vertices of the two triangles of a Star of David in Hosoya's triangle with c as its interior point, then*
(1) $a_1 a_2 a_3 = b_1 b_2 b_3$,
(2) $\gcd(a_1, a_2, a_3) = \gcd(b_1, b_2, b_3) = 1$,
(3) $\gcd(a_1, b_2) \gcd(b_1, a_2) = c$.

Proof. We prove parts (2) and (3). It is easy to see that part (1) is true. We take a_1, a_2, a_3 and b_1, b_2, b_3 as the vertices of the two triangles of the Star of David in Figure 2 part (a). The proof for the Star of David in Figure 2 part (b) is similar and we omit it.

We choose $a_1 = H(r,k) = F_k F_{r-k+1}$. So,

$$\begin{aligned}
a_2 &= H(r+2, k+1) = F_{k+1} F_{r-k+2}, & a_3 &= H(r+1, k+2) = F_{k+2} F_{r-k}, \\
b_1 &= H(r, k+1) = F_{k+1} F_{r-k}, & b_2 &= H(r+2, k+2) = F_{k+2} F_{r-k+1}, \\
b_3 &= H(r+1, k) = F_k F_{r-k+2}, & c &= H(r+1, k+1) = F_{r-k+1} F_{k+1}.
\end{aligned}$$

Proof of part (2). We first prove that $\gcd(a_1, a_2, a_3) = 1$. Thus,

$$\begin{aligned}\gcd(a_1,a_2,a_3) &= \gcd(F_kF_{r-k+1}, F_{k+1}F_{r-k+2}, F_{k+2}F_{r-k})\\ &= \gcd(\gcd(F_kF_{r-k+1}, F_{k+1}F_{r-k+2}), F_{k+2}F_{r-k}).\end{aligned}$$

Lemma 3.1 part (2) with $m = k$, $s = r - k + 1$, $n = k + 1$ and $t = r - k + 2$ implies that

$$\begin{aligned}\gcd(F_kF_{r-k+1}, F_{k+1}F_{r-k+2}) &= \gcd(F_k, F_{r-k+2})\gcd(F_{r-k+1}, F_{k+1})\\ &= F_{\gcd(k,r-k+2)}F_{\gcd(r-k+1,k+1)}.\end{aligned}$$

Thus,

$$\gcd(\gcd(F_kF_{r-k+1}, F_{k+1}F_{r-k+2}), F_{k+2}F_{r-k}) = \gcd(F_{\gcd(k,r-k+2)}F_{\gcd(r-k+1,k+1)}, F_{k+2}F_{r-k}).$$

It is easy to see that

$$\gcd(F_{\gcd(k,r-k+2)}, F_{k+2}) = F_{\gcd(k,r-k+2,k+2)} = F_1 \text{ or } F_2$$

and

$$\gcd(F_{\gcd(r-k+1,k+1)}, F_{r-k}) = F_{\gcd(r-k+1,k+1,r-k)} = F_1 = 1.$$

Lemma 2.1 and Proposition 2.2 part (2) imply that

$$\begin{aligned}\gcd(a_1,a_2,a_3) &= \gcd(F_{\gcd(k,r-k+2)}F_{\gcd(r-k+1,k+1)}, F_{k+2}F_{r-k})\\ &= \gcd(F_{\gcd(k,r-k+2)}, F_{k+2})\gcd(F_{\gcd(r-k+1,k+1)}, F_{r-k})\\ &= F_{\gcd(k,r-k+2,k+2)}F_{\gcd(r-k+1,k+1,r-k)}\\ &= F_1F_1 \text{ or } F_2F_1\\ &= 1.\end{aligned}$$

The proof of $\gcd(b_1, b_2, b_3) = 1$ is similar and we omit it. This proves part (2).

