MEMBERS OF LUCAS SEQUENCES WHOSE EULER FUNCTION IS A POWER OF 2

MOHAMED TAOUFIQ DAMIR, BERNADETTE FAYE, FLORIAN LUCA, AND AMADOU TALL

Abstract. Here, we show that if $u_0 = 0$, $u_1 = 1$, and $u_{n+2} = ru_{n+1} + su_n$ for all $n \geq 0$ is the Lucas sequence with $s \in \{\pm 1\}$, then there are only finitely many effectively computable n such that $\phi(|u_n|)$ is a power of 2, where ϕ is the Euler function. We illustrate our general result by a few specific examples. This generalizes prior results of the third author and others which dealt with the above problem for the particular Lucas sequences of the Fibonacci and Pell numbers.

1. Introduction

Let $\phi(m)$ be the Euler function of the positive integer m. It is well-known that for $m \geq 3$, the regular polygon with m sides is constructible with the ruler and the compass if and only if $\phi(m)$ is a power of 2. This happens exactly when m is the product of a power of 2 and a square free number all whose prime factors are Fermat primes; i.e., prime numbers of the form $2^{2^n} + 1$ for some $n \geq 0$. For more information on Fermat numbers, see [1].

In [2], Luca found all the Fibonacci numbers whose Euler function is a power of 2. In [3], Luca and Stănică found all the Pell numbers whose Euler function is a power of 2. Here, we prove a more general result which contains the results of [2] and [3] as particular cases. Namely, we consider the Lucas sequence $(u_n)_{n\geq 0}$, with $u_0 = 0,\ u_1 = 1$ and

$$u_{n+2} = ru_{n+1} + su_n \qquad \text{for all} \qquad n \geq 0,$$

where $s \in \{\pm 1\}$ and $r \neq 0$ is an integer. Let $\Delta = r^2 + 4s$ and assume that $\Delta \neq 0$, so, in particular, $(r, s) \neq (\pm 2, -1)$. It is then well-known that if we let

$$(\gamma, \delta) = \left(\frac{r + \sqrt{\Delta}}{2}, \frac{r - \sqrt{\Delta}}{2} \right),$$

then the so-called Binet formula

$$u_n = \frac{\gamma^n - \delta^n}{\gamma - \delta} \qquad \text{holds for all} \qquad n \geq 0. \tag{1.1}$$

We assume that γ/δ is not a root of 1, which happens if $(r, s) \neq (\pm 1, -1)$. Observe that this condition implies that $\Delta = r^2 + 4s > 0$. So, γ and δ are real. If $r < 0$, we may replace (r, s) by $(-r, s)$, whose effect is that it replaces the pair (γ, δ) by the pair $(-\delta, -\gamma)$, so, in particular, u_n by $(-1)^{n-1}u_n$. Such a transformation does not change $|u_n|$. Thus, we may assume that $r > 0$. In this case, we have $\gamma > 1$ and $\delta = -s\gamma^{-1} \in \{-\gamma^{-1}, \gamma^{-1}\}$. Furthermore, $u_n > 0$ for all $n \geq 1$. In fact, we have $u_{n+1} \geq u_n$ for all $n \geq 0$ with the inequality being strict for $n \geq 2$. This is clear if $r = 1$, because then $s = 1$ and so $u_n = F_n$, the nth Fibonacci number, while if

Research supported in part by Projects PAPIIT IN104512, CONACyT Mexico–France 193539, CONACyT Mexico–India 163787, and a Marcos Moshinsky Fellowship.

$r \geq 2$, then, by induction on $n \geq 0$, we have

$$u_{n+2} \geq 2u_{n+1} - u_n = u_{n+1} + (u_{n+1} - u_n) > u_{n+1}.$$

We have the following theorem.

