THE ORDER OF APPEARANCE OF PRODUCT OF CONSECUTIVE FIBONACCI NUMBERS

DIEGO MARQUES

Abstract. Let F_n be the nth Fibonacci number. The order of appearance $z(n)$ of a natural number n is defined as the smallest natural number k such that n divides F_k. For instance, $z(F_n) = n$, for all $n \geq 3$. In this paper, among other things, we prove that

$$z(F_nF_{n+1}F_{n+2}) = \frac{n(n+1)(n+2)}{2},$$

for all even positive integers n.

1. Introduction

Let $(F_n)_{n\geq 0}$ be the Fibonacci sequence given by $F_{n+2} = F_{n+1} + F_n$, for $n \geq 0$, where $F_0 = 0$ and $F_1 = 1$. These numbers are well-known for possessing amazing properties (consult [4] together with its very extensive annotated bibliography for additional references and history). In 1963, the Fibonacci Association was created to provide enthusiasts an opportunity to share ideas about these intriguing numbers and their applications. We cannot go very far in the lore of Fibonacci numbers without encountering its companion Lucas sequence $(L_n)_{n\geq 0}$ which follows the same recursive pattern as the Fibonacci numbers, but with initial values $L_0 = 2$ and $L_1 = 1$.

The study of the divisibility properties of Fibonacci numbers has always been a popular area of research. Let n be a positive integer number, the *order (or rank) of appearance* of n in the Fibonacci sequence, denoted by $z(n)$, is defined as the smallest positive integer k, such that $n|F_k$ (some authors also call it *order of apparition*, or *Fibonacci entry point*). There are several results about $z(n)$ in the literature. For instance, $z(n) < \infty$ for all $n \geq 1$. The proof of this fact is an immediate consequence of the Théorème Fondamental of Section XXVI in [10, p. 300]. Indeed, $z(m) < m^2 - 1$, for all $m > 2$ (see [15, Theorem, p. 52]) and in the case of a prime number p, one has the better upper bound $z(p) \leq p + 1$, which is a consequence of the known congruence $F_{p-(\frac{p}{5})} \equiv 0 \pmod p$, for $p \neq 2, 5$, where $(\frac{a}{q})$ denotes the Legendre symbol of a with respect to a prime $q > 2$.

In recent papers, the author [6, 7, 8] found explicit formulas for the order of appearance of integers related to Fibonacci numbers, such as $F_m \pm 1, F_{mk}/F_k$ and F_n^k. In particular, it was proved that $z(F_{4m} \pm 1) = 8m^2 - 2$, if $m > 1$, and $z(F_n^{k+1}) = nF_n^k$, for all non-negative integers k and $n > 3$ with $n \not\equiv 3 \pmod 6$.

In this paper, we continue this program and study the order of appearance of product of consecutive Fibonacci numbers. Our main results are the following.

Theorem 1.1. *We have*

(i) *For $n \geq 3$,*

$$z(F_nF_{n+1}) = n(n+1).$$

Research supported in part by FAP-DF, FEMAT-Brazil and CNPq-Brazil.

(ii) *For* $n \geq 2$,

$$z(F_nF_{n+1}F_{n+2}) = \begin{cases} n(n+1)(n+2), & \text{if} \quad n \equiv 1 \pmod 2, \\ \frac{n(n+1)(n+2)}{2}, & \text{if} \quad n \equiv 0 \pmod 2. \end{cases}$$

(iii) *For* $n \geq 1$,

$$z(F_nF_{n+1}F_{n+2}F_{n+3}) = \begin{cases} \frac{n(n+1)(n+2)(n+3)}{2}, & \text{if} \quad n \not\equiv 0 \pmod 3, \\ \frac{n(n+1)(n+2)(n+3)}{3}, & \text{if} \quad n \equiv 0, 9 \pmod{12}, \\ \frac{n(n+1)(n+2)(n+3)}{6}, & \text{if} \quad n \equiv 3, 6 \pmod{12}. \end{cases}$$

We recall that the *Fibonacci factorial* of n (also called *Fibonorial*), denoted by $n_F!$, is defined as the product of the first n nonzero Fibonacci numbers (sequence A003266 in OEIS [13]). In the search for $z(n_F!)$, we found that $n_F!|F_{n!}$ (and thus $z(n_F!)|n!$) for $n = 1, ..., 10$ (see Table 1).