Proof of part (3). We prove that $\gcd(a_1, b_2)\gcd(b_1, a_2) = c$. The choice of a_1, a_2, b_1, and b_2, and Lemma 2.1 imply that

$$\begin{aligned}\gcd(a_1,b_2) &= \gcd(F_kF_{r-k+1}, F_{k+2}F_{r-k+1})\\ &= F_{r-k+1}\gcd(F_k, F_{k+2})\\ &= F_{r-k+1}F_{\gcd(k,k+2)}\\ &= F_{r-k+1}.\end{aligned}$$

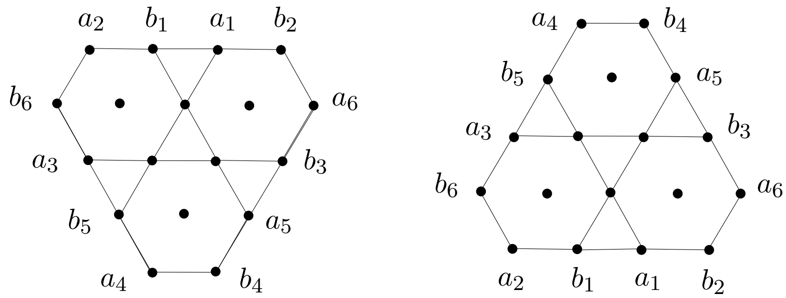
Similarly we obtain that $\gcd(b_1, a_2) = F_{k+1}$. Thus, $\gcd(a_1, b_2)\gcd(b_1, a_2) = F_{r-k+1}F_{k+1} = c$. This proves part (3). □

Proposition 3.3. *If $a_1, \ldots, a_5$ and $b_1, \ldots, b_5$ are points in Hosoya's triangle as in Figure* 3*, then* $\gcd(a_1, a_2, a_3, a_4, a_5) = \gcd(b_1, b_2, b_3, b_4, b_5) = 1$.

Proof. We prove this proposition for the points in Hosoya's triangle as in Figure 3 part (a), the proof for the points as in Figure 3 part (b) is similar and we omit it.

We prove $\gcd(a_1, a_2, a_3, a_4, a_5) = 1$. The proof of $\gcd(b_1, b_2, b_3, b_4, b_5) = 1$ is similar. We choose $a_1 = H(r, k) = F_kF_{r-k+1}$. So,

$$\begin{aligned}a_2 &= H(r, k-2) = F_{k-2}F_{r-k+3}, & a_3 &= H(r+2, k-1) = F_{k-1}F_{r-k+4},\\ a_4 &= H(r+4, k+1) = F_{k+1}F_{r-k+4}, & a_5 &= H(r+3, k+2) = F_{k+2}F_{r-k+2}.\end{aligned}$$



(a) (b)

FIGURE 3

We know that $\gcd(a_2, a_1, a_3, a_4, a_5, a_6) = \gcd(\gcd(a_1, a_2), \gcd(a_3, a_4), a_5)$. This is equal to

$$\gcd(\gcd(F_kF_{r-k+1}, F_{k-2}F_{r-k+3}), \gcd(F_{k-1}F_{r-k+4}, F_{k+1}F_{r-k+4}), F_{k+2}F_{r-k+2}). \tag{3.1}$$

Lemma 3.1 part (2) with $m = k$, $s = r - k + 1$, $n = k - 2$ and $t = r - k + 3$ implies that

$$\gcd(F_kF_{r-k+1}, F_{k-2}F_{r-k+3}) = F_{\gcd(k,r-k+3)}F_{\gcd(r-k+1,k-2)}. \tag{3.2}$$

It easy to see that $\gcd(F_{k-1}F_{r-k+4}, F_{k+1}F_{r-k+4}) = F_{r-k+4}\gcd(F_{k-1}, F_{k+1})$. This and Lemma 3.1 part (1) imply

$$\gcd(F_{k-1}F_{r-k+4}, F_{k+1}F_{r-k+4}) = F_{r-k+4}.$$

From this and the equalities, (3.1) and (3.2) we obtain that

$$\begin{aligned}\gcd(a_1, a_2, a_3, a_4, a_5) &= \gcd(F_{\gcd(k,r-k+3)}F_{\gcd(r-k+1,k-2)}, F_{r-k+4}, F_{k+2}F_{r-k+2})\\ &= \gcd(F_{\gcd(k,r-k+3)}F_{\gcd(r-k+1,k-2)}, F_{k+2}F_{r-k+2}, F_{r-k+4}).\end{aligned} \tag{a}$$