Theorem 1.1. *Assume $s = \pm 1$, $r > 0$ be an integer, $(r, s) \neq (2, -1)$, $(1, -1)$. Suppose $n > 0$ is such that $\phi(u_n)$ is a power of 2. Then writing $n = 2^{a_0} p_1^{a_1} \cdots p_k^{a_k}$, where $3 \leq p_1 < \cdots < p_k$ are distinct primes and a_0, $a_1, \ldots, a_k$ are nonnegative integers, we have that $a_0 \leq 4$ and $p_i^{a_i} < 2(r^2+3)^2$ for all $i = 1, \ldots, k$.*

Example 1.2. *Consider the case when $u_n = F_n$ is the Fibonacci sequence and assume that $\phi(F_n)$ is a power of 2. We have $r = 1$, therefore $p_i^{a_i} < 32$ for $i = 1, \ldots, k$. Since the Euler functions of F_7, F_{11}, F_{13}, F_{17}, F_{19}, F_{23}, F_{25}, F_{27}, F_{29}, F_{31} are not powers of 2, it follows that $p_1^{a_1} \cdots p_k^{a_k}$ is a divisor of $3^2 \times 5$. Finally, since the Euler function of F_8 is not a power of 2, it follows that n is a divisor of $2^2 \times 3^2 \times 5$, and now a very quick calculation shows that $n \in \{1, 2, 3, 4, 5, 6, 9\}$, which is the main result from* [2].

Example 1.3. *Consider the case when $u_n = P_n$, the Pell sequence and assume that $\phi(P_n)$ is a power of 2. Then $r = 2$, so $p_i^{a_i} < 98$ for $i = 1, \ldots, k$. A quick calculation shows that of all odd prime power values of $p^a < 98$, the Euler function of P_{p^a} is a power of 2 only for $p^a = 3$. Further, the Euler function of P_{16} is not a power of 2, so n is a divisor of $2^3 \times 3$. Computing the remaining values, we get that the only values for n are in $\{1, 2, 3, 4, 8\}$, which is the main result in* [3].

2. Preliminary Results

For a nonzero integer m we write $\nu_2(m)$ for the exponent of 2 in the factorization of m. We let $\{v_n\}_{n \geq 0}$ for the companion Lucas sequence of $\{u_n\}_{n \geq 0}$ given by $v_0 = 2$, $v_1 = r$ and $v_{n+2} = rv_{n+1} + sv_n$. Its Binet formula is

$$v_n = \gamma^n + \delta^n \qquad \text{for all} \qquad n \geq 0. \tag{2.1}$$

We have the following results. Recall that $s \in \{\pm 1\}$.

Lemma 2.1. *We have the following relations:*

i) *If $r \equiv 0 \pmod 2$, then*

$$\nu_2(u_n) = \begin{cases} 0 & \text{if } n \equiv 1 \pmod 2, \\ \nu_2(r) + \nu_2(n) - 1 & \text{if } n \equiv 0 \pmod 2, \end{cases}$$

and

$$\nu_2(v_n) = \begin{cases} \nu_2(r) & \text{if } n \equiv 1 \pmod 2, \\ 1 & \text{if } n \equiv 0 \pmod 2. \end{cases}$$

ii) *If $r \equiv 1 \pmod 2$, then*

$$\nu_2(u_n) = \begin{cases} 0 & \text{if } n \not\equiv 0 \pmod 3, \\ \nu_2(r^2+s) & \text{if } n \equiv 3 \pmod 6, \\ \nu_2(r^2+s) + \nu_2(r^2+3s) + \nu_2(n) - 1 & \text{if } n \equiv 0 \pmod 6, \end{cases}$$

and

$$\nu_2(v_n) = \begin{cases} 0 & \text{if } n \not\equiv 0 \pmod 3, \\ v_2(r^2+3s) & \text{if } n \equiv 3 \pmod 6, \\ 1 & \text{if } n \equiv 0 \pmod 6. \end{cases}$$

Proof. i) Say r is even. If $\{w_n\}_{n\geq 0}$ is any binary recurrent sequence of recurrence $w_{n+2} = rw_{n+1} + sw_n$, then $w_{n+2} \equiv w_n \pmod 2$. In particular, w_n has the same parity as w_0 or w_1 if n is even or odd, respectively. Since $v_0 = 2,\ v_1 = r$ are even, it follows that v_n is always even. If $n = 2k$ is even, then

$$v_n = \gamma^{2k} + \delta^{2k} = (\gamma^k + \delta^k)^2 - 2(\gamma\delta)^k = v_k^2 \pm 2$$