It is therefore reasonable to conjecture that $z(n_F!)|n!$ and so $F_1 \cdots F_n|F_{n!}$, for all positive integers n. However, this is not true, because $z(110_F!) \nmid 110!$ In fact, $\nu_{11}(F_1 \cdots F_{110}) = 12 > 11 = \nu_{11}(F_{110!})$. Hence, motivated by this fact, we prove that

n	1	2	3	4	5	6	7	8	9	10
$z(n_F!)$	$1!$	$1!$	$\frac{3!}{2}$	$\frac{4!}{2}$	$\frac{5!}{2}$	$\frac{6!}{12}$	$\frac{7!}{12}$	$\frac{8!}{48}$	$\frac{9!}{144}$	$\frac{10!}{288}$

Table 1. The order of appearance of $n_F!$, for $1 \leq n \leq 10$.

Theorem 1.2. *For all* $p \in \{2, 3, 5, 7\}$ *and* $n \geq 1$, *we have*

$$\nu_p(F_1 \cdots F_n) \leq \nu_p(F_{n!}).$$

Here $\nu_p(n)$ *denotes the* p*-adic order of* n *which, as usual, is the exponent of the highest power of* p *dividing* n.

We organize the paper as follows. In Section 2, we will recall some useful properties of Fibonacci numbers such as d'Ocagne's identity and a result concerning the p-adic order of F_n. The last two sections will be devoted to the proof of theorems.

2. Auxiliary Results

Before proceeding further, we recall some facts on Fibonacci numbers for the convenience of the reader.

Lemma 2.1. *We have*

(a) $F_n|F_m$ *if and only if* $n|m$.
(b) $\gcd(F_n, F_m) = F_{\gcd(n,m)}$.
(c) $2F_n|F_{2n}$, *for all* $n \equiv 0 \pmod 3$.
(d) *(d'Ocagne's identity)* $(-1)^nF_{m-n} = F_mF_{n+1} - F_nF_{m+1}$.
(e) $F_{p-(\frac{p}{5})} \equiv 0 \pmod p$, *for all primes* p.

Most of the previous items can be proved by using induction together with the well-known Binet's formula:

$$F_n = \frac{\alpha^n - \beta^n}{\alpha - \beta}, \quad \text{for} \quad n \geq 0,$$

where $\alpha = (1 + \sqrt{5})/2$ and $\beta = (1 - \sqrt{5})/2$. We refer the reader to [1, 3, 4, 11] for more details and additional bibliography.

The second lemma is a consequence of the previous one.

Lemma 2.2. *We have*

(a) *If* $F_n|m$, *then* $n|z(m)$.
(b) *If* $n|F_m$, *then* $z(n)|m$.

Proof. For (a), since $F_n|m|F_{z(m)}$, by Lemma 2.1 (a), we get $n|z(m)$. In order to prove (b), we write $m = z(n)q + r$, where q and r are integers, with $0 \leq r < z(n)$. So, by Lemma 2.1 (d), we obtain

$$(-1)^{z(n)q} F_r = F_m F_{z(n)q+1} - F_{z(n)q} F_{m+1}.$$

Since n divides both F_m and $F_{z(n)q}$, then it also divides F_r implying $r = 0$ (keep in mind the range of r). Thus, $z(n)|m$. □

The p-adic order of Fibonacci and Lucas numbers was completely characterized, see [2, 9, 12, 14]. For instance, from the main results of Lengyel [9], we extract the following result.

Lemma 2.3. *For* $n \geq 1$, *we have*

$$\nu_2(F_n) = \begin{cases} 0, & \text{if } n \equiv 1, 2 \pmod{3}; \\ 1, & \text{if } n \equiv 3 \pmod{6}; \\ 3, & \text{if } n \equiv 6 \pmod{12}; \\ \nu_2(n) + 2, & \text{if } n \equiv 0 \pmod{12}, \end{cases}$$

$$\nu_5(F_n) = \nu_5(n), \text{ and if } p \text{ is prime} \neq 2 \text{ or } 5, \text{ then}$$

$$\nu_p(F_n) = \begin{cases} \nu_p(n) + \nu_p(F_{z(p)}), & \text{if } n \equiv 0 \pmod{z(p)}; \\ 0, & \text{if } n \not\equiv 0 \pmod{z(p)}. \end{cases}$$

A proof of this result can be found in [9].

