REPRESENTING POSITIVE INTEGERS AS A SUM OF LINEAR RECURRENCE SEQUENCES

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Abstract. The Zeckendorf representation, using sums of Fibonacci numbers, is widely known. Fraenkel generalized to recurrence sequences $u_n = a_1 u_{n-1} + \cdots + a_h u_{n-h}$ provided $a_1 \geq a_2 \geq \cdots \geq a_h > 0$. We remove this restriction, but do assume $a_i \geq 0$, and show that a unique representation of every positive integer is possible with digit strings composed of certain blocks which are lexicographically less than $a_1 a_2 \cdots a_h$.

1. Introduction

A well-known property of the Fibonacci numbers is the Zeckendorf representation. Every positive integer N is uniquely represented as a sum of Fibonacci numbers with the condition that no two consecutive Fibonacci numbers are used [8]. Note that using the combinatorial Fibonacci numbers $\{1, 2, 3, 5, \dots\}$ saves complications which would arise using $\{0, 1, 1, 2, 3, 5, \dots\}$.

Generalizations to other second order recurrence sequences and to some higher order recurrences have been examined [1, 2, 3, 4, 5, 6, 7]. Even the common base 10 and base 2 representations are examples using first order recurrences $u_n = 10u_{n-1}$ and $u_n = 2u_{n-1}$.

Indeed, if $\{u_n\}$ is pretty much any sequence of positive integers we could study representations of the form

$$N = \sum_{i=0}^{m} d_i u_i \tag{1.1}$$

where the d_i are "digits" of the representation. We will assume that $u_0 = 1$, the u_i are strictly increasing, and the $d_i \geq 0$. If so, every positive N has a unique representation by simply using the greedy algorithm.

Informally, this means simply subtracting the largest possible u_i at each step. Formally, if $u_m \leq N < u_{m+1}$, then $d_i = 0$ for $i > m$. For $j \leq m$ let $N_j = N - \sum_{i>j} d_i u_i$. Then $d_j = \lfloor N_j / u_j \rfloor$ where $\lfloor \cdot \rfloor$ denotes the floor function.

The property which sets the representations using recurrence sequences apart is that the resulting strings of digits $d_m \cdots d_0$ are easily described. For other sequences, the digits do not appear to satisfy any easy to describe rule. The two properties we would like in representations are uniqueness and a simple rule to describe the digits. For base 10, the rule is $0 \leq d_i < 10$; for Zeckendorf it is $0 \leq d_i \leq 1$ and $d_i d_{i-1} \neq 11$.

In [2], Fraenkel looks at recurrences of any order:

$$u_n = a_1 u_{n-1} + a_2 u_{n-2} + \cdots + a_h u_{n-h}. \tag{1.2}$$

He requires that the a_i be decreasing, $a_1 \geq a_2 \geq \cdots \geq a_h$.

More recently, the authors [7] and Kologler, Kopp, Miller, and Wang [10, 13] have noted that the decreasing condition on the a_i is not necessary.

Specifically, let $\mathcal{S} = a_1a_2 \dots a_h$ be called the signature of recurrence (1.2). Also, define an associated set of strings $\boldsymbol{S} = \{S_0, S_1, \dots, S_{A-1}\}$ where $A = a_1 + a_2 + \cdots + a_h$, $S_0 = 0$, and for $i > 0$ the S_i consist of all strings of length l, $1 \le l \le h$ such that $a_1 \cdots a_{l-1} \le S_i < a_1 \cdots a_l$ in lexicographic order. Thus, the signature of the recurrence $u_n = 2u_{n-1} + 4u_{n-2} + u_{n-4}$ is $S = 2401$ and the associated strings are $\boldsymbol{S} = \{0, 1, 20, 21, 22, 23, 2400\}$. Similarly the strings associated with 10052 are $\{0, 1000, 1001, 1002, 1003, 1005, 10050, 10051\}$.

Theorem 1.1. *If $\{u_n\}$ satisfies the recurrence $u_n = a_1u_{n-1}+a_2u_{n-2}+\cdots+a_hu_{n-h}$ with $a_i \ge 0$ and $a_1 \ne 0$, then every positive integer N can be uniquely expressed as $\sum d_iu_i$ where the string of digits $d_md_{m-1}\cdots d_0$ is composed of blocks of digits in the set $\mathcal{S}$ described previously.*

A recurrence of the type in Theorem 1.1 will be called a positive recurrence.

