ON THE NUMBER OF COMPLEX HORADAM SEQUENCES WITH A FIXED PERIOD

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ABSTRACT. The Horadam sequence is a direct generalization of the Fibonacci numbers in the complex plane, depending on a family of four complex parameters: two recurrence coefficients and two initial conditions. Here the Horadam sequences with a given period are enumerated. The result generates a new integer sequence whose representation involves some well-known functions such as Euler's totient function φ and the number of divisors function ω.

1. Introduction

A Horadam sequence $\{w_n\}_{n=0}^{\infty} = \{w_n(a, b; p, q)\}_{n=0}^{\infty}$ is defined by the recurrence

$$w_{n+2} - pw_{n+1} + qw_n = 0, \quad w_0 = a, w_1 = b, \tag{1.1}$$

where the parameters a, b, p, q are complex numbers. It is well-known to deliver many long-standing and familiar sequences as particular instances, and has been the object of study in its general form since the 1960's (see the survey article [3]). Periodic orbits of complex Horadam sequences have been characterized in [1], and arise when zeros of the characteristic equation

$$x^2 - px + q = 0 \tag{1.2}$$

(called generators) are roots of unity; we denote the form of such roots for convenience as $z_1 = z_1(p, q) = e^{2\pi i p_1/k_1}$ and $z_2 = z_2(p, q) = e^{2\pi i p_2/k_2}$ where p_1, p_2, k_1, k_2 are positive integers.

For equal roots $z_1 = z_2$ of (1.2), the general term of Horadam's sequence $\{w_n\}_{n=0}^{\infty}$ is

$$w_n = \left[a + \left(\frac{b}{z} - a\right)n\right] z^n. \tag{1.3}$$

In this case the sequence can only be periodic when $b = az$ and z is a root of unity.

For distinct roots $z_1 \neq z_2$ of (1.2), the general term of Horadam's sequence $\{w_n\}_{n=0}^{\infty}$ is

$$w_n = Az_1^n + Bz_2^n, \tag{1.4}$$

where the constants A and B can be obtained from the initial condition, as

$$A = \frac{az_2 - b}{z_2 - z_1}, \quad B = \frac{b - az_1}{z_2 - z_1}. \tag{1.5}$$

When $AB = 0$, at least one of the generators z_1 and z_2 does not appear explicitly in w_n, and the orbit of the sequence degenerates to either a regular polygon centered in 0, or to a point. For $AB \neq 0$, the sequence is periodic when the distinct generators z_1 and z_2 are roots of unity.

Here we investigate the number of distinct Horadam sequences which (for arbitrary initial conditions) have a fixed period, giving enumeration formulas in both degenerate and non-degenerate cases. The question of precisely how many such sequences exist is interesting from a theoretical point of view, and is but one of a number of problems highlighted as worthy of study in the analysis of Horadam sequence cyclicity.

2. Theory and Results

Let $k \geq 2$ be a positive integer. The enumerating function for the number of Horadam sequences $\{w_n\}_{n=0}^{\infty}$ having period k is denoted by $H_P(k)$. Clearly, this number depends on the generators z_1, z_2 and the initial conditions a, b. Throughout this paper the notations (k_1, k_2) or $\gcd(k_1, k_2)$ are used for the greatest common divisor and $[k_1, k_2]$ or $\operatorname{lcm}(k_1, k_2)$ for the least common multiple of the positive integers k_1 and k_2.

There are two types of (degenerate and non-degenerate) periodic orbits for which to account.

2.1. Degenerate orbits.

This case covers periodic sequences producing a degenerated orbit (regular polygon centered in 0 or point). As detailed in [1], this happens when the Horadam sequences $\{w_n\}_{n=0}^{\infty}$ given by (1.3) or (1.4) depend on only one of the generators and this is a root of unity, say $z_1 = e^{2\pi i p_1/k_1}$. The number of distinct sequences having period k is given by

$$H_P(k) = \sharp\{(p_1, k_1) : \ (p_1, k_1) = 1, \ k_1 = k\} = \varphi(k), \tag{2.1}$$

where φ is Euler's well-known totient function [4].