Since $\gcd(k, r - k + 3, k + 2) = 1$ or 2 and $\gcd(r - k + 1, k - 2, r - k + 2) = 1$, we have that

$$\gcd(F_{\gcd(k,r-k+3)}, F_{k+2}) = F_{\gcd(k,r-k+3,k+2)} = 1$$

and

$$\gcd(F_{\gcd(r-k+1,k-2)}, F_{r-k+2}) = F_{\gcd(r-k+1,k-2,r-k+2)} = 1.$$

This and Proposition 2.2 part (2), imply that

$$\begin{aligned}\gcd(F_{\gcd(k,r-k+3)}F_{\gcd(r-k+1,k-2)}, F_{k+2}F_{r-k+2}) &= F_{\gcd(k,r-k+3,r-k+2)}F_{\gcd(r-k+1,k-2,k+2)}\\ &= F_1F_{\gcd(r-k+1,k-2,k+2)}.\end{aligned}$$

This and (a) imply that

$$\gcd(a_2, a_1, a_3, a_4, a_5) = \gcd(F_{\gcd(r-k+1,k-2,k+2)}, F_{r-k+4}) = F_{\gcd(r-k+1,k-2,k+2,r-k+4)}. \tag{b}$$

We now suppose that

$$d = \gcd(r - k + 1, k - 2, k + 2, r - k + 4) = \gcd(\gcd(k - 2, k + 2), \gcd(r - k + 1, r - k + 4)).$$

Thus, $d \mid \gcd(k - 2, k + 2)$ and $d \mid \gcd(r - k + 1, r - k + 4)$. Since $\gcd(k - 2, k + 2)$ is $1, 2$ or 4 and $\gcd(r - k + 1, r - k + 4)$ is 1 or 3, we have that $d = 1$. This and (b) imply that $\gcd(a_2, a_1, a_3, a_4, a_5) = F_1 = 1$. □

Any set of the form $\{F_kF_j, F_kF_{j+1}, \ldots, F_kF_{j+l}\}$ where $l \geq 1$, is called a *subdiagonal* of Hosoya's triangle. It is clear that any subdiagonal of Hosoya's triangle is included in either $\mathrm{S}(F_k)$ or $\mathrm{B}(F_k)$. In Theorem 3.4 we generalize Theorem 3.2 part (3).

Theorem 3.4. *Let P be a polygon in Hosoya's triangle. If D_1 and D_2 are two diagonals of P such that*
(1) D_1 and D_2 are subdiagonals of Hosoya's triangle,
(2) $D_1 \cap D_2 = \{c\}$,
(3) $|D_1| \geq 3$ and $|D_2| \geq 3$,
then $\gcd(D_1 - \{c\}) \gcd(D_2 - \{c\}) = c$.

Proof. First we suppose that $|D_1| = |D_2| = 3$ and prove $\gcd(D_1 - \{c\}) \gcd(D_2 - \{c\}) = c$. There are 9 relative positions for the diagonals D_1 and D_2 as shown in Figure 4. We prove cases (a) and (e). The proofs of other seven cases are similar to the proof of case (e). The proof of case (a) follows from Theorem 3.2 part (3).

Proof of case (e). We choose $c = H(r, k) = F_kF_{r-k+1}$. So,

$$\begin{aligned} a_1 &= H(r-1, k) = F_kF_{r-k}, & a_2 &= H(r+1, k) = F_kF_{r-k+2}, \\ b_1 &= H(r+1, k+1) = F_{k+1}F_{r-k+1}, & b_2 &= H(r+2, k+2) = F_{k+2}F_{r-k+1}. \end{aligned}$$

Therefore,

$$\begin{aligned} \gcd(D_1 - \{c\}) \gcd(D_2 - \{c\}) &= \gcd(a_1, a_2) \gcd(b_1, b_2) \\ &= \gcd(F_kF_{r-k}, F_kF_{r-k+2}) \gcd(F_{k+1}F_{r-k+1}, F_{k+2}F_{r-k+1}) \\ &= F_k \gcd(F_{r-k}, F_{r-k+2}) F_{r-k+1} \gcd(F_{k+1}, F_{k+2}) \\ &= F_kF_{r-k+1} = c. \end{aligned}$$

This proves case (e).

