ESTIMATING THE APÉRY NUMBERS

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Abstract. We use a crude method to find the dominant term in the asymptotic expansion of the nth Apéry number.

1. Introduction

In 1983, Apéry stunned the mathematical world by proving that $\zeta(3)$ is irrational. Alf van der Poorten has given an entertaining account of Apéry's presentation [1].

Apéry's proof involves the eponymous numbers,

$$A_n = \sum_{k=0}^{n} \binom{n}{k}^2 \binom{n+k}{k}^2,$$

that satisfy the recurrence

$$(n+1)^3 A_{n+1} = (34n^3 + 51n^2 + 27n + 5)A_n - n^3 A_{n-1}$$

together with $A_0 = 1$, $A_1 = 5$.

The purpose of this note is to demonstrate how one can use a crude yet effective method to find the dominant term in the asymptotic expansion of A_n. Indeed, we will find that

$$A_n \sim \frac{1}{2^{\frac{9}{4}}(\pi n)^{\frac{3}{2}}}(\sqrt{2}+1)^{4n+2} \quad \text{as } n \to \infty.$$

2. The Calculation

Let

$$u_k = \binom{n}{k}^2 \binom{n+k}{k}^2.$$

The maximum u_k is found by solving the equation $u_k = u_{k+1}$. This yields

$$\frac{(n-k)^2(n+k+1)^2}{(k+1)^2} = 1,$$

$$(n-k)(n+k+1) = (k+1)^2,$$

or roughly speaking,

$$n^2 - k^2 = k^2,$$

$$k = \frac{n}{\sqrt{2}}.$$

Let N be the integer closest to $\frac{n}{\sqrt{2}}$. The maximum term, roughly speaking, is

$$\begin{aligned}
u_N &= \binom{n}{N}^2 \binom{n+N}{N}^2 \\
&= \frac{(n+N)!^2}{N!^4(n-N)!^2} \\
&\approx \frac{2\pi n\left(1+\frac{1}{\sqrt{2}}\right)\left(\frac{n(\sqrt{2}+1)}{\sqrt{2}e}\right)^{n(2+\sqrt{2})}}{2\pi^2 n^2\left(\frac{n}{\sqrt{2}e}\right)^{2n\sqrt{2}} \cdot 2\pi n\left(1-\frac{1}{\sqrt{2}}\right)\left(\frac{n(\sqrt{2}-1)}{\sqrt{2}e}\right)^{n(2-\sqrt{2})}} \\
&= \frac{1}{2\pi^2 n^2}\frac{\sqrt{2}+1}{\sqrt{2}-1} \cdot \frac{(\sqrt{2}+1)^{(2+\sqrt{2})n}}{(\sqrt{2}+1)^{-(2-\sqrt{2})n}} \\
&= \frac{1}{2\pi^2 n^2}(\sqrt{2}+1)^{4n+2} \\
&= H.
\end{aligned}$$

Nearby (think of k as up to $n^{\frac{3}{4}}$), we have, again making small approximations,

$$\begin{aligned}
u_{N+k} &= \frac{(n+N+k)!^2}{(N+k)!^4(n-N-k)!^2} \\
&= \frac{(n+N)!^2}{N!^4(n-N)!^2} \cdot \frac{(n-N)^2 \cdots (n-N-k+1)^2(n+N+1)^2 \cdots (n+N+k)^2}{(N+1)^4 \cdots (N+k)^4} \\
&= H \cdot \frac{(n-N)^{2k}(n+N)^{2k}}{N^{4k}} \\
&\quad \cdot \frac{\left(1-\frac{1}{n-N}\right)^2 \cdots \left(1-\frac{k-1}{n-N}\right)^2\left(1+\frac{1}{n+N}\right)^2 \cdots \left(1+\frac{k}{n+N}\right)^2}{\left(1+\frac{1}{N}\right)^4 \cdots \left(1+\frac{k}{N}\right)^4} \\
&\approx H\left(\frac{n^2-N^2}{N^2}\right)^{2k} \exp\left\{-2\left(\frac{1}{n-N}+\frac{2}{n-N}+\cdots+\frac{k-1}{n-N}\right)\right. \\
&\qquad + 2\left(\frac{1}{n+N}+\frac{2}{n+N}+\cdots+\frac{k}{n+N}\right) \\
&\qquad \left. - 4\left(\frac{1}{N}+\frac{2}{N}+\cdots+\frac{k}{N}\right)\right\} \\
&\approx H\exp\left\{-k^2\left(\frac{1}{n-N}-\frac{1}{n+N}\right)-2k^2 \cdot \frac{1}{N}\right\} \\
&= H\exp\left\{-k^2\left(\frac{2N}{n^2-N^2}+\frac{2}{N}\right)\right\} \\
&\approx H\exp\left\{-k^2\left(\frac{n\sqrt{2}}{\frac{1}{2}n^2}+\frac{2\sqrt{2}}{n}\right)\right\} \\
&= H\exp\left\{-\frac{4\sqrt{2}}{n}k^2\right\}.
\end{aligned}$$

This calculation was carried out for k positive, but a similar calculation works for k negative.

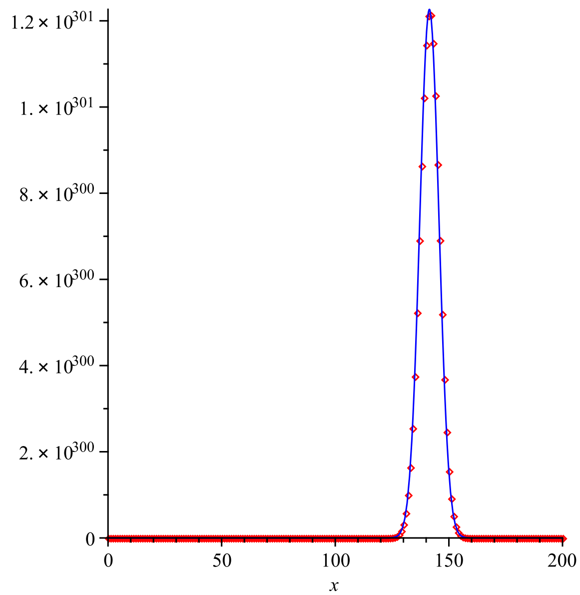

FIGURE 1. $n = 200$ - Values of terms in the sum for the 200th Apéry number and approximating function.

It follows that

$$A_n = \sum u_k \approx H \int_{-\infty}^{\infty} \exp\left\{ -\frac{4\sqrt{2}}{n} k^2 \right\} dk = H \cdot \sigma\sqrt{2\pi},$$

where

$$2\sigma^2 = \frac{n}{4\sqrt{2}}.$$

Finally,

$$\sigma = \frac{\sqrt{n}}{2^{\frac{7}{4}}}, \quad \sigma\sqrt{2\pi} = \frac{\sqrt{\pi n}}{2^{\frac{5}{4}}},$$

and

$$A_n \sim \frac{1}{2^{\frac{9}{4}}(\pi n)^{\frac{3}{2}}} (\sqrt{2}+1)^{4n+2}.$$

References

[1] A. van der Poorten, *A proof the Euler missed ...*, The Mathematical Intelligencer, **1.4** (1979), 195–203.

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