If no generator appears explicitly in the formulas (1.3) or (1.4) (this is when $z_1 \neq z_2$, $A = 0$, $B = 0$ or $z_1 = z_2 = z$, $a = 0$, $b = 0$), the periodic sequence is constant and the number of generator configurations leading to periodicity $k \geq 2$ is therefore zero.

2.2. Non-Degenerate orbits.

Here we cover periodic sequences producing non-degenerated orbits. In this case the generators are distinct roots of unity $z_1 = e^{2\pi i p_1/k_1}$ and $z_2 = e^{2\pi i p_2/k_2}$, and the arbitrary initial conditions a, b are such that $AB \neq 0$ for A, B defined in (1.5).

As established in [1], the period of the Horadam sequence delivered by a generator pair z_1, z_2 is $[\operatorname{ord}(z_1), \operatorname{ord}(z_2)] = \operatorname{lcm}(\operatorname{ord}(z_1), \operatorname{ord}(z_2))$ (where $\operatorname{ord}(z)$ is the order of z). Representing the pair (z_1, z_2) by the quadruple (p_1, k_1, p_2, k_2), we want to select those producing a sequence having period k. To ensure that the enumeration formula generates all the distinct periodic sequences, we shall assume w.l.o.g. that z_1, z_2 are primitive roots of unity and $k_1 \leq k_2$.

The number of distinct sequences having period k can be enumerated from the quadruples

$$H_P(k) = \sharp\{(p_1, k_1, p_2, k_2) : \ (p_1, k_1) = (p_2, k_2) = 1, \ [k_1, k_2] = k, \ k_1 \leq k_2\}. \tag{2.2}$$

Some formulas for this expression are identified, based on the properties of pairs (k_1, k_2) satisfying $[k_1, k_2] = k$, and their corresponding generators $z_1 = e^{2\pi i p_1/k_1}$ and $z_2 = e^{2\pi i p_2/k_2}$.

2.3. A first formula for $H_P(k)$.

To derive this formula we first generate the pairs (k_1, k_2) satisfying $[k_1, k_2] = k$ and then count the pairs (p_1, p_2) such that (p_1, k_1, p_2, k_2) satisfies (2.2).

The first lemma counts the quadruples (p_1, k_1, p_2, k_2) in (2.2) for which $k_1 = k_2$.

Lemma 2.1. *If $k_1 = k_2$ and $[k_1, k_2] = k$ then $k_1 = k_2 = k$.*

The result is not difficult to prove, and shows that the only pair (k_1, k_2) s.t. $k_1 = k_2$ is (k, k). The number of quadruples (p_1, k, p_2, k) fulfilling (2.2) produced in this case is

$$H'_P(k) = \sharp\{(p_1, p_2) : \ (p_1, k) = (p_2, k) = 1, p_1 < p_2\} = \frac{1}{2}\, \varphi(k) \left(\varphi(k) - 1\right), \tag{2.3}$$

as the number of choices for each of p_1 and p_2 is $\varphi(k)$ and $p_1 < p_2$.

The second lemma counts the quadruples (p_1, k_1, p_2, k_2) when $k_1 \neq k_2$ and $[k_1, k_2] = k$.

Lemma 2.2. *If $[k_1, k_2] = k$ and $k_1 \neq k_2$, the number of quadruples (p_1, k_1, p_2, k_2) produced is*

$$H''_P(k) = \sharp\{(p_1, k_1, p_2, k_2) : \ (p_1, k_1) = (p_2, k_2) = 1, [k_1, k_2] = k\} = \varphi(k_1)\varphi(k_2).$$

Proof. As $k_1 \neq k_2$ the primitive roots z_1 and z_2 are distinct for all combinations p_1 and p_2. This means that any combination pairs $(p_1, k_1) = (p_2, k_2) = 1$ may be considered. There are $\varphi(k_1)$ pairs (p_1, k_1) and $\varphi(k_2)$ pairs (p_2, k_2), therefore the result. $\square$

Theorem 2.3. *The number of distinct Horadam sequences of period $k \geq 2$ is equal to*

$$H_P(k) = \sum_{[k_1,k_2]=k,\, k_1<k_2} \varphi(k_1)\varphi(k_2) + \frac{1}{2}\,\varphi(k)\left(\varphi(k) - 1\right). \tag{2.4}$$

To evaluate this formula one needs to generate all ordered pairs (k_1, k_2), whose l.c.m is k. Special versions of the formula are computed for periods with particular prime decompositions.