Lemma 2.4. *For any integer* $k \geq 1$ *and* p *prime, we have*

$$\frac{k}{p-1} - \left\lfloor \frac{\log k}{\log p} \right\rfloor - 1 \leq \nu_p(k!) \leq \frac{k-1}{p-1}, \tag{2.1}$$

where, as usual, $\lfloor x \rfloor$ *denotes the largest integer less than or equal to* x.

Proof. Recall the well-known Legendre's formula [5]:

$$\nu_p(k!) = \frac{k - s_p(k)}{p-1}, \tag{2.2}$$

where $s_p(k)$ is the sum of digits of k in base p. Since k has $\lfloor \log k / \log p \rfloor + 1$ digits in base p, and each digit is at most $p - 1$, we get

$$1 \leq s_p(k) \leq (p-1)\left(\left\lfloor \frac{\log k}{\log p} \right\rfloor + 1\right). \tag{2.3}$$

Therefore, the inequality in (2.1) follows from (2.2) and (2.3). □

Now, we are ready to deal with the proof of theorems.

3. The Proof of Theorem 1.1

3.1. **Proof of (i).** For $\epsilon \in \{0,1\}$, one has that $F_{n+\epsilon}|F_nF_{n+1}$ and so Lemma 2.2 (a) yields $n+\epsilon|z(F_nF_{n+1})$. But, $\gcd(n,n+1)=1$ and therefore $n(n+1)|z(F_nF_{n+1})$. On the other hand, $F_{n+\epsilon}|F_{n(n+1)}$ (Lemma 2.1 (a)) and hence, $F_nF_{n+1}|F_{n(n+1)}$, since $\gcd(F_n,F_{n+1})=1$. Now, by using Lemma 2.2 (b), we conclude that $z(F_nF_{n+1})|n(n+1)$. In conclusion, we have $z(F_nF_{n+1})=n(n+1)$. □

We remark that the same idea can be used to prove that if $m_1,\dots,m_k$ are positive integers, such that $\gcd(m_i,m_j)=1$ or 2, for all $i\neq j$, then

$$z(F_{m_1}\cdots F_{m_k})|m_1\cdots m_k.$$

If $m_1,\dots,m_k$ are pairwise relatively prime, then

$$z(F_{m_1}\cdots F_{m_k})=m_1\cdots m_k.$$

3.2. **Proof of (ii).** For $\epsilon\in\{0,1,2\}$, we have $F_{n+\epsilon}|F_{n(n+1)(n+2)}$. By Lemma 2.1 (b), the numbers F_n,F_{n+1},F_{n+2} are pairwise coprime. In fact, if $\epsilon_1,\epsilon_2\in\{0,1,2\}$ are distinct, then $\gcd(n+\epsilon_1,n+\epsilon_2)=1$ or 2. In any case, we get

$$\gcd(F_{n+\epsilon_1},F_{n+\epsilon_2})=F_{\gcd(n+\epsilon_1,n+\epsilon_2)}=1.$$

Thus, $F_nF_{n+1}F_{n+2}|F_{n(n+1)(n+2)}$ and so

$$z(F_nF_{n+1}F_{n+2})|n(n+1)(n+2). \tag{3.1}$$

Now, we use $F_{n+\epsilon}|F_nF_{n+1}F_{n+2}$, to conclude that $n+\epsilon$ divides $z(F_nF_{n+1}F_{n+2})$. So, the proof splits in two cases:

Case 1: If n is odd, then $n,n+1,n+2$ are pairwise coprime. Therefore, $n(n+1)(n+2)|z(F_nF_{n+1}F_{n+2})$. This fact, together with (3.1), yields the result in this case.