is congruent to 2 modulo 4 because $2 \mid v_k$. If $n = 2k+1$, then

$$v_{2k+1} = (\gamma+\delta)\left(\frac{\gamma^{2k+1}+\delta^{2k+1}}{\gamma+\delta}\right) = rw_k \quad \text{where} \quad w_k = c(\gamma^2)^k + d(\delta^2)^k,$$

where $c = \gamma/r,\ d = \delta/r$. Thus, $\{w_k\}_{k\geq 0}$ is a binary recurrent sequence of roots $\gamma^2,\ \delta^2$, whose sum is $\gamma^2+\delta^2 = v_2$ is even and whose product is $\gamma^2\delta^2 = 1$. By the remark at the beginning of the proof, w_k has the same parity as w_0 or w_1 if k is even and odd, respectively, and since $w_0 = 1,\ w_1 = \gamma^2+\delta^2-\gamma\delta = v_2 \pm 1$ is also odd, it follows that w_k is always odd. This shows that $\nu_2(v_{2k+1}) = \nu_2(r)$ and takes care of the parity of v_n. For u_n, since $u_0 = 0,\ u_1 = 1$, it follows that u_n is even or odd according to whether n is even or odd, respectively. If n is even and we write $n = 2^k\ell$ with $k \geq 1$ and ℓ odd, then

$$u_n = u_{2^k\ell} = \frac{\gamma^{2^k}-\delta^{2^k}}{\gamma-\delta}\left(\frac{(\gamma^{2^k})^\ell-(\delta^{2^k})^\ell}{\gamma^{2^k}-\delta^{2^k}}\right) = v_1v_2\cdots v_{2^{k-1}}\left(\frac{(\gamma^{2^k})^\ell-(\delta^{2^k})^\ell}{\gamma^{2^k}-\delta^{2^k}}\right).$$

Since $v_1 = r$, and v_{2^i} is congruent to 2 modulo 4 for all $i = 1,\ldots,k-1$, the part about $\nu_2(u_n)$ when n is even follows provided that we show that the factor in the parenthesis above is odd. But this is w_ℓ, where now $\{w_n\}_{n\geq 0}$ is the Lucas sequence of roots γ^{2^k} and δ^{2^k}, the sum of which is v_{2^k} which is even and the product of which is $(\gamma\delta)^{2^k} = 1$, and now the fact that w_ℓ is odd when ℓ is odd follows by the argument at the beginning of the proof, because $w_1 = 1$ is odd. This takes care of (i).

ii) Say r is odd. Then $u_{n+2} \equiv u_{n+1} + u_n \pmod 2$ and the same is true for $\{v_n\}_{n\geq 0}$. Since $v_0 \equiv u_0 \equiv 0 \pmod 2$ and $v_1 \equiv u_1 \equiv 1 \pmod 2$, it follows that both u_n and v_n have the same parity as F_n, the nth Fibonacci number, which is even if and only if $3 \mid n$. This takes care of ii) when $3 \nmid n$. Now take $n = 3k$. Then

$$u_n = \frac{\gamma^{3k}-\delta^{3k}}{\gamma-\delta} = \frac{\gamma^3-\delta^3}{\gamma-\delta}\left(\frac{(\gamma^3)^k-(\delta^3)^k}{\gamma^3-\delta^3}\right) = (r^2+s)w_k,$$

where $\{w_n\}_{n\geq 0}$ is the Lucas sequence of roots $\gamma^3 + \delta^3$ the sum of which is $r(r^2+3s)$, which is even and for which $\nu_2(r(r^2+3s)) = \nu_2(r^2+3s)$ and the product of which is $(\gamma\delta)^3 = -s^3$. Similarly,

$$v_n = (\gamma^3)^k + (\delta^3)^k$$

is the companion Lucas sequence of $\{w_n\}_{n\geq 0}$. Since this new Lucas sequence has the property that its sum of roots (namely, its corresponding "r") is r^2+3s which is even, the results from i) apply to w_k and its companion and give ii). $\square$