The first few terms of the number sequence $H_P(k)$

$$1, 1, 3, 5, 10, 11, 21, 22, 33, 34, 55, 46, 78, 69, 92, 92, 136, 105, \ldots$$

are not currently indexed in the OEIS [5], suggesting that this is a new number sequence.

Example 1. Prime numbers. When k is a prime number we have $\varphi(k) = k - 1$. For this number we just have two divisor pairs

$$(k_1, k_2) \in \{(1, k), (k, k)\},$$

with multiplicities $\varphi(1)\varphi(k) = k - 1$ and $\varphi(k)(\varphi(k) - 1)/2 = (k - 1)(k - 2)/2$, giving the formula

$$H_P(k) = k(k - 1)/2. \tag{2.5}$$

For example, when $k = 23$ there is a total of $23 \cdot 22/2 = 253$ distinct solutions, while for $k = 11$ there is a total of $11 \cdot 10/2 = 55$ distinct solutions. Explicitly, for $k = 5$ there are 10 solutions given by the fraction pairs

$$\begin{aligned}\left(\frac{p_1}{k_1}, \frac{p_2}{k_2}\right) \in \Bigg\{ &\left(\frac{1}{1}, \frac{1}{5}\right), \left(\frac{1}{1}, \frac{2}{5}\right), \left(\frac{1}{1}, \frac{3}{5}\right), \left(\frac{1}{1}, \frac{4}{5}\right), \left(\frac{1}{5}, \frac{2}{5}\right), \\ &\left(\frac{1}{5}, \frac{3}{5}\right), \left(\frac{1}{5}, \frac{4}{5}\right), \left(\frac{2}{5}, \frac{3}{5}\right), \left(\frac{2}{5}, \frac{4}{5}\right), \left(\frac{3}{5}, \frac{4}{5}\right) \Bigg\}.\end{aligned}$$

Example 2. Powers of a prime number. When $k = p^m$ with p a prime number and $m \geq 2$ we have $\varphi(k) = p^m(1 - 1/p) = p^m - p^{m-1}$. For this number we have the divisor pairs $(k_1, k_2) \in \{(1, k), (p, k), \ldots, (p^{m-1}, k), (k, k)\}$, with multiplicities $\varphi(p^j)\varphi(k)$ for $j = 0, \ldots, m-1$ and $\varphi(k)(\varphi(k) - 1)/2 = (k - k/p)(k - k/p - 1)/2$. Summing we obtain a telescopic sum in which the consecutive terms (up to the last two) cancel out

$$\begin{aligned} H_P(k) &= \left(1 + (p - 1) + (p^2 - p) + \cdots + (p^{m-1} - p^{m-2}) + (p^m - p^{m-1} - 1)/2\right)\varphi(k) \\ &= \frac{k^2 - k^2/p^2 - k + k/p}{2} = \frac{\varphi(k)[2k - \varphi(k) - 1]}{2}. \end{aligned} \tag{2.6}$$

For example, when $k = 9 = 3^2$ one obtains $H_P(k) = \frac{6[18-6-1]}{2} = 33$ while for $k = 4$ one obtains $H_P(k) = \frac{2[8-2-1]}{2} = 5$ and the distinct solutions are

$$\left(\frac{p_1}{k_1}, \frac{p_2}{k_2}\right) \in \left\{ \left(\frac{1}{1}, \frac{1}{4}\right), \left(\frac{1}{1}, \frac{3}{4}\right), \left(\frac{1}{2}, \frac{1}{4}\right), \left(\frac{1}{2}, \frac{3}{4}\right), \left(\frac{1}{4}, \frac{3}{4}\right) \right\}.$$