Case 2: For n even, we have that $F_n|F_{\frac{n(n+1)(n+2)}{2}}$ and so $z(F_nF_{n+1}F_{n+2})$ divides $n(n+1)(n+2)/2$. We already know that $n+\epsilon|z(F_nF_{n+1}F_{n+2})$ and $\gcd(n,n+2)=2$. If $n\equiv 0 \pmod 4$, then $n,n+1,(n+2)/2$ are pairwise coprime. In the case of $n\equiv 2\pmod 4$, the numbers $n/2,n+1,n+2$ are pairwise coprime. Thus, in any case, we have $n(n+1)(n+2)/2|z(F_nF_{n+1}F_{n+2})$ and the desired result is proved. □

3.3. **Proof of (iii).** By the same arguments as before, we conclude that

$$n+\epsilon|z(F_nF_{n+1}F_{n+2}F_{n+3}),\ \text{for } \epsilon\in\{0,1,2,3\}. \tag{3.2}$$

Assume first that $n\not\equiv 0\pmod 3$. Then there exists only one pair among $(n,n+2)$ and $(n+1,n+3)$ whose greatest common divisor is 2. Without loss of generality, we suppose that $\gcd(n,n+2)=2$. Again, as in the previous item, we can see that $n/2^a,n+1,(n+2)/2^b,n+3$ are pairwise coprime, for distinct $a,b\in\{0,1\}$ suitably chosen depending on the class of n modulo 4. Thus,

$$\frac{n(n+1)(n+2)(n+3)}{2}=\frac{n(n+1)(n+2)(n+3)}{2^{a+b}}\mid z(F_nF_{n+1}F_{n+2}F_{n+3}).$$

Since there are two even numbers among $n,n+1,n+2,n+3$, we have that $F_{n+\epsilon}|F_{\frac{n(n+1)(n+2)(n+3)}{2}}$. However, $\gcd(F_n,F_{n+3})=F_{\gcd(n,n+3)}=1$, because $3\nmid n$. Thus, $F_n,F_{n+1},F_{n+2},F_{n+3}$ are pairwise coprime yielding that $F_nF_{n+1}F_{n+2}F_{n+3}|F_{\frac{n(n+1)(n+2)(n+3)}{2}}$. We apply Lemma 2.2 to get

$$z(F_nF_{n+1}F_{n+2}F_{n+3})|\frac{n(n+1)(n+2)(n+3)}{2}.$$

This finishes the proof in this case. Now suppose that $n \equiv 0 \pmod 3$. If $9|n$, then $\gcd(n,(n+3)/3) = 1$, while $\gcd(n/3, n+3) = 1$ when $9 \nmid n$. In any case, for a suitable choice of $a,b,c,d,e,f \in \{0,1\}$, where $a \neq b$ and only one among c,d,e,f is 1, we obtain that

$$\frac{n}{2^c 3^a}, \frac{n+1}{2^d}, \frac{n+2}{2^e}, \frac{n+3}{2^f 3^b}$$

are pairwise coprime. Here the sets $\{a,b\}$ and $\{c,d,e,f\}$ depend on the class of n modulo 4 and 9, respectively. Hence, by (3.2), we get

$$\frac{n(n+1)(n+2)(n+3)}{6} = \frac{n(n+1)(n+2)(n+3)}{2^{c+d+e+f}3^{a+b}} | z(F_n F_{n+1} F_{n+2} F_{n+3}), \tag{3.3}$$

since $a+b = c+d+e+f = 1$.

Note that there are 2 even numbers among $n, n+1, n+2, n+3$ and also 3 divides both n and $n+3$. Thus $F_{n+\epsilon} | F_{\frac{n(n+1)(n+2)(n+3)}{6}}$, for $\epsilon \in \{0,1,2,3\}$. Since $\gcd(F_n, F_{n+3}) = 2$ and $\gcd(F_{n+3}, F_{n+1}F_{n+2}) = 1$, then $\gcd(F_n F_{n+1} F_{n+2}, F_{n+3}) = 2$. Now, we use that F_n, F_{n+1}, F_{n+2} are pairwise coprime to ensure that $F_n F_{n+1} F_{n+2} | F_{\frac{n(n+1)(n+2)(n+3)}{6}}$. Since F_{n+3} also divides $F_{\frac{n(n+1)(n+2)(n+3)}{6}}$, we get