Lemma 2.2. *We have the following relations:*

i) *If* $r \equiv 0 \pmod 2$ *and* $k \geq 2$, *then*

$$\nu_2(v_{2^k}-2) = v_2(r^2+4s) + 2v_2(r) + 2k - 4.$$

ii) *If* $r \equiv 1 \pmod 2$ *and* $k \geq 2$, *then*

$$v_{2^k} \equiv 7 \pmod 8.$$

Proof. For $k \geq 2$, we write

$$v_{2^k} - 2 = \gamma^{2^k} + \delta^{2^k} - 2 = (\gamma^{2^{k-1}} - \delta^{2^{k-1}})^2 = \Delta u_{2^{k-1}}^2, \tag{2.2}$$

where $\Delta = r^2 + 4s = (\gamma - \delta)^2$. Thus, if r is even, we get, by Lemma 2.1, that

$$\nu_2(v_{2^k} - 2) = \nu_2(\Delta) + 2\nu_2(u_{2^{k-1}}) = \nu_2(r^2 + 4s) + 2\nu_2(r) + 2(k-2).$$

If r is odd, then $\Delta = r^2 + 4s \equiv 5 \pmod 2$ and $u_{2^{k-1}}$ is odd, by Lemma 2.1, so that the right-hand side of formula (2.2) is congruent to $5 \pmod 8$, which yields $v_{2^k} \equiv 7 \pmod 8$. □

Lemma 2.3. *Let a, b be nonnegative integers with $a \equiv b \pmod 2$. Then*

$$u_a - u_b = \begin{cases} u_{(a-b)/2} v_{(a+b)/2} & \textit{if} \quad s = 1 \quad \textit{or} \quad a \equiv b \pmod 4, \\ u_{(a+b)/2} v_{(a-b)/2} & \textit{if} \quad s = -1 \quad \textit{and} \quad a \equiv b + 2 \pmod 4). \end{cases}$$

Proof. Straightforward verification using Binet's formulas (1.1) and (2.1). □

3. Proof of Theorem 1.1

We use the fact that if $\phi(m)$ is a power of 2 and d is a divisor of m, then $\phi(d)$ is a power of 2 as well. We assume that $n > 1$, $\phi(u_n)$ is a power of 2 and $p^a \| n$ and we want to bound p^a. We proceed in various steps.

Case 1. *p is odd and $p \mid \Delta$.*

It is well-known that $p \mid u_n$. Furthermore, if $p^2 \mid n$, then $p^2 \mid u_n$. Since $\phi(u_n)$ is a power of 2, it follows that it is not possible that $p^2 \mid n$, therefore $a \leq 1$. Thus, in this case

$$p^a \leq p \leq \Delta = r^2 + 4s < (r^2 + 3)^2.$$

Case 2. *$p \geq 5$ and $p \nmid \Delta$.*

We consider the number $u_{p^a}/u_{p^{a-1}}$, which is a divisor of u_n. Since it is also a divisor of u_{p^a} and $p \geq 5$, it follows, by Lemma 2.1, that $u_{p^a}/u_{p^{a-1}}$ is an odd number larger than 1 because $u_{m+1} > u_m$ for all $m \geq 2$. Since the Euler function of the odd number $u_{p^a}/u_{p^{a-1}} > 1$ is a power of 2, it can be written as

$$\frac{u_{p^a}}{u_{p^{a-1}}} = q_1 q_2 \cdots q_t, \quad \text{where} \quad q_i = 2^{2^{n_i}} + 1 \quad \text{is prime for} \quad 1 \leq i \leq t. \tag{3.1}$$

We assume that $n_1 < \cdots < n_t$. We look at the smallest prime factor q_1 of $u_{p^a}/u_{p^{a-1}}$. Since $p \nmid \Delta$, it follows that q_1 is *primitive* for u_{p^a}. In particular, $q_1 \equiv \pm 1 \pmod{p^a}$. If $q_1 \equiv 1 \pmod{p^a}$, then, since $q_1 = 2^{2^{n_1}} + 1$, it follows that $2^{2^{n_1}} + 1 \equiv 1 \pmod{p^a}$. Thus, $p \mid 2^{2^{n_1}}$, which is false. Hence, $q_1 \equiv -1 \pmod{p^a}$, therefore

$$2^{2^{n_1}} + 1 = -1 + p^a \ell \qquad \text{for some integer} \quad \ell. \tag{3.2}$$