Example 3. Products of two prime numbers. When $k = pq$ $(p < q)$ is the product of two prime numbers, $\varphi(k) = \varphi(p)\varphi(q)$. For this number we have five divisor pairs

$$(k_1, k_2) \in \{(1, k), (p, q), (p, k), (q, k), (k, k)\},$$

with multiplicities $\varphi(1)\varphi(k)$, $\varphi(p)\varphi(q)$, $\varphi(p)\varphi(k)$, $\varphi(q)\varphi(k)$ and $\varphi(k)(\varphi(k) - 1)/2$, which gives

$$H_P(k) = (p - 1)(q - 1)(pq + p + q)/2.$$

For example, when $k = 6 = 2 \cdot 3$ the solutions are

$$\left(\frac{p_1}{k_1}, \frac{p_2}{k_2}\right) \in \left\{ \left(\frac{1}{1}, \frac{1}{6}\right), \left(\frac{1}{1}, \frac{5}{6}\right), \left(\frac{1}{2}, \frac{1}{3}\right), \left(\frac{1}{2}, \frac{2}{3}\right), \left(\frac{1}{2}, \frac{1}{6}\right), \right.$$
$$\left. \left(\frac{1}{2}, \frac{5}{6}\right), \left(\frac{1}{3}, \frac{1}{6}\right), \left(\frac{1}{3}, \frac{5}{6}\right), \left(\frac{2}{3}, \frac{1}{6}\right), \left(\frac{2}{3}, \frac{5}{6}\right), \left(\frac{1}{6}, \frac{5}{6}\right) \right\}, \tag{2.7}$$

for a total of 11 distinct solutions. Some of the orbits realized for $k = 6$ are plotted in Figure 1. One can notice the geometric variety of shapes produced even for small values of k, which range from regular polygons in Figure 1 (a) to more complex orbits in Figure 1 (b), (c), or (d).

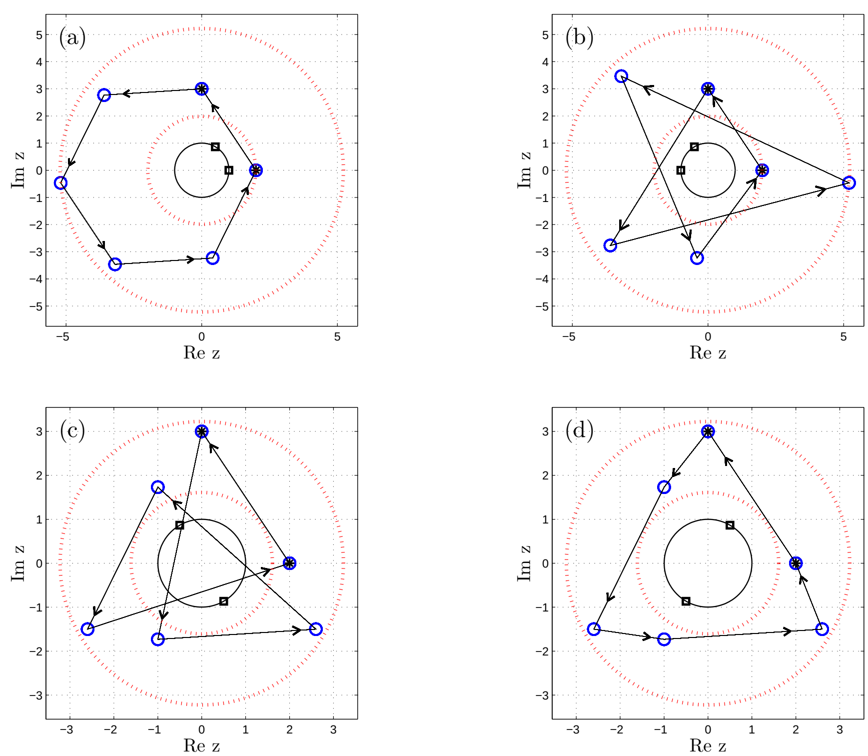

FIGURE 1. The terms of sequence $\{w_n\}_{n=0}^N$ obtained from (1.4) for the pairs $(\frac{p_1}{k_1}, \frac{p_2}{k_2})$ *(a)* $(\frac{1}{1}, \frac{1}{6})$; *(b)* $(\frac{1}{2}, \frac{1}{3})$; *(c)* $(\frac{1}{3}, \frac{5}{6})$; *(d)* $(\frac{2}{3}, \frac{1}{6})$ when $a = 2$ and $b = 3i$ (stars). Arrows indicate the direction of the orbit $w_0, w_1, \ldots, w_6 = w_0$ (circles). Also plotted are generators z_1, z_2 (squares), unit circles (solid line) and boundaries of the annulus $U(0, ||A| - |B||, |A| + |B|)$ (dotted line) with A, B from (1.5).