$$F_n F_{n+1} F_{n+2} F_{n+3} | 2F_{\frac{n(n+1)(n+2)(n+3)}{6}} | F_{\frac{n(n+1)(n+2)(n+3)}{3}},$$

where we used Lemma 2.1 (c). Thus, Lemma 2.2 (b) yields

$$z(F_n F_{n+1} F_{n+2} F_{n+3}) | \frac{n(n+1)(n+2)(n+3)}{3}. \tag{3.4}$$

Combining (3.3) and (3.4), we get

$$z(F_n F_{n+1} F_{n+2} F_{n+3}) \in \left\{ \frac{n(n+1)(n+2)(n+3)}{6}, \frac{n(n+1)(n+2)(n+3)}{3} \right\} \tag{3.5}$$

holds for all positive integers $n \equiv 0 \pmod 3$. In order to complete the proof, it suffices to prove that

$$F_n F_{n+1} F_{n+2} F_{n+3} \nmid F_{\frac{n(n+1)(n+2)(n+3)}{6}}, \text{ for all } n \equiv 0, 9 \pmod{12} \tag{3.6}$$

and

$$F_n F_{n+1} F_{n+2} F_{n+3} \mid F_{\frac{n(n+1)(n+2)(n+3)}{6}}, \text{ for all } n \equiv 3, 6 \pmod{12}. \tag{3.7}$$

We claim that (3.6) is true. In fact, if $n \equiv 0 \pmod{12}$, then $n+3 \equiv 3 \pmod 6$. On the one hand, by setting $n = 12\ell$ and by Lemma 2.3, we have

$$\nu_2(F_n F_{n+1} F_{n+2} F_{n+3}) = \nu_2(F_n) + \nu_2(F_{n+3}) = \nu_2(n) + 3 = \nu_2(\ell) + 5.$$

On the other, $\frac{n(n+1)(n+2)(n+3)}{6} = 12\ell(12\ell+1)(6\ell+1)(4\ell+1)$ and so

$$\nu_2(F_{\frac{n(n+1)(n+2)(n+3)}{6}}) = \nu_2(12\ell(12\ell+1)(6\ell+1)(4\ell+1)) + 2 = \nu_2(\ell) + 4.$$

This means that $\nu_2(F_n F_{n+1} F_{n+2} F_{n+3}) > \nu_2(F_{\frac{n(n+1)(n+2)(n+3)}{6}})$ which is enough to prove (3.6). A similar argument holds for the case $n \equiv 9 \pmod{12}$.

Now to prove (3.7), we shall show that

$$\nu_p(F_n F_{n+1} F_{n+2} F_{n+3}) \le \nu_p(F_{\frac{n(n+1)(n+2)(n+3)}{6}}),$$

for all primes p. In fact,

Case 1: If $p = 5$, then

$$\nu_5(F_nF_{n+1}F_{n+2}F_{n+3}) = \nu_5(F_{n+j}) = \nu_5(n+j) = \nu_5(F_{\frac{n(n+1)(n+2)(n+3)}{6}}).$$

Case 2: If $p \neq 2$ or 5, then p divides at most one among $F_n, F_{n+1}, F_{n+2}, F_{n+3}$. Suppose that this is the case (otherwise $\nu_p(F_nF_{n+1}F_{n+2}F_{n+3}) = 0$ and we are done). Then, let j be the integer belonging to $\{0,1,2,3\}$ such that $p|F_{n+j}$. Thus,

$$\nu_p(F_nF_{n+1}F_{n+2}F_{n+3}) = \nu_p(F_{n+j}) = \nu_p(n+j) + \nu_p(F_{z(p)}).$$

On the other hand,

$$\nu_p(F_{\frac{n(n+1)(n+2)(n+3)}{6}}) = \nu_p(n(n+1)(n+2)(n+3)) - \nu_p(6) + \nu_p(F_{z(p)}).$$