We will give two short proofs of this theorem, one of which shows how the desired representation is produced by a greedy algorithm.

Next, we will show how this type of representation can exist for recurrences where a_1 need not be positive and where some of the a_i may be negative.

If we allow arbitrary initial values, special rules are needed for d_0. The best choice is to use the sequence $\{u_n\}$ which counts the number of ways to tile a $1 \times n$ rectangle using a_i different kinds of tiles of length i. This choice generally produces nice combinatorial properties and identities.

Example 1.2. *Let $u_n = u_{n-1} + 3u_{n-2}$, so including $u_0 = 1, \{u_n\}_{n\ge0} = \{1, 1, 4, 7, 19, \dots\}$. Here $\mathcal{S} = 13$ and $\mathbf{S} = \{0, 10, 11, 12\}$. The desired digital representation for integers $N \le 19$ are:*

$u_4 = 19$	$u_3 = 7$	$u_2 = 4$	$u_1 = 1$	$u_0 = 1$	N
			1	0	1
			1	1	2
			1	2	3
		1	0	0	4
		1	1	0	5
		1	2	0	6
	1	0	0	0	7
	1	0	1	0	8
	1	0	1	1	9
	1	0	1	2	10
	1	1	0	0	11
	1	1	1	0	12
	1	1	1	1	13
	1	1	1	2	14
	1	2	0	0	15
	1	2	1	0	16
	1	2	1	1	17
	1	2	1	2	18
1	0	0	0	0	19

Since the strings of digits in $\mathbf{S}$ are the strings of digits used in expressing N, another way to think of this is to define a new sequence $\{v_{i,n}\}$ consisting of blocks of linear combinations of the u_n with coefficients corresponding to the strings in $\mathbf{S}$.

If $S_i \in \mathbf{S}$, $S_i = a_1a_2\dots a_{k-1}m$ where $0 \le m < a_k$, define $v_{i,n} = a_1u_n + a_2u_{n-1} + \cdots + a_{k-1}\ u_{n-k+2} + mu_{n-k+1}$. Then $v_{1,n} < v_{2,n} < \cdots < v_{A-1,n} < v_{1,n+1}$. Since u_{-1} does not correspond to a digit, any $v_{i,n}$ which would involve u_{-1}, is not defined. Let V_n be the sequence of all the $v_{i,n}$ arranged in increasing order. The representation of N can be computed by expressing N as a sum of the V_n using the greedy algorithm, and then replacing the V_n by the corresponding u_n. What do the resulting strings of digits look like?

Each V_n is a combination of u_j with coefficient strings from $\mathbf{S}$. Moreover, if the greedy algorithm uses the term $v_{i,n}$ when expressing some N_j then $v_{1,n} \le N_j < v_{i+1,n}$ (where $v_{A,n} = v_{1,n+1}$).

If, in this case, $v_{i,n} = a_1u_n + \cdots + mu_{n-k+1}$, then $N_j - v_{i,n} < u_{n-k+1} = v_{1,n-k+1}$. Hence, no u_i is present in any two V_n terms when applying the greedy algorithm. Also, $N_j - v_{i,n}$ can be any number less than $v_{1,n-k+1}$ so any preceding term in the sequence is a possible next term in the representation.

Therefore, the string of digits $d_md_{m-1}\cdots d_0$ consists of blocks of digits from $\mathbf{S}$.

Example 1.2 (*continued*) *Since* $S = \{0, 10, 11, 12\}$ *and* u_{-1} *is not defined,* $v_{i,0}$ *is not defined. Then* $v_{1,1} = u_1 + 0u_0$*,* $v_{2,1} = u_1 + u_0$*,* $v_{3,1} = u_1 + 2u_0$ *and* $\{V_n\} = \{v_{i,n}\}_{n\ge1} = \{1, 2, 3; 4, 5, 6; 7, 11, 15; 19, 26, 33; \dots\}$.