We now prove that $\gcd(D_1 - \{c\}) \gcd(D_2 - \{c\}) = c$ when $|D_1| > 3$ or $|D_2| > 3$. Thus, there are D_1', D_2', A and B such that $D_1 = D_1' \cup A$ and $D_2 = D_2' \cup B$ where
(1) D_1' and D_2' are subdiagonals of Hosoya's triangle,
(2) $D_1' \cap D_2' = \{c\}$,
(3) $|D_1'| = |D_2'| = 3$.
It is easy to see that $\gcd(D_1 - \{c\}) = \gcd(D_1' - \{c\})$ and $\gcd(D_2 - \{c\}) = \gcd(D_2' - \{c\})$. Since $|D_1'| = 3$ and $|D_2'| = 3$, $\gcd(D_1' - \{c\}) \gcd(D_2' - \{c\}) = c$. Therefore, we have that $\gcd(D_1 - \{c\}) \gcd(D_2 - \{c\}) = c$. This proves the theorem. □

4. The GCD Property in a Polygon

Hosoya [5, 7] proved several interesting properties for his sequence using algebra and the geometry involved in Hosoya's triangle. Some of those properties are algebraic identities that show the relationship between Hosoya's sequence and the Fibonacci and Lucas sequences. In his work there are algebraic identities that show the relationship between the algebra of points and their location in Hosoya's triangle. Koshy [7, Chapter 15] has a complete chapter dedicated to the study of Hosoya's triangle and the arithmetic of points located in a rhomboid or in a triangle.

The main purpose of this section is the study of the GCD properties for a special configuration of points in any $n \times n$ rhombus of Hosoya's triangle. The last corollary of this section

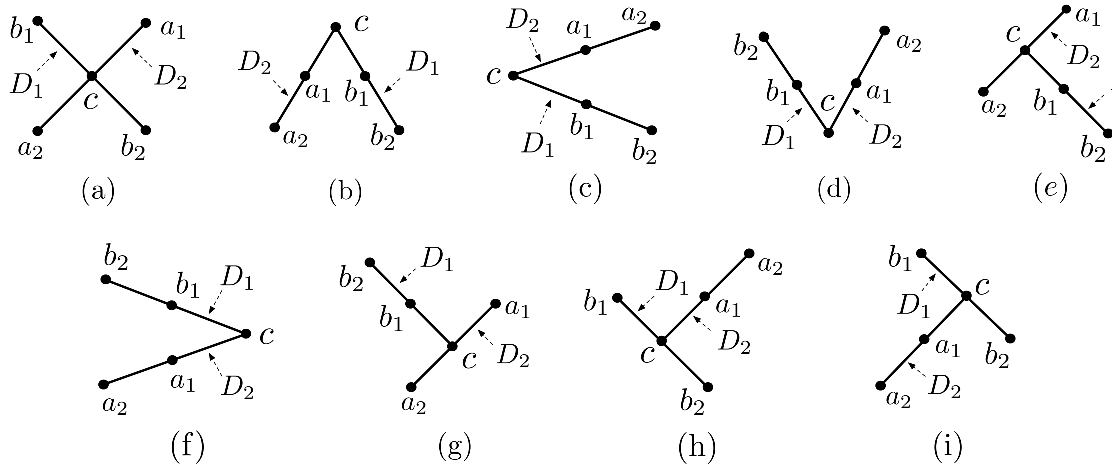


FIGURE 4. Diagonals in a polygon with $c = H(r,k) = F_k F_{r-k+1}$.

shows that the GCD of a particular configuration of n points in any polygon is always 1 or 2. Theorem 3.2 part (2) can be also proved using Theorem 4.3.