If $p > 3$, $\nu_p(6) = 0$ and the desired inequality follows. In the case of $p = 3$, we have

$$\begin{aligned}\nu_3(F_{\frac{n(n+1)(n+2)(n+3)}{6}}) &= \nu_3(n) + \nu_3(n+3) - 1 + \nu_p(F_{z(p)})\\ &\geq v_3(n+j) + \nu_3(F_{z(p)}) = \nu_3(F_nF_{n+1}F_{n+2}F_{n+3}).\end{aligned}$$

Case 3: When $p = 2$, we use that $n \equiv 3, 6 \pmod{12}$. Let us suppose that $n \equiv 3 \pmod{12}$ (the other case can be handled in the same way). Then $n+3 \equiv 6 \pmod{12}$ and by Lemma 2.3, we obtain

$$\nu_2(F_nF_{n+1}F_{n+2}F_{n+3}) = \nu_2(F_n) + \nu_2(F_{n+3}) = 4.$$

However, if $n = 12\ell + 3$, then

$$\begin{aligned}\nu_2(F_{\frac{n(n+1)(n+2)(n+3)}{6}}) &= \nu_2(F_{12(4\ell+1)(3\ell+1)(12\ell+5)(2\ell+1)})\\ &= \nu_2(12(4\ell+1)(3\ell+1)(12\ell+5)(2\ell+1)) + 2 = 4.\end{aligned}$$

The proof is then complete. □

4. The Proof of Theorem 1.2

For $p = 5$, we have

$$\nu_5(F_1 \cdots F_n) = \sum_{j=1}^{n} \nu_5(F_j) = \sum_{j=1}^{n} \nu_5(j) = \nu_5(n!) = \nu_5(F_{n!}).$$

In the case of $p = 2$, we first use *Mathematica* to see that the result holds for all $1 \leq n \leq 48$. So, we shall assume $n \geq 49$. Now, we note that $\nu_2(F_n) \neq 0$ if and only if $3|n$. Thus, the only terms with non-zero 2-adic order among $F_1, \ldots, F_n$ are $F_3, \ldots, F_{3\lfloor n/3 \rfloor}$ and so

$$\begin{aligned}\nu_2(F_1 \cdots F_n) &= [\nu_2(F_3) + \nu_2(F_6) + \nu_2(F_9)] + \nu_2(F_{12})\\ &+ [\nu_2(F_{15}) + \nu_2(F_{18}) + \nu_2(F_{21})] + \nu_2(F_{24})\\ &+ \cdots + \nu_2(F_{12\lfloor n/12 \rfloor}) + \ell,\end{aligned}$$

where $\ell \in \{0,1,4,5\}$ and depends on the residue class of n modulo 12. By Lemma 2.3, each bracketed term in the above sum is 5 and thus,

$$\nu_2(F_1 \cdots F_n) = 5\left\lfloor \frac{n}{12} \right\rfloor + \sum_{j=1}^{\lfloor n/12 \rfloor} \nu_2(F_{12j}) + \ell = 9\left\lfloor \frac{n}{12} \right\rfloor + \nu_2(\left\lfloor \frac{n}{12} \right\rfloor !) + \ell.$$

We now apply Lemma 2.4 (with $p = 2$) together with the fact that $\ell \leq 5$, to get the bound

$$\nu_2(F_1 \cdots F_n) \leq \frac{5n}{6} + 4. \tag{4.1}$$

On other other hand, since $12|n!$ (because $n > 3$), Lemma 2.3 yields

$$\nu_2(F_{n!}) = \nu_2(n!) + 2.$$

Again, we use Lemma 2.4 to obtain

$$\nu_2(F_{n!}) \geq n - \left\lfloor \frac{\log n}{\log 2} \right\rfloor + 1. \tag{4.2}$$

The proof of this case finishes by noting that the right-hand side of (4.2) is greater than $5n/6 + 4$, for $n \geq 49$.