Since $p \geq 5$, it follows that $n_1 \geq 2$. Further, reducing the above relation modulo 4, we get that $2 \| \ell$. Thus, we have that

$$p^a \leq \frac{2^{2^{n_1}} + 2}{\ell} \leq 2^{2^{n_1} - 1} + 1.$$

Since the number $2^{2^{n_1} - 1} + 1$ is a multiple of 3 and $p \geq 5$, the above inequality implies that in fact

$$p^a < 2^{2^{n_1} - 1}. \tag{3.3}$$

We now use a 2-adic argument to bound n_1 in terms of p. Namely, performing the multiplication on the right-hand side of (3.1) above, we get that the right-hand side of (3.1) is congruent to $1 + 2^{2^{n_1}} \pmod{2^{2^{n_1}+1}}$. Hence,

$$2^{n_1} = \nu_2(u_{p^a}/u_{p^{a-1}} - 1) = \nu_2((u_{p^a} - u_{p^{a-1}})/u_{p^{a-1}}). \tag{3.4}$$

Since $u_{p^{a-1}}$ is odd, we get that

$$2^{n_1} = \nu_2(u_{p^a} - u_{p^{a-1}}).$$

By Lemma 2.3, we get that

$$u_{p^a} - u_{p^{a-1}} = u_{p^{a-1}(p+\varepsilon)/2} v_{p^{a-1}(p-\varepsilon)/2} \qquad \text{for some} \qquad \varepsilon \in \{\pm 1\}.$$

Since $p \ge 5$, exactly one of $p^{a-1}(p \pm 1)/2$ is even and the other is odd, and exactly one is a multiple of 3 and the other is not. Invoking Lemma 2.1, we get that

$$2^{n_1} = \nu_2(u_{p^a} - u_{p^{a-1}}) \le \max\{\nu_2(u_{(p+\varepsilon)/2}), \nu_2(v_{(p-\varepsilon)/2})\} + \nu_2(r). \tag{3.5}$$

The extra term $\nu_2(r)$ in fact appears only when r is even and $(p+\varepsilon)/2$ is also even. Let $A = \nu_2(r) + \nu_2(r^2+s) + \nu_2(r^2+3s)$. Note that $A \ge 1$. We distinguish two cases.

Case 2.1 *The maximum on the right–hand side of* (3.5) *is at most* A.

In this case, $2^{2^{n_1}-1} \le 2^{A+\nu_2(r)-1}$. If r is even, then $A = \nu_2(r)$, and therefore $2^{A+\nu_2(r)-1} \le r^2/2$. If r is odd, then $A = \nu_2(r^2+s) + \nu_2(r^2+3s)$, and since $(r^2+3s) - (r^2+s) = 2s = \pm 2$, it follows that $\min\{\nu_2(r^2+s), \nu_2(r^2+3s)\} = 1$. Hence,

$$2^{A+\nu_2(r)-1} \le \max\{r^2/2, r^2+3s, r^2+s\} \le r^2+3. \tag{3.6}$$

By inequality (3.3), we get that

$$p^a < 2^{2^{n_1}-1} \le 2^{A+\nu_2(r)-1} \le r^2+3. \tag{3.7}$$

Case 2.2 *The maximum on the right-hand side of* (3.5) *exceeds* A.

A quick look at Lemma 2.1, shows that this case occurs only if the above maximum is at $\nu_2(u_{(p+\varepsilon)/2})$. Further, the condition $\nu_2(u_{(p+\varepsilon)/2}) > A$ implies that $\nu_2((p+\varepsilon)/2) \ge 2$. Thus,

$$p + \varepsilon = 2^{\alpha+1}k \quad \text{holds with some odd number} \quad k \quad \text{and some} \quad \alpha \ge 2, \tag{3.8}$$

and relation (3.5) and Lemma 2.1 give

$$2^{n_1} = B + \alpha - 1 \tag{3.9}$$

for some $1 \le B \le A + \nu_2(r)$. In fact, it is easy to deduce that $B = A + \nu_2(r)$, but we shall not need this precise information. Thus, also using relation (3.2), we get

$$-2 + p^a\ell = 2^{2^{n_1}} = 2^{B+\alpha-1} = 2^{B-1} \times 2^\alpha = 2^{B-1}\left(\frac{p+\varepsilon}{2k}\right).$$