An $n \times n$ *rhombus* R in Hosoya's triangle is an arrangement of n^2 points forming a rhombus. Formally, we denote by $R_n(t,k)$ the $n \times n$ rhombus

$$R_n(t,k) = \bigcup_{i,j=1}^{n} \mathrm{B}(F_{t+i}) \cap \mathrm{S}(F_{k+j}) \text{ where } k,t \in \mathbb{N}.$$

Figure 5 depicts the rhombus $R_4(2,1)$.

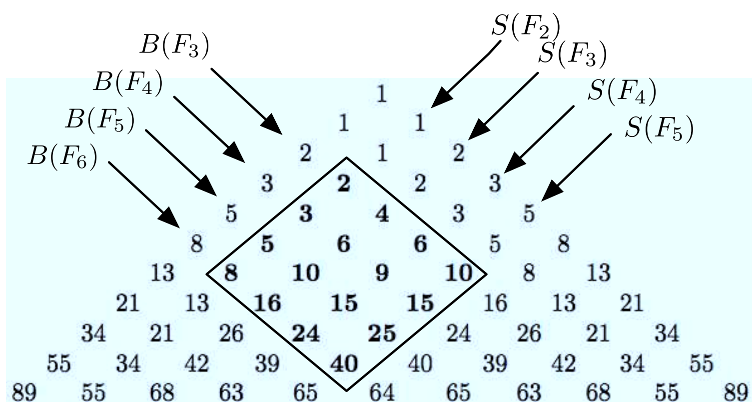


FIGURE 5. The rhombus $R_4(2,1)$.

We say that $A = \{a_1, \ldots, a_m\}$ is a collection of *non-attacking* points of Hosoya's triangle if no two distinct points in A may be in the same slash diagonal or backslash diagonal. This resembles the definition of non-attacking rooks on a board (see [9, pages 36-37 and 46-54] or [10]).

Let $a_1, \dots, a_n$ be n non-attacking points in a $n \times n$ rhombus of Hosoya's triangle. Theorems 4.3 and 4.5 show that $\gcd(a_1, \dots, a_n)$ is always 1 or 2, and give a complete characterization for when either case occurs. We give a combinatorial proof for the first theorem and an algebraic proof for the second one. The proof of Lemma 4.1 is straightforward and we omit it.

We recall from Section 2 that if $A \in \mathbb{Z}^k$, then the number $n_p(A)$ represents the cardinality of $\{i \in \{1, \dots, k\} \mid p \text{ divides } A(i)\}$. We also recall that $\mathrm{S}(F_m)$ and $\mathrm{B}(F_m)$ denote the slash and backslash diagonal, respectively.

Lemma 4.1. *Suppose that $x_1 < x_2 < \dots < x_n$ are integer numbers with $n \geq 3$. If p is a prime that divides at most one integer of $\{x_i, x_{i+1}, x_{i+2}\}$ where $1 \leq i \leq n-2$, then*

$$n_p(x_1, x_2, \dots, x_n) \leq \left\lceil \frac{n}{3} \right\rceil.$$

Lemma 4.2. *Suppose that $F_{l+1}, F_{l+2}, \dots, F_{l+n}$ are Fibonacci numbers where $n \geq 3$ and $l \geq 1$. If p is a prime, then*

$$n_p(F_{l+1}, F_{l+2}, \dots, F_{l+n}) \leq \left\lceil \frac{n}{3} \right\rceil.$$

Proof. We know that $\gcd(F_m, F_t) = 1$ for any two m and t with $|m - t| \leq 2$. This implies that p divides, at most, one Fibonacci number in the set $\{F_{l+i}, F_{l+i+1}, F_{l+i+2}\}$ for every i. Since $F_{l+1} < F_{l+2} < \cdots < F_{l+n}$, we can apply Lemma 4.1 to the sequence $F_{l+1}, F_{l+2}, \dots, F_{l+n}$. Thus, $n_p(F_{l+1}, F_{l+2}, \dots, F_{l+n}) \leq \lceil \frac{n}{3} \rceil$. □

