THE NUMBER OF SEQUENCES OF n TOSSES OF A COIN WITH k PAIRS OF CONSECUTIVE HEADS

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Abstract. We solve the problem of finding the number of sequences of n tosses of a coin with k pairs of consecutive heads.

1. Introduction

The problem of finding the number, $U(n,k)$, of sequences of n tosses of a coin with k pairs of consecutive heads has been tackled at least twice recently [1, 2]. In [1], a recurrence was obtained, while in [2], the exponential generating function was used to find $U(n,k)$ for $k = 1, 2, 3$, and an indication given as to how to proceed to find further formulas. In this note, I follow the prescription I gave in [4] in my comments on [1] (completely overlooked by the author of [2]), for finding $U(n,k)$ from the ordinary generating function.

I found [4, p. 152], that

$$\sum_{n \geq k+1} U(n,k)x^n = \frac{x^{k+1}(1-x)^{k-1}}{(1-x-x^2)^{k+1}}$$

and then [4, p. 153], said

"It can be shown via partial fractions that

$$U(n,1) = \frac{(n-1)L_n + 2F_{n-1}}{5}$$

where the $\{L_n\}$ are the Lucas numbers, $L_n = F_{n+1} + F_{n-1}$.

Also,

$$U(n,2) = \frac{1}{5}\binom{n-3}{2}F_{n-2} + \frac{3}{25}\binom{n-4}{1}L_{n-3} + \frac{6}{25}F_{n-4} + \frac{1}{5}\binom{n-2}{1}L_{n-1} + \frac{2}{5}F_{n-2}$$

and there is a similar formula for $U(n,k)$ for each value of k.

In order to find these formulas, write $1 - x = x^2 + (1 - x - x^2)$ in the generating function, expand by the binomial theorem, write $1 - x - x^2 = -(x+\alpha)(x+\beta)$, and use the formula

$$\frac{1}{u^n v^n} = (-1)^n \sum_{k=0}^{n-1} \frac{\binom{n-1+k}{k}}{c^{n+k}u^{n-k}} + \sum_{k=0}^{n-1} (-1)^k \frac{\binom{n-1+k}{k}}{c^{n+k}v^{n-k}}$$

with $u = x + \alpha$, $v = x + \beta$, $c = u - v = \alpha - \beta = \sqrt{5}$. (See Hirschhorn [3] for a proof of this formula.) Then write $\frac{1}{x+\alpha} = \frac{-\beta}{1-\beta x}$, $\frac{1}{x+\beta} = \frac{-\alpha}{1-\alpha x}$. The rest is fairly straightforward."

Following this recipe, I find that

$$U(n,k) = \sum_{t=0}^{k}\sum_{l=0}^{t}\binom{n-2k-t+2l+1}{k-t}\binom{k-1}{l}\binom{k+t-2l}{t-l}$$
$$\times \frac{1}{\sqrt{5}^{k+t-2l+1}}\left(\alpha^{n-2k-t+2l+2} - (-1)^{k+t-2l}\beta^{n-2k-t+2l+2}\right).$$

2. The Calculations

For $k \geq 1$,

$$\sum_{n\geq 0} U(n,k)x^n = \frac{x^{k+1}(1-x)^{k-1}}{(1-x-x^2)^{k+1}}$$
$$= \frac{x^{k+1}(x^2+(1-x-x^2))^{k-1}}{(1-x-x^2)^{k+1}}$$
$$= x^{k+1}\frac{\sum_{l=0}^{k-1}\binom{k-1}{l}(x^2)^{k-1-l}(1-x-x^2)^l}{(1-x-x^2)^{k+1}}$$
$$= x^{k+1}\sum_{l=0}^{k-1}\binom{k-1}{l}\frac{x^{2k-2l-2}}{(1-x-x^2)^{k-l+1}}$$
$$= x^{k+1}\sum_{l=0}^{k-1}\binom{k-1}{l}\frac{x^{2k-2l-2}}{(-(x+\alpha)(x+\beta))^{k-l+1}}$$
$$= x^{k+1}\sum_{l=0}^{k-1}\binom{k-1}{l}(-1)^{k-l+1}x^{2k-2l-2}\cdot\frac{1}{((x+\alpha)(x+\beta))^{k-l+1}}$$
$$= x^{k+1}\sum_{l=0}^{k-1}\binom{k-1}{l}(-1)^{k-l+1}x^{2k-2l-2}$$
$$\times\left((-1)^{k-l+1}\sum_{m=0}^{k-l}\frac{\binom{k-l+m}{m}}{\sqrt{5}^{k-l+1+m}(x+\alpha)^{k-l+1-m}}\right.$$
$$\left.+\sum_{m=0}^{k-l}(-1)^m\frac{\binom{k-l+m}{m}}{\sqrt{5}^{k-l+1+m}(x+\beta)^{k-l+1-m}}\right)$$
$$= x^{k+1}\sum_{l=0}^{k-1}\sum_{m=0}^{k-l}x^{2k-2l-2}\binom{k-1}{l}\binom{k-l+m}{m}\frac{1}{\sqrt{5}^{k-l+1+m}}$$
$$\times\left(\frac{1}{(x+\alpha)^{k-l+1-m}} - (-1)^{k-l+m}\frac{1}{(x+\beta)^{k-l+1-m}}\right)$$