Example 4. More general numbers. The formula for $k = 12$ involves the divisor pairs

$$(k_1, k_2) \in \{(1, 12), (2, 12), (3, 4), (3, 12), (4, 6), (4, 12), (6, 12), (12, 12)\},$$

with multiplicities $\varphi(p)\varphi(q)$ for each pair (p, q) in the list s.t. $p < q$, and $\varphi(12)(\varphi(12) - 1)/2$ for the pair $(12, 12)$. This gives the formula

$$H_P(12) = 4 + 4 + 4 + 8 + 4 + 8 + 8 + 4 \cdot 3/2 = 46.$$

Even in this example, the number of divisor pairs for periods with more complicated prime decomposition was high. An equivalent (but more direct) formula for $H_P(k)$, which does not require the generation of all quadruples (p_1, k_2, p_2, k_2) is proposed below.

2.4. **A second formula for $H_P(k)$.** One can prove that φ satisfies the formula

$$\varphi(\gcd(k_1, k_2)) \cdot \varphi(\operatorname{lcm}(k_1, k_2)) = \varphi(k_1) \cdot \varphi(k_2), \tag{2.8}$$

which can be used to derive an algorithmic version of (2.4), in the following steps:

- Choose a divisor d of k, s.t. $1 \le d < k$,
- Estimate how many pairs k_1, k_2 satisfy $d = (k_1, k_2)$ and $k = [k_1, k_2]$,
- Sum all the terms $\varphi(d)\varphi(k)$ over d, with the corresponding multiplicity.

Formula (2.4) becomes

$$H_P(k) = \left[\sum_{d|k,\, d<k} \varphi(d) GL(d, k)\right] \varphi(k) + \frac{1}{2}\, \varphi(k)\left(\varphi(k) - 1\right), \tag{2.9}$$

where the arithmetic function $GL(d, k)$ is computed in the following lemma.

Lemma 2.4. *Let $d < k$ be two natural numbers s.t. $d|k$, whose prime decomposition is*

$$d = p_1^{d_1} p_2^{d_2} \cdots p_n^{d_n}, \quad k = p_1^{m_1} p_2^{m_2} \cdots p_n^{m_n}, \quad (1 \le d_i \le m_i).$$

The number of pairs of natural numbers k_1, k_2 which satisfy $d = (k_1, k_2)$ and $k = [k_1, k_2]$ is

$$GL(d, k) = \sharp\{(k_1, k_2) :\ d = (k_1, k_2) \text{ and } k = [k_1, k_2]\} = 2^{\omega(k/d) - 1}, \tag{2.10}$$

where $\omega(x)$ represents the number of distinct prime divisors for the integer x.

Proof. Let the numbers k_1 and k_2 be written as

$$k_1 = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_n^{\alpha_n}, \quad k_2 = p_1^{\beta_1} p_2^{\beta_2} \cdots p_n^{\beta_n}.$$

When $d = (k_1, k_2)$ and $k = [k_1, k_2]$, for each index $i \in \{1, \ldots, n\}$, we have

$$\min\{\alpha_i, \beta_i\} = d_i, \quad \max\{\alpha_i, \beta_i\} = m_i.$$

First, numbers k_1 and k_2 are distinct or otherwise $d = k$. There are two possibilities.

When $d_i = m_i$, one has $\alpha_i = \beta_i = d_i = m_i$. Each choice of $i \in I = \{i \in \{1, \ldots, n\} : d_i < m_i\}$ generates two possible pairs $(\alpha_i, \beta_i) \in \{(d_i, m_i), (m_i, d_i)\}$, hence in total there are $2^{|I|}$ distinct pairs of powers. The number of pairs (k_1, k_2) s.t. $k_1 < k_2$ is therefore $2^{|I|-1}$.