Theorem 4.3. *Let $n \geq 3$ be an integer with $n \neq 4$. If $a_1, \dots, a_n$ are distinct non-attacking points in a $n \times n$ rhombus R of Hosoya's triangle, then $\gcd(a_1, \dots, a_n) = 1$.*

Proof. We prove this theorem by contradiction. We assume that $\gcd(a_1, \dots, a_n) > 1$. Thus, there is a prime number p such that

$$p \mid a_i \text{ for every } i \in \{1, \dots, n\}. \tag{4.1}$$

Since R is a $n \times n$ rhombus, there are integers t and k such that $R = R_n(t, k)$. Thus,

$$R_n(t,k) = \bigcup_{i,j=1}^{n} \mathrm{B}(F_{t+i}) \cap \mathrm{S}(F_{k+j}). \tag{4.2}$$

If we identify each point a_i with its corresponding ordered pair of Fibonacci numbers, then $a_i = F_{s_i} F_{r_i} \longleftrightarrow (F_{s_i}, F_{r_i})$ for $i \in \{1, \dots, n\}$. This and (4.1) imply that

$$\begin{array}{ccc} p \mid F_{s_1} & \text{or} & p \mid F_{r_1}, \\ p \mid F_{s_2} & \text{or} & p \mid F_{r_2}, \\ \vdots & & \vdots \\ p \mid F_{s_n} & \text{or} & p \mid F_{r_n}. \end{array} \tag{4.3}$$

Since $\{a_1, \dots, a_n\}$ is a collection of non-attacking points, equation (4.2) implies that

$$\{F_{s_1}, \dots, F_{s_n}\} = \{F_{t+1}, \dots, F_{t+n}\} \text{ and } \{F_{r_1}, \dots, F_{r_n}\} = \{F_{k+1}, \dots, F_{k+n}\}.$$

This and (4.3) imply that the $2n$–tuple $(F_{t+1}, \dots, F_{t+n}, F_{k+1}, \dots, F_{k+n})$ contains at least n entries divisible by p. Thus, it is clear that

$$\begin{aligned} n \leq n_p(F_{t+1}, \dots, F_{t+n}, F_{k+1}, \dots, F_{k+n}) &= n_p(F_{t+1}, \dots, F_{t+n}) + n_p(F_{k+1}, \dots, F_{k+n}) \\ &\leq \left\lceil \frac{n}{3} \right\rceil + \left\lceil \frac{n}{3} \right\rceil. \end{aligned}$$

Therefore, $\lceil \frac{n}{3} \rceil + \lceil \frac{n}{3} \rceil \geq n$. That is a contradiction, because $n \geq 3$ and $n \neq 4$. This proves the theorem. □

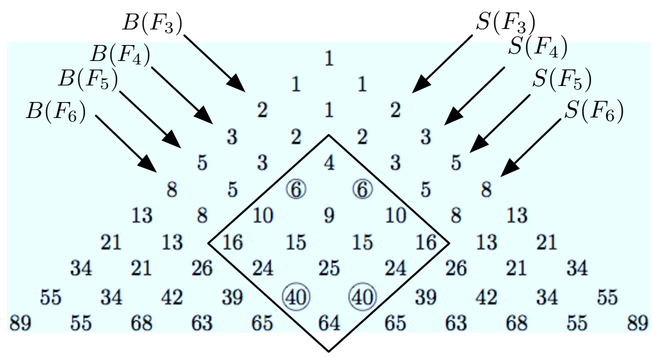


FIGURE 6. A 4×4 rhombus with 4 non-attacking points.

The conclusion of Theorem 4.3 is not always true when $n = 4$. For example, if $a_1 = 6$, $a_2 = 6$, $a_3 = 40$ and $a_4 = 40$ for the rhombus shown in Figure 6, then $\gcd(a_1, a_2, a_3, a_4) = 2$. This is not a coincidence; we prove that for any configuration of non-attacking points a_1, a_2, a_3 and a_4 in a 4×4 rhombus, $\gcd(a_1, a_2, a_3, a_4) = 1$ or 2.