Thus, we get that

$$p(2k\ell p^{a-1} - 2^{B-1}) = 4k + \varepsilon 2^{B-1}. \tag{3.10}$$

Assume first that the left-hand side of the formula (3.10) above is 0. Then $2k\ell p^{a-1} = 2^{B-1}$. Since k is odd, $2\|\ell$, the only possibility is $\ell = 2$, $a = 1$, $k = 1$, $B = 3$. We then get $2^{2^{n_1}} + 1 = -1 + 2p$, therefore $p = 2^{2^{n_1}-1} + 1$, which is a multiple of 3, a contradiction. Thus, the left-hand side of equation (3.10) is nonzero. If $k \ge 2^{B-2}$, then $2k\ell \ge 4k \ge 2^B$, so $2k\ell p^{a-1} - 2^{B-1} \ge 2k$, so (3.10) gives

$$p \le \frac{4k + 2^{B-1}}{2k} = 2 + \frac{2^{B-2}}{k} \le 3,$$

a contradiction. Thus, $k < 2^{B-2}$, so, by (3.10) again,

$$p < 4 \cdot 2^{B-2} + 2^{B-1} \le 3 \times 2^{B-1} \le 3 \times 2^{A+\nu_2(r)-1} \le 3(r^2+3),$$

where for the last inequality we have used inequality (3.6). Thus,

$$2^\alpha = \frac{p+\varepsilon}{2k} < 2(r^2+3),$$

therefore,

$$2^{2^{n_1}} = 2^{B+\alpha-1} \le 2^{A+\nu_2(r)-1} 2^\alpha \le (r^2+3) \times (2(r^2+3)) = 2(r^2+3)^2,$$

getting, by (3.3), that

$$p^a < 2^{2^{n_1}-1} \le (r^2+3)^2,$$

which is what we wanted to prove.

Case 3. $p = 3$ *and* $p \nmid \Delta$.

Up to some minor particularities, this case is similar to Case 2. We work again with $u_{3^a}/u_{3^{a-1}}$. If $a = 1$, then $3^a = 3 < (r^2+3)^2$, which is what we wanted. Suppose that $a \ge 2$. If r is even by Lemma 2.1, it follows that u_{3^a} is odd, so $u_{3^a}/u_{3^{a-1}}$ is also odd. If r is odd, then $\nu_2(u_{3^a}) = \nu_2(r^2+s) = \nu_2(u_{3^{a-1}})$, so $u_{3^a}/u_{3^{a-1}}$ is also odd and it is larger than 1. We again write equation (3.1), as well as its conclusion (3.2). If $n_1 = 1$, we get $-1 + 3^a\ell = 2^{2^1} + 1 = 5$, showing that $3^a \mid 6$, so $a = 1$, which is not the case we are treating. Thus, $n_1 \ge 2$, and (3.3) gives

$$3^a < 2^{2^{n_1}-1}. \tag{3.11}$$

Equation (3.3) is

$$2^{n_1} = \nu_2(u_{3^a}/u_{3^{a-1}} - 1) = \nu_2((u_{3^a} - u_{3^{a-1}})/u_{3^{a-1}}).$$

Since $3^a \equiv 3^{a-1} + 2 \pmod 4$, we have, by Lemma 2.3,

$$u_{3^a} - u_{3^{a-1}} = \begin{cases} u_{3^{a-1}} v_{2\times 3^{a-1}} & \text{if} \quad s = 1 \\ u_{2\times 3^{a-1}} v_{3^{a-1}} & \text{if} \quad s = -1. \end{cases}$$

In particular,

$$\frac{u_{3^a} - u_{3^{a-1}}}{u_{3^{a-1}}} = v_{2\times 3^{a-1}} \quad \text{or} \quad v_{3^{a-1}}^2$$

according to whether $s = 1$ or $s = -1$. If r is even, we deduce, by Lemma 2.1, that $2^{n_1} \le 2A$. If r is odd, then, again by Lemma 2.1, we deduce that $2^{n_1} \le 2\nu_2(r^2+3s) \le 2A - 2$. Hence, at any rate, $2^{n_1} \le 2A$, therefore

$$2^{2^{n_1}} \le 2^{2A} = 4 \times (2^{A-1})^2 \le 4(r^2+3)^2,$$

where we used again inequality (3.6). By (3.11), we get

$$3^a < 2^{2^{n_1}-1} \le 2(r^2+3)^2,$$

which is what we wanted.