$$
\begin{aligned}
&= x^{k+1}\sum_{l=0}^{k-1}\sum_{m=0}^{k-l} x^{2k-2l-2}\binom{k-1}{l}\binom{k-l+m}{m}\frac{1}{\sqrt{5}^{k-l+1+m}}\\
&\quad\times\left(\left(\frac{-\beta}{1-\beta x}\right)^{k-l-m+1}-(-1)^{k-l+m}\left(\frac{-\alpha}{1-\alpha x}\right)^{k-l-m+1}\right)\\
&= x^{k+1}\sum_{l=0}^{k-1}\sum_{m=0}^{k-l} x^{2k-2l-2}\binom{k-1}{l}\binom{k-l+m}{m}\frac{1}{\sqrt{5}^{k-l+1+m}}\\
&\quad\times\left(\left(\frac{\alpha}{1-\alpha x}\right)^{k-l-m+1}-(-1)^{k-l+m}\left(\frac{\beta}{1-\beta x}\right)^{k-l-m+1}\right)\\
&= x^{k+1}\sum_{l=0}^{k-1}\sum_{m=0}^{k-l} x^{2k-2l-2}\binom{k-1}{l}\binom{k-l+m}{m}\frac{1}{\sqrt{5}^{k-l+1+m}}\\
&\quad\times\Bigg(\alpha^{k-l-m+1}\sum_{N\geq 0}\binom{k-l-m+N}{N}\alpha^N x^N\\
&\qquad -(-1)^{k-l+m}\beta^{k-l-m+1}\sum_{N\geq 0}\binom{k-l-m+N}{N}\beta^N x^N\Bigg)\\
&= \sum_{N\geq 0}\sum_{l=0}^{k-1}\sum_{m=0}^{k-l} x^{N+3k-2l-1}\binom{N+k-l-m}{N}\binom{k-1}{l}\binom{k-l+m}{m}\\
&\quad\times\frac{1}{\sqrt{5}^{k-l+m+1}}\left(\alpha^{N+k-l-m+1}-(-1)^{k-l+m}\beta^{N+k-l-m+1}\right).
\end{aligned}
$$

With $N+3k-2l-1=n$, this yields

$$
\begin{aligned}
U(n,k) &= \sum_{l=0}^{k-1}\sum_{m=0}^{k-l}\binom{n-2k+l-m+1}{k-l-m}\binom{k-1}{l}\binom{k-l+m}{m}\\
&\quad\times\frac{1}{\sqrt{5}^{k-l+m+1}}\left(\alpha^{n-2k+l-m+2}-(-1)^{k-l+m}\beta^{n-2k+l-m+2}\right).
\end{aligned}
$$

If we now write $l+m=t$,

$$
\begin{aligned}
U(n,k) &= \sum_{t=0}^{k}\sum_{l=0}^{t}\binom{n-2k-t+2l+1}{k-t}\binom{k-1}{l}\binom{k+t-2l}{t-l}\\
&\quad\times\frac{1}{\sqrt{5}^{k+t-2l+1}}\left(\alpha^{n-2k-t+2l+2}-(-1)^{k+t-2l}\beta^{n-2k-t+2l+2}\right).
\end{aligned}
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References

[1] M. N. Despande and J. P. Shiwalkar, *The number of HH's in a coin-tossing experiment and the Fibonacci sequence*, Mathematical Gazette, **92** (2008), 147–150.
[2] M. Griffiths, *Fibonacci expressions arising from a coin-tossing scenario involving pairs of consecutive heads*, The Fibonacci Quarterly, **49.3** (2011), 249–254.
[3] M. D. Hirschhorn, *A note on partial fractions*, Australian Mathematical Society Gazette, **30** (2003), 81.
[4] M. D. Hirschhorn, *On 92.27*, Mathematical Gazette, **93** (2009), 151–154.

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