Example 1.3. *Let* $u_n = u_{n-1} + 3u_{n-3} + 2u_{n-4}$ *so* $\{u_n\}_{n\ge0} = \{1, 1, 1, 4, 9, 14, 28, 63, \dots\}$*. Also* $\mathcal{S} = 1032$ *and so* $\mathbf{S} = \{0, 100, 101, 102, 1030, 1031\}$*. Note that* $\mathbf{S}$ *contains no two digit string since no such* S_i *would satisfy* $1 \le S_i < 10$*. In defining the* $v_{i,n}$*, the terms* $v_{i,0}$ *and* $v_{i,1}$ *are undefined,*

$v_{1,2} = u_2 + 0u_1 + 0u_0 = u_2$, $\quad v_{2,2} = u_2 + 0u_1 + u_0$, $\quad v_{3,2} = u_2 + 0u_1 + 2u_0$*, but* $v_{4,2}$ *and* $v_{5,2}$ *are undefined.*

Therefore, $\{V_n\} = \{1, 2, 3;\ 4, 5, 6, 7, 8;\ 9, 10, 11, 12, 13;\ 14, 18, 22, 26, 27;\ 28, \dots\}$.

Thus, for example, $3 = V_3 = v_{3,2} = u_2 + 0u_1 + 2u_0 = 1 + 0 + 2$, would have digits 102. Also, $24 = 22 + 2 = V_{16} + V_2 = v_{3,5} + v_{2,2} = (u_5 + 2u_3) + (u_2 + u_0) = 14 + 2 \cdot 4 + 1 + 1$ would have digits 102101, composed of the strings 102 and 101.

The same procedure can be used for any N and any sequence $\{u_n\}$ which satisfies the conditions described above, namely all $a_i \ge 0$ and $a_1 > 0$.

By the definition of $\mathbf{S}$, the blocks of digits are a prefix-free code, i.e. no block is an initial string of any other block. Hence, any string composed of such blocks is easily parsed into the appropriate blocks. In Example 2 above, the string 101|0|0|1031|102|100 parses correctly into blocks and is an allowable string. However, 101|0|0|11001 is not allowed since in parsing the string we encounter a block beginning with 11. No such block is in $\mathbf{S}$. Thus we have proved Theorem 1.

If we allow alternate initial values, the description of digits needs to be adjusted allowing for large values of $u_0 = 1$. This is not necessary if u_n counts the number of ways to tile a $1 \times n$ rectangle using a_i types of tiles of length i.

There is another connection with this interpretation of u_n. Create blocks of digits $\mathcal{B}_i$ as follows. Let $\mathcal{B}_0 = 0$, and for $i > 0$, $\mathcal{B}_i$ is the digit i followed by zeros, with a total of a_i blocks of length i. Define $B(i)$ to be the length of block $\mathcal{B}_i$. Thus, $B(0) = 1, B(i) = 1$ for $0 \le i < a_1$, $B(i) = 2$ for $a_1 \le i \le a_1 + a_2$, etc. Clearly, u_n counts the number of n digit strings composed of the blocks $\mathcal{B}_i$, since this is equivalent to a standard tiling problem.

Arrange these strings in lexicographic, or equivalently numerical order. (Not including the string of all zeros.) Let $w_j(k)$ = number in this ordering of the string consisting of j followed by k zeros. Since the first string of the form $j0\cdots 0$ has $B(j)$ digits, or $B(j)-1$ zeros, $w_j(k)$ is defined for $k \geq B(j)-1$.

Example 1.2 (*continued*)

$$\{\mathcal{B}_i\} = \{0, 10, 20, 30\}$$
$$B(0) = 1, B(1) = B(2) = B(3) = 2.$$

1.	10	8.	1010	15.	3000	
2.	20	9.	1020	16.	3010	
3.	30	10.	1030	17.	3020	
4.	100	11.	2000	18.	3030	$\cdots$
5.	200	12.	2010	19.	10000	
6.	300	13.	2020	20.	10010	
7.	1000	14.	2030	21.	10020	

Thus, $w_1(1) = 1 = u_1$, $w_2(1) = 2 = v_{2,1}$, $w_3(1) = 3 = v_{3,1}$, $w_1(2) = 4 = u_{1,2} \ldots$, $w_1(3) = 7 = u_{1,3}$. In general, $w_1(n) = u_n = v_{1,n}$, $w_2(n) = u_n + u_{n-1} = v_{2,n}$, $w_3(n) = u_n + 2u_{n-1} = v_{3,n}$.