Case 4. $p = 2$.

In this case, $u_{2^a} \mid u_n$. Assume that $a \ge 5$. Then

$$u_{2^a} = v_1 v_2 \cdots v_{2^{a-1}}.$$

Assume that r is odd. Lemma 2.2 shows that both v_4 and v_8 are congruent to 7 (mod 8). Since the only Fermat prime which is congruent to 3 modulo 4 is 3, and each one of v_4 and

v_8 is a product of distinct Fermat primes, it follows easily that $3 \mid v_4$ and $3 \mid v_8$, so $9 \mid u_n$, a contradiction. So, in fact, $a \leq 4$ in this case.

Assume next that r is even and $a \geq 5$. Then $v_4 v_8 v_{16}$ is a divisor of u_{2^a} and in particular its Euler function is a power of 2. By Lemma 2.2, we have

$$\nu_2(v_4 - 2) = \nu_2(r^2 + 4s) + 2\nu_2(r),$$
$$\nu_2(v_8 - 2) = \nu_2(r^2 + 4s) + 2\nu_2(r) + 2,$$
$$\nu_2(v_{16} - 2) = \nu_2(r^2 + 4s) + 2\nu_2(r) + 4.$$

Writing $b = \nu_2(r^2 + 4s) + 2\nu_2(r)$, we get that

$$v_4 = 2q_1 \cdots q_t \quad \text{with} \quad q_i = 2^{2^{n_i}} + 1 \quad \text{where} \quad n_1 < \cdots < n_t,$$
$$v_8 = 2q'_1 \cdots q'_{t'} \quad \text{with} \quad q'_i = 2^{2^{n'_i}} + 1 \quad \text{where} \quad n'_1 < \cdots < n'_{t'},$$
$$v_{16} = 2q''_1 \cdots q''_{t''} \quad \text{with} \quad q''_i = 2^{2^{n''_i}} + 1 \quad \text{where} \quad n''_1 < \cdots < n''_{t''},$$

and where furthermore $2^{n_1} = b,\ 2^{n'_1} = b + 2,\ 2^{n''_1} = b + 4$ and the sets

$$\{n_1, \ldots, n_t\}, \quad \{n'_1, \ldots, n'_{t'}\} \quad \text{and} \quad \{n''_1, \ldots, n''_{t''}\}$$

are mutually disjoint. Hence, $2^{n_1} + 2^{n''_1} = 2^{n'_1+1} (= 2b + 4)$, with distinct $n_1,\ n'_1,\ n''_1$, which is impossible by the uniqueness of the base 2 representation. This contradiction shows that in fact $a \leq 4$.

4. Acknowledgements

We thank the referee for a careful reading of the paper and for comments and suggestions which improved its quality. This paper was written during a visit of F. L. at the AIMS-Senegal in February of 2013. He thanks the people of this institution for their hospitality.

References

[1] M. Křížek, F. Luca, and L. Somer, *17 Lectures on Fermat Numbers. From Number Theory to Geometry, with a foreword by Alena Šolcová*, CMS Books in Mathematics/Ouvrages de Mathématiques de la SMC, **9**, Springer-Verlag, New York, 2001.

[2] F. Luca, *Equations involving arithmetic functions of Fibonacci numbers*, The Fibonacci Quarterly, **38.1** (2000), 49–55.

[3] F. Luca and P. Stănică, *Equations with arithmetic functions of Pell numbers*, Preprint, 2012.

MSC2010: 11A25, 11B39

AIMS-Sénégal, Km 2 route de Joal (Centre IRD Mbour), BP: 64566 Dakar-Fann, Sénégal
E-mail address: m.taoufiq.damir@aims-senegal.org

AIMS-Sénégal, Km 2 route de Joal (Centre IRD Mbour), BP: 64566 Dakar-Fann, Sénégal
E-mail address: bernadette@aims-senegal.org

Centro de Ciencias Matemáticas UNAM, Ap. Postal 61-3 (Xangari), CP 58089, Morelia, Michoacán, MEXICO
E-mail address: fluca@matmor.unam.mx

AIMS-Sénégal, Km 2 route de Joal (Centre IRD Mbour), BP: 64566 Dakar-Fann, Sénégal
E-mail address: amadou.tall@aims-senegal.org