As the prime decomposition of k/d is

$$k/d = p_1^{m_1 - d_1} p_2^{m_2 - d_2} \cdots p_n^{m_n - d_n} = \prod_{i \in I} p_i^{m_i - d_i},$$

one obtains that $|I| = \omega(k/d)$. This ends the proof. □

Theorem 2.5. *Using formula (2.9), $H_P(k)$ can be written more compactly as*

$$H_P(k) = \left[\sum_{d|k,\, d<k} \varphi(d) 2^{\omega(k/d)} + \varphi(k) - 1\right] \frac{\varphi(k)}{2}. \tag{2.11}$$

Example 4. revisited using (2.11). The divisors of $12 = 2^2 \cdot 3$ smaller than 12 are

$$1,\ 2 = 2^1,\ 3 = 3^1,\ 4 = 2^2,\ 6 = 2 \cdot 3.$$

Writing the terms in formula (2.11) explicitly one obtains

$$\left[\varphi(1)2^1 + \varphi(2)2^1 + \varphi(3)2^0 + \varphi(4)2^0 + \varphi(6)2^0\right] \varphi(12) + \frac{\varphi(12)\left(\varphi(12) - 1\right)}{2} = 46. \tag{2.12}$$

Example 5. Square-free numbers. When k is a square-free positive number $k = p_1 p_2 \ldots p_m$ for $m \geq 2$ and $p_1, \ldots, p_m$ prime numbers, a compact formula for $H_P(k)$ can be obtained. Each divisor d of k is given by a product $p_{i_1} p_{i_1} \ldots p_{i_j}$, where $1 \leq i_1 \leq i_2 \leq \ldots \leq i_j \leq m$ for $j = 0, \ldots, m$. The corresponding term in formula (2.11) can further be written as

$$\varphi(d) 2^{\omega(k/d)} = \varphi(p_{i_1}) \varphi(p_{i_2}) \cdots \varphi(p_{i_j}) 2^{m-j}.$$

Summing over all possible divisors d of k one obtains the formula

$$\begin{aligned} H_P(k) &= \left[\sum_{j=0}^{m-1} \left(\sum_{1 \leq i_1 \leq i_2 \leq \cdots \leq i_j \leq m} \varphi(p_{i_1}) \varphi(p_{i_2}) \cdots \varphi(p_{i_j}) \right) 2^{m-j} + \varphi(p_1) \cdots \varphi(p_m) - 1\right] \frac{\varphi(k)}{2} \\ &= \left[(\varphi(p_1) + 2) \cdots (\varphi(p_m) + 2) - 1\right] \frac{\varphi(k)}{2} \\ &= \left[(p_1 + 1) \cdots (p_m + 1) - 1\right] \frac{(p_1 - 1) \cdots (p_m - 1)}{2}, \end{aligned} \tag{2.13}$$

where we have used that $\varphi(k) = \varphi(p_1) \cdots \varphi(p_m)$ and $\varphi(p) = p - 1$ for any prime number p.

For example, when $k = 30 = 2 \cdot 3 \cdot 5$ the number of periodic orbits is

$$H_P(k) = \left[3 \cdot 4 \cdot 6 - 1\right] \frac{1 \cdot 2 \cdot 4}{2} = 284.$$

Remark 2.6. An alternative result for $H_P(k)$ can be obtained using the generator pairs $z_1 = e^{2\pi i p_1/k}$ and $z_2 = e^{2\pi i p_2/k}$ with $1 \leq p_1 < p_2 \leq k$, when these are not necessarily primitive roots of unity. Clearly, $\mathrm{ord}(z_1) = k/(p_1, k)$ and $\mathrm{ord}(z_2) = k/(p_2, k)$. The sequence generated by z_1 and z_2 has period k if $[\mathrm{ord}(z_1), \mathrm{ord}(z_2)] = k$. Using the well-known property $[x, y](x, y) = xy$ (for $x, y \in \mathbb{N}$) for the positive integers $\mathrm{ord}(z_1)$ and $\mathrm{ord}(z_2)$, one obtains the condition

$$k \left(\frac{k}{(p_1, k)}, \frac{k}{(p_2, k)} \right) = \frac{k}{(p_1, k)} \frac{k}{(p_2, k)} \Longleftrightarrow (p_1, k)(p_2, k) \left(\frac{k}{(p_1, k)}, \frac{k}{(p_2, k)} \right) = k. \tag{2.14}$$

From the property $x(y, z) = (xy, xz)$ (for $x, y, z \in \mathbb{N}$), the above relations are equivalent to

$$\Big((p_2, k)k, (p_1, k)k \Big) = k \Longleftrightarrow ((p_1, k), (p_2, k)) = 1.$$

The periodic orbits can therefore be generated from the pairs (p_1, p_2) satisfying the condition

$$H_P(k) = \sharp\{(p_1, p_2) :\ ((p_1, k), (p_2, k)) = 1,\ 1 \leq p_1 < p_2 \leq k\}. \tag{2.15}$$

When written explicitly, this formula yields a result similar to (2.4).