There are 4! ways in which we can choose 4 non-attacking points in a 4×4 rhombus (see [9, 10]). We divide these 24 configurations into two types. A 4×4 rhombus with four non-attacking points is called a *configuration of type I* if at least one of the non-attacking points is in a corner, it is *a configuration of type II* if there are no non-attacking points in any corner of the rhombus. That is, a configuration is of type II if it is not of type I.

It is easy to see that each configuration of type I contains a configuration of three non-attacking points in a 3×3 rhombus. This can be done by "ignoring" one of the non-attacking points on the corner of the 4×4 rhombus (see Figure 7). Note that there are exactly 4 configurations of type II. These configurations are depicted in Figure 8.

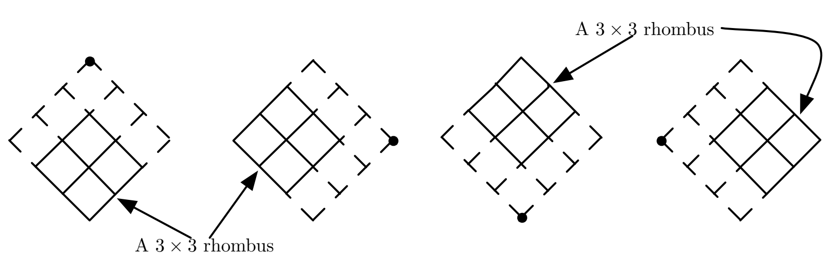


FIGURE 7. Some configurations of type I.

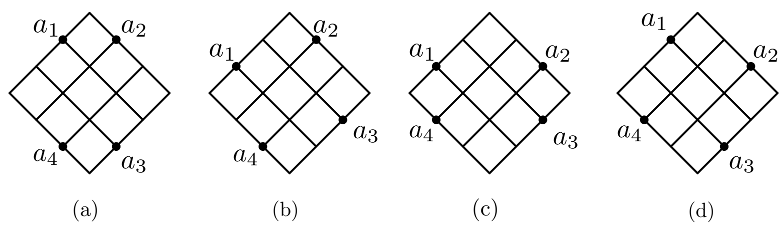


FIGURE 8. All configurations of type II.

Lemma 4.4. *Let R be a configuration of type II with a_1, a_2, a_3 and a_4 its non-attacking points. If there are t and k in $\mathbb{N}$ such that $R = R_4(t,k)$, then* $\gcd(a_1, a_2, a_3, a_4) = F_{\gcd(k+1,k+4,t+1,t+4)}$.

Proof. We prove that $\gcd(a_1, a_2, a_3, a_4) = F_{\gcd(k+1,k+4,t+1,t+4)}$ for the configuration in Figure 8 part (a); the proofs for the other three configurations are similar. From the positions of a_1, a_2, a_3, and a_4 in $R = R_4(t,k)$ shown in the Figure 8 part (a), we can see that $a_1 = F_{k+1}F_{t+2}$, $a_2 = F_{k+2}F_{t+1}$, $a_3 = F_{k+4}F_{t+3}$ and $a_4 = F_{k+3}F_{t+4}$. Therefore,

$$\begin{aligned}\gcd(a_1, a_2, a_3, a_4) &= \gcd(F_{k+1}F_{t+2}, F_{k+2}F_{t+1}, F_{k+4}F_{t+3}, F_{k+3}F_{t+4})\\ &= \gcd(\gcd(F_{k+1}F_{t+2}, F_{k+2}F_{t+1}), \gcd(F_{k+4}F_{t+3}, F_{k+3}F_{t+4})). \qquad \text{(a)}\end{aligned}$$

Lemma 3.1 part (2) implies that (a) is equal to

$$\gcd\left(F_{gcd(k+1,t+1)}F_{gcd(t+2,k+2)}, F_{gcd(k+4,t+4)}F_{gcd(t+3,k+3)}\right). \qquad \text{(b)}$$

Since $\gcd(k+1, k+3) = 1$ or 2 and $\gcd(t+1, t+3) = 1$ or 2, $\gcd(k+1, k+3, t+1, t+3) = 1$ or 2. This and Lemma 2.1 imply that

$$\gcd\left(F_{\gcd(k+1,t+1)}, F_{\gcd(t+3,k+3)}\right) = F_{\gcd(k+1,t+1,t+3,k+3)} = F_1 \text{ or } F_2.$$