When $p = 3$ or 7, we again use Lemma 2.3 to get

$$\begin{aligned} \nu_p(F_1 \cdots F_n) &= \sum_{j=1}^{\lfloor n/z(p) \rfloor} (\nu_p(z(p)j) + \nu_p(F_{z(p)})) \\ &= \left\lfloor \frac{n}{z(p)} \right\rfloor \nu_p(F_{z(p)}) + \nu_p\left(\left\lfloor \frac{n}{z(p)} \right\rfloor !\right), \end{aligned}$$

where we used that $\nu_p(z(p)) = 0$, since by Lemma 2.1 (e), $p|F_{p\pm 1}$ and so by Lemma 2.2 (b), one has that $z(p)|(p \pm 1)$. Now, we apply Lemma 2.4 to obtain

$$\nu_p(F_1 \cdots F_n) \leq \left\lfloor \frac{n}{z(p)} \right\rfloor \nu_p(F_{z(p)}) + \frac{\left\lfloor \frac{n}{z(p)} \right\rfloor - 1}{p-1}. \tag{4.3}$$

On the other hand, $\nu_p(F_{n!}) = \nu_p(n!) + \nu_p(F_{z(p)})$ and hence, again by Lemma 2.4, we have

$$\nu_p(F_{n!}) \geq \frac{n}{p-1} - \left\lfloor \frac{\log n}{\log p} \right\rfloor - 1 + \nu_p(F_{z(p)}). \tag{4.4}$$

By combining (4.3) and (4.4), it suffices to prove for $p = 3$ that

$$n \geq 3 \left\lfloor \frac{n}{4} \right\rfloor + 2 \left\lfloor \frac{\log n}{\log 3} \right\rfloor$$

and for $p = 7$ that

$$n \geq 7 \left\lfloor \frac{n}{8} \right\rfloor + 6 \left\lfloor \frac{\log n}{\log 7} \right\rfloor + 4.$$

However, both these inequalities hold for all $n \geq 123$. For the remaining cases, we use a simple *Mathematica* routine to check that $\nu_p(F_1 \cdots F_n) \leq \nu_p(F_{n!})$ is also valid for $n = 1, \ldots, 122$. This completes the proof. $\square$

5. Acknowledgement

The author is grateful to Pavel Trojovský for nice discussions on the subject.

References

[1] A. Benjamin and J. Quinn, *The Fibonacci numbers–exposed more discretely*, Math. Mag., **76.3** (2003), 182–192.

[2] J. H. Halton, *On the divisibility properties of Fibonacci numbers*, The Fibonacci Quarterly, **4.3** (1966), 217–240.

[3] D. Kalman and R. Mena, *The Fibonacci numbers–exposed*, Math. Mag., **76.3** (2003), 167–181.

[4] T. Koshy, *Fibonacci and Lucas Numbers with Applications*, Wiley, New York, 2001.

[5] A. M. Legendre, *Theorie des Nombres*, Firmin Didot Freres, Paris, 1830.

[6] D. Marques, *On integer numbers with locally smallest order of appearance in the Fibonacci sequence*, Internat. J. Math. Math. Sci., Article ID 407643 (2011), 4 pages.

[7] D. Marques, *On the order of appearance of integers at most one away from Fibonacci numbers*, (to appear in The Fibonacci Quarterly).
[8] D. Marques, *On the order of appearance of powers of Fibonacci and Lucas numbers*, Preprint.
[9] T. Lengyel, *The order of the Fibonacci and Lucas numbers*, The Fibonacci Quarterly, **33.3** (1995), 234–239.
[10] E. Lucas, *Théorie des fonctions numériques simplement périodiques*, Amer. J. Math., **1** (1878), 184–240, 289-321.
[11] P. Ribenboim, *My Numbers, My Friends: Popular Lectures on Number Theory*, Springer-Verlag, New York, 2000.
[12] D. W. Robinson, *The Fibonacci matrix modulo m*, The Fibonacci Quarterly, **1.2** (1963), 29–36.
[13] N. J. A. Sloane (Ed.), The On-Line Encyclopedia of Integer Sequences, 2010. `http://www.research.att.com/∼njas/sequences/`.
[14] J. Vinson, *The relation of the period modulo m to the rank of apparition of m in the Fibonacci sequence,* The Fibonacci Quarterly, **1.2** (1963), 37–45.
[15] N. N. Vorobiev, *Fibonacci Numbers*, Birkhäuser, Basel, 2003.

MSC2010: 11B39

Departamento de Matemática, Universidade de Brasília, Brasília, DF, 70910-900, Brazil
E-mail address: `diego@mat.unb.br`