2.5. **Computational comparison of the two formulas for $H_P(k)$.** To evaluate $H_P(k)$ using (2.4), one has to enumerate the ordered pairs of positive integers (k_1, k_2) s.t. $[k_1, k_2] = k$. In the notations of Lemma 2.4, $[k_1, k_2] = k$ becomes $\max\{\alpha_i, \beta_i\} = m_i$ for all $i \in \{1, \ldots, n\}$. As $0 \leq \alpha_i, \beta_i \leq m_i$, there are $(m_i + 1)^2$ pairs (α_i, β_i), of which m_i^2 satisfy $0 \leq \alpha_i, \beta_i \leq m_i - 1$. The number of pairs (α_i, β_i) satisfying $\max\{\alpha_i, \beta_i\} = m_i$ is $(m_i + 1)^2 - m_i^2 = 2m_i + 1$. Considering $i \in \{1, \ldots, n\}$, the number of all divisor pairs (k_1, k_2) is $(2m_1 + 1)(2m_2 + 1) \cdots (2m_n + 1)$. Apart from (k, k) each pair appeared twice, so the number of ordered pairs in formula (2.4) is

$$[(2m_1 + 1)(2m_2 + 1) \cdots (2m_n + 1) + 1]/2.$$

In formula (2.9) one just needs to identify all the distinct divisors d of k, which are exactly

$$(m_1 + 1)(m_2 + 1) \cdots (m_n + 1),$$

and multiply them by the appropriate weights $GL(d, k)$. This suggests that for numbers with many different prime divisors the second formula provides the value $H_P(k)$ in fewer steps.

3. Upper and lower bounds for $H_P(k)$

The first few terms of the sequence $H_P(k)$ are plotted in Figure 2 (a), along with some lower and upper boundaries given by the expressions

$$\frac{\varphi(k)k}{2} \leq H_P(k) \leq \frac{(k-1)k}{2}, \tag{3.1}$$

which can be derived from formulas (2.11) and (2.15) as detailed below.

Formula $k(k-1)/2$ represents the number of pairs (p_1, p_2) satisfying $1 \leq p_1 < p_2 \leq k$ in (2.15), so this is an upper bound for $H_P(k)$ (with equality attained when k is prime (2.5)). As suggested by the referee, the upper bound is attained only for k prime. Whenever $H_P(k) = k(k-1)/2$, all the pairs (p_1, p_2) such that $1 \leq p_1 < p_2 \leq k$ have to satisfy the relation $((p_1, k), (p_2, k)) = 1$ (as shown in (2.15)). If k has a proper divisor $1 < d < k$, then the pair $(p_1, p_2) = (d, k)$ has the property $((d, k), (k, k)) = d$, which is a contradiction.