Example 1.3 (*continued*) *As above, with* $\mathcal{S} = 1032$,

$$\{\mathcal{B}_i\} = \{0, 100, 200, 300, 4000, 5000\}$$
$$B(0) = 1, B(1) = 3, B(2) = 3, B(3) = 3, B(4) = 4, B(5) = 4.$$

Listing the strings in order:

1.	100	9.	10000	17.	100300	
2.	200	10.	20000	18.	200000	
3.	300	11.	30000	19.	200100	
4.	1000	12.	40000	20.	200200	
5.	2000	13.	50000	21.	200300	$\cdots$
6.	3000	14.	100000	22.	300000	
7.	4000	15.	100100	23.	300100	
8.	5000	16.	100200	24.	300200	

$w_1(2) = 1 = u_2 = v_{1,2}$; $w_2(2) = 2 = u_2 + u_0 = v_{2,2}$; $w_3(2) = 3 = u_2 + 2u_0 = v_{3,2}$ but $w_4(n)$ and $w_5(n)$ are defined for $n \geq 3$.

For any sequence, $w_1(n)$ is preceded by all strings of length $\leq n$ so $w_1(n) = u_n$. For $j \geq 2$, the strings preceding $w_j(n)$ are those preceding $w_{j-1}(n)$ plus the block B_{j-1} of length $B(j-1)$ followed by any string of length $n - B(j-1) + 1$. Thus,

$$w_j(n) = w_{j-1}(n) + u_{n-B(j-1)+1}. \tag{1.3}$$

Iterating equation (1.3):

$$w_j(k) = \sum_{i=0}^{j-1} u_{k-B(i)}. \tag{1.4}$$

Recall from the definition of $B(i)$ that $B(i) = j$ for a_j values of i. So collecting like terms in equation (1.4), the sums we obtain have the form:

$$w_j(k) = a_1 u_k + a_2 u_{k-1} + \cdots + a_{r-1} u_{k-r+2} + m u_{k-r+1} \tag{1.5}$$

where $0 \leq m < a_r$. Hence, the sequence of numbers $w_j(k)$ is the same as the $v_{j,k}$.

This procedure offers a way to order the strings composed of the blocks $\mathcal{B}_i$ (or equivalently the associated tiling) and to associate each $N > 0$ with such a string. By associating N with its string we can relate to expressing N as a sum of $w_j(k)$, using the greedy algorithm.

If $w_j(k) \le N < w_{j+1}(k)$ then the string associated with N begins with the block $\mathcal{B}_j$ and has length $k+1$. The rest of the string is the one associated with the remainder $N - w_j(k)$ which has length at most $k+1-B(j)$ since $\mathcal{B}_j$ has length $B(j)$. When $w_j(k)$ is expressed as a sum of u_n, by equation (1.4), the terms $u_k, u_{k-1}, \ldots, u_{k-B(j-1)+1}$ are present.

In the remainder $N - w_j(k)$, the largest u_n appearing is $u_{k-B(j)}$, so there is no overlap in the u_n terms. So the representation of N consists of disjoint blocks of digits as described by equation (1.5), which is the same as Theorem 1.1.

2. Nonpositive Recurrences

An open question is whether some similar representation, with digit strings satisfying a simple pattern, is possible if some of the coefficients a_j are negative.

If a sequence $\{u_n\}$ satisfies the recurrence (1.2) the associated characteristic polynomial is $f(x) = x^h - a_1\, x^{h-1} - \cdots - a_h$. If $g(x)$ is any polynomial, then u_n also satisfies the recurrence whose characteristic polynomial is $f(x)g(x)$. This is equivalent to using the original recurrence to replace some number of occurrences of various u_j. The recurrence of minimal order and corresponding characteristic polynomial is the minimal recurrence and polynomial.