Similarly, we get that $\gcd(k+2, k+4, t+2, t+4) = 1$ or 2. This and Lemma 2.1 imply that

$$\gcd\left(F_{\gcd(t+2,k+2)}, F_{\gcd(k+4,t+4)}\right) = F_{\gcd(t+2,k+2,t+4,k+4)} = F_1 \text{ or } F_2.$$

These and Proposition 2.2 part (2) shows that (b) is equal to

$$\gcd\left(F_{gcd(k+1,t+1)}, F_{gcd(k+4,t+4)}\right)\gcd\left(F_{gcd(t+3,k+3)}, F_{gcd(t+2,k+2)}\right).$$

This, (a), (b) and $\gcd\left(F_{gcd(t+3,k+3)}, F_{gcd(t+2,k+2)}\right) = 1$ prove the lemma. □

Theorem 4.5. *Let a_1, a_2, a_3 and a_4 be non-attacking points in a 4×4 rhombus R. If there are t and k in $\mathbb{N}$ such that $R = R_4(t,k)$, then*

(1) if R is of type I, then $\gcd(a_1, a_2, a_3, a_4) = 1$.
(2) if R is of type II and $\gcd(k+1, t+1, 3) = 1$, *then* $\gcd(a_1, a_2, a_3, a_4) = 1$.
(3) R is of type II and $\gcd(k+1, t+1, 3) = 3$ *if and only if* $\gcd(a_1, a_2, a_3, a_4) = 2$.

Proof. We prove part (1). We assume that R is of type I. We suppose that a_4 is in a corner of R. Thus, a_1, a_2 and a_3 form a configuration of 3 non-attacking points in a 3×3 rhombus. This and Theorem 4.3 imply that $\gcd(a_1, a_2, a_3) = 1$. Therefore, $\gcd(a_1, a_2, a_3, a_4) = 1$. This proves part (1).

We now prove part (2). We assume that R is of type II and $\gcd(k+1, t+1, 3) = 1$. If $d = \gcd(k+1, k+4, t+1, t+4)$, then $d \mid 3$. This and $\gcd(k+1, t+1, 3) = 1$ imply that $d = 1$.

Therefore, Lemma 4.4 implies that $\gcd(a_1, a_2, a_3, a_4) = F_{\gcd(k+1,k+4,t+1,t+4)} = F_1 = 1$. This proves part (2).

We prove part (3). For the sufficient condition we assume that R is of type II and that $\gcd(k+1, t+1, 3) = 3$. If $d = \gcd(k+1, k+4, t+1, t+4)$, then $d \mid 3$. This and $\gcd(k+1, t+1, 3) = 3$ imply that $3 \mid d$. So, $d = 3$. Therefore, Lemma 4.4 implies that

$$\gcd(a_1, a_2, a_3, a_4) = F_{\gcd(k+1,k+4,t+1,t+4)} = F_3 = 2.$$

This proves the sufficient condition.

Conversely, if $\gcd(a_1, a_2, a_3, a_4) = 2$, then parts (1) and (2) imply that R is not a configuration of type I and $\gcd(k + 1, t + 1, 3) \neq 1$. Thus, R is of type II and $\gcd(k + 1, t + 1, 3) = 3$. This proves part (3). □

We collect the results of Theorems 4.3 and 4.5 in the following corollary. So, the proof is a direct application of these two theorems.

Corollary 4.6. *Let P be a polygon in Hosoya's triangle and let $A = \{a_1, \ldots, a_n\}$ be a collection of non-attacking points in P with $n \geq 3$. If there are t and k in $\mathbb{N}$ such that $A \subset R_n(t, k)$, then*

$$\gcd(a_1, \ldots, a_n) = \begin{cases} 2, & \text{if } n = 4,\ R \text{ is of type II and } \gcd(k+1, t+1, 3) = 3; \\ 1, & \text{otherwise.} \end{cases}$$

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