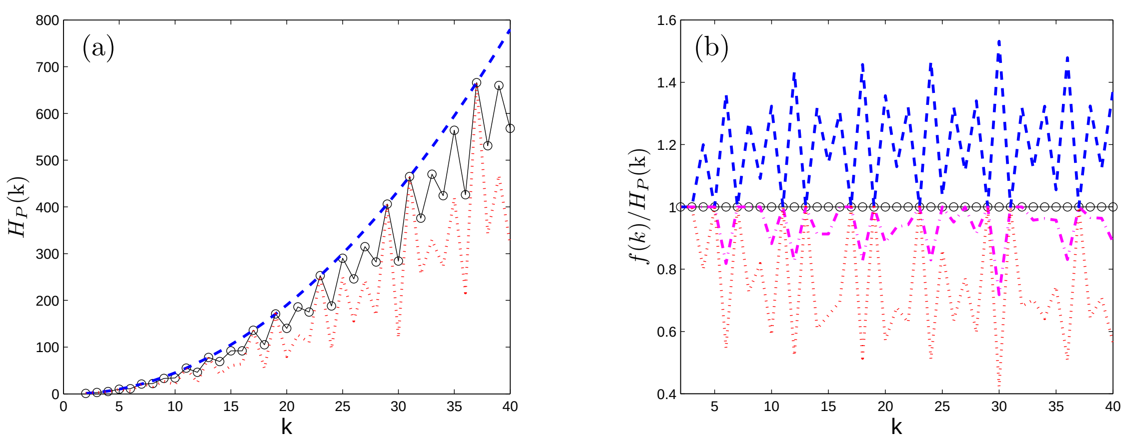

Figure 2. First 40 terms of the sequences (a) $H_P(k)$ (circles), $(k-1)k/2$ (dashed) and $\frac{\varphi(k)k}{2}$ (dotted); (b) $f(k)/H_P(k)$, where $f(k)$ is $H_P(k)$ (circles), $(k-1)k/2$ (dashed), $\frac{\varphi(k)k}{2}$ (dotted) and $\frac{\varphi(k)[2k-\varphi(k)-1]}{2}$ (dash-dotted).

The lower bound can be obtained from (2.11) by writing

$$\begin{aligned} H_P(k) &= \left[\varphi(1)\left(2^{\omega(k)}-1\right) + \sum_{d|k,\,1<d<k} \varphi(d)\left(2^{\omega(k/d)}-1\right) + \sum_{d|k,\,d<k} \varphi(d) + \varphi(k) - 1\right]\frac{\varphi(k)}{2} \\ &\geq \left[1 + \sum_{d|k,\,1<d<k} \varphi(d)\left(2^{\omega(k/d)}-1\right)\right]\frac{\varphi(k)}{2} + (k-1)\frac{\varphi(k)}{2} \geq \frac{\varphi(k)}{2}k. \end{aligned}$$

In the proof we have used the well-known relation $\sum_{d|k,\,1<d<k} \varphi(d) = k$ and that $2^{\omega(k)} \geq 2$. From (2.5), the lower bound is attained when k is a prime number.

As illustrated in Figure 2 (b), a better lower bound for $H_P(k)$ seems to be given by formula $\frac{\varphi(k)[2k-\varphi(k)-1]}{2}$ (2.6), which attains equality when k is a prime power. Here we prove that this is a lower bound for $H_P(k)$ whenever $k = p_1 p_2 \cdots p_m$ is square-free, by using the inequality

$$(p_1+1)(p_2+1)\dots(p_m+1) + (p_1-1)(p_2-1)\dots(p_m-1) \geq 2p_1p_2\cdots p_m, \qquad (3.2)$$

which is clearly true because the terms with negative signs cancel out. This can be written as

$$(p_1+1)(p_2+1)\dots(p_m+1) \geq 2k - \varphi(k), \qquad (3.3)$$

which by using (2.13) gives the following inequality, valid when k is square-free

$$H_P(k) = \left[(p_1+1)\cdots(p_m+1) - 1\right]\frac{(p_1-1)\cdots(p_m-1)}{2} \geq \frac{\varphi(k)[2k-\varphi(k)-1]}{2}. \qquad (3.4)$$

More analysis is required to establish whether this is a lower bound for $H_P(k)$ in general. Other bounds can be obtained using the inequalities for $\varphi(k)$ detailed in [4].

4. Summary

In this article we have identified all possible Horadam sequences with a given period $H_P(k)$. Two equivalent formulas have been proposed for $H_P(k)$, which involved the totient function. The first formula enumerated all the generator pairs, while the second used the divisors of the period to directly count the number of solutions. A new number sequence has been identified, together with some upper and lower bounds.

The research in this paper can be extended towards finding better upper and lower bounds for $H_P(k)$. The complexity analysis of the two formulas can also be examined in more detail. A much more challenging problem is counting the number of periodic general order linear recursions, whose characterization has been done by the authors [2].

The issue of cyclic sequence counting also arises in some other work (to be published at a later date) whereby a method is proposed—driven by the roots of so called Catalan polynomials—to automatically generate sets of periodic Horadam sequences which collectively have an easily identified upper bound in number.

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