For the Fibonacci numbers, the minimal polynomial is $x^2 - x - 1$. Since $(x^2 - x - 1)(x^2 - 2) = x^4 - x^3 - 3x^2 + 2x + 2$, F_n also satisfies $F_{n+4} - F_{n+3} - 3F_{n+2} + 2F_{n+1} + 2F_n = 0$. However, the corresponding recurrence is not positive and so Theorem 1 is not applicable. But F_n also satisfies $F_{n+6} = F_{n+5} + 3F_{n+1} + F_n$ and Theorem 1 is applicable. Thus, we also have unique representations where the string of digits is composed of the blocks $\{0,\ 10000,\ 10001,\ 10002,\ 100030,\ 100031\}$. Thus, the Zeckendorf representation is just one of many ways to express integers uniquely as sums of Fibonacci numbers such that the resulting digital strings are easily described as all concatenations of a set of allowable blocks.

Although Theorem 1.1 is not applicable to nonpositive recurrences, the remarks above suggest that if the minimal recurrence of u_n is nonpositive, it may also satisfy a higher order positive recurrence. We will call such a recurrence sequence convertible.

In such a case, Theorem 1 would yield a well behaved, unique representation for all positive integers. Is this always possible? In general there will be many possible choices for a higher order positive recurrence. As always, special rules may be needed for the coefficients of u_n for small values of n, depending on the initial values chosen.

One type of nonpositive recurrence is when $a_1 = 0$.

Example 2.1. $u_0 = 1,\ u_1 = 2,\ u_2 = 3$ *and* $u_n = u_{n-2} + u_{n-3}$ *for* $n \ge 4$.

$$\{u_n\} = \{1, 2, 3, 3, 5, 6, 8, 11, 14, 19, 25, 33, 44, \ldots\}.$$

Since $u_0 = 1$, no choice of initial values for this recurrence yields a strictly increasing sequence. We could in this instance require that u_3 is never used.

Since $u_n - u_{n-2} - u_{n-3} = 0$, we can add or subtract any multiple of this for any value of n. An easy way to express the process is in a table.

	u_{n-1}	u_{n-2}	u_{n-3}	u_{n-4}	
	0	1	1		
$u_n =$	1	0	-1	-1	
		-1	0	1	1
	1	0	0	0	1

Thus, $u_n = u_{n-1} + u_{n-5}$. Hence, $\mathcal{S} = 10001$ $\boldsymbol{S} = \{0, 10000\}$.

Every N is then uniquely representable in the form (1.1) with $d_i = 0$ or 1 and all 1's separated by at least four 0's.

The order of a converted recurrence may be very large. Here is an example where the order is only a little larger.

Example 2.2. $u_n = 2u_{n-2} + u_{n-3} + u_{n-4}$

	u_{n-1}	u_{n-2}	u_{n-3}	u_{n-4}	u_{n-5}	u_{n-6}	u_{n-7}	u_{n-8}	u_{n-9}	u_{n-10}
	0	2	1	1						
	1	0	-2	-1	-1					
		-2	0	4	2	2				
$u_n =$			1	0	-2	-1	-1			
				-4	0	8	4	4		
					1	0	-2	-1	-1	
						-1	0	2	1	1
	1	0	0	0	0	8	1	5	0	1

Thus, u_n also satisfies the positive order recurrence $u_n = u_{n-1} + 8u_{n-6} + u_{n-7} + 5u_{n-8} + u_{n-10}$. It may also be the case that some of the a_i are negative.

Example 2.3. $u_n = 3u_{n-1} - 2u_{n-2} + 2u_{n-3}$

	u_{n-1}	u_{n-2}	u_{n-3}	u_{n-4}	
	3	-2	2		
$u_n =$	-2	6	-4	4	
		-2	6	-4	4
	1	2	4	0	4

Thus, u_n is convertible to $u_n = u_{n-1} + 2u_{n-2} + 4u_{n-3} + 4u_{n-5}$. Again we emphasize the converted recurrence is not unique.

The following questions remain to be answered.

(1) Are there conditions on the a_i which are necessary or sufficient to guarantee the existence of a converted positive recurrence? Simple conditions for arbitrary recurrences

seem unlikely. Conditions for low order recurrence such as second and third order will be addressed in a future manuscript.

(2) If u_n is not convertible to a positive recurrence can positive integers N be uniquely represented as in (1.1) where the strings of digits still satisfy a simple description similar to Theorem 1.1?

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