NOTE ON KUMMER'S CONGRUENCES FOR EULER NUMBERS

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Abstract. It is shown how Kummer's congruences for Euler numbers follow directly from the corresponding congruences for generalized Bernoulli numbers.

Euler numbers E_n can be defined by the generating function

$$\frac{2}{e^t + e^{-t}} = \sum_{n=0}^{\infty} E_n \frac{t^n}{n!} \qquad (|t| < \pi).$$

These numbers, particularly their arithmetic properties, have been studied quite intensely in recent times; see, e.g., [3, 4, 6] and the references given therein. It is well-known that E_n is an integer and it vanishes for all odd indices n. The sequence of nonzero E_n begins with $1, -1, 5, -61, \ldots$.

Among fundamental properties of Euler numbers are the Kummer congruences which in their simple form relate E_n to $E_{n+\phi(p^h)}$ modulo p^h, where p is an odd prime, $h \geq 1$, and ϕ denotes the Euler totient function. The general form of Kummer congruences will be presented below. In recent literature, these congruences (in varying forms) are often proved by different methods (e.g. [3, 4]). These are interesting in their own right, but we find it worth observing that the congruences are special cases of the corresponding congruences for the generalized Bernoulli numbers, and the latter have a very natural source in the theory of p-adic L-functions (see the last paragraph). The aim of this note is to remind of this connection.

The generalized Bernoulli numbers $B_n(\chi)$ belonging to a given (primitive) Dirichlet character χ modulo f are defined by the generating function

$$\sum_{a=1}^{f} \frac{\chi(a) t e^{at}}{e^{ft} - 1} = \sum_{n=0}^{\infty} B_n(\chi) \frac{t^n}{n!}.$$

Let χ_4 be the unique nontrivial character mod 4. Its nonzero values are $\chi_4(a) = (-1)^{(a-1)/2}$ (with a odd), so that the generating function of $B_n(\chi_4)$ is

$$\frac{te^t - te^{3t}}{e^{4t} - 1} = -\frac{te^t}{e^{2t} + 1} = -\frac{t}{e^t + e^{-t}}.$$

Comparing this with the generating function of E_n we see that

$$-\frac{t}{e^t + e^{-t}} = -\frac{1}{2} \sum_{n=0}^{\infty} E_n \frac{t^{n+1}}{n!} = -\frac{1}{2} \sum_{n=0}^{\infty} n E_{n-1} \frac{t^n}{n!}.$$

Thus we have the following result.

Proposition. *For $n \geq 1$,*

$$\frac{B_n(\chi_4)}{n} = -\frac{1}{2} E_{n-1}.$$

Remark. This result is certainly not new. It is given, for example, in [3] and [4], although in a different form by means of values of Euler (and Bernoulli) polynomials. Observe that the value of the Euler polynomial $E_n(X)$ at $X = \frac{1}{2}$ equals $2^{-n}E_n$.

The Kummer congruences for $B_n(\chi)$ read

$$\Delta_c^k(1-\chi(p)p^{n-1})\frac{B_n(\chi)}{n} \equiv 0 \pmod{p^{hk}},$$

whenever $n \geq 1$, $k \geq 1$, $h \geq 1$ and c is divisible by $\phi(p^h)$; in the case of characters χ modulo p-power there are moreover some further restrictions on n. Here Δ_c denotes the difference operator with span $c(\geq 1)$, that is, $\Delta_c x_n = x_{n+c} - x_n$. Note that operating k times by Δ_c gives

$$\Delta_c^k x_n = \sum_{t=0}^{k}(-1)^{k-t}\binom{k}{t}x_{n+tc}.$$

For references and additional information on the above congruences, see the final paragraphs below.

By Proposition, our congruences with $\chi = \chi_4$ imply that

$$\Delta_c^k\bigl(1-(-1)^{(p-1)/2}p^n\bigr)E_n \equiv 0 \pmod{p^{hk}}$$

for any $n \geq 0$, $k \geq 1$, $h \geq 1$, c divisible by $\phi(p^h)$. This is the general form of Kummer's congruences for Euler numbers, provided $p > 2$. In the particular case $k = 1$ one has the "short" Kummer congruences appearing most commonly in the literature:

$$E_{n+c} \equiv \bigl(1-(-1)^{(p-1)/2}p^n\bigr)E_n \pmod{p^h} \qquad (n \geq 0,\ h \geq 1,\ \phi(p^h)|c).$$

Indeed, observe that $c \geq \phi(p^h) \geq h$ and so $p^{n+c} \equiv 0 \pmod{p^h}$.

As a first simple application, the reader is invited to write down this congruence for $n = 0$, to get a result derived in many papers by another method (e.g., [2, 6]).

Note in passing that the more famous Kummer congruences for (ordinary) Bernoulli numbers B_n are obtained by choosing $\chi = 1$, the trivial character mod 1. The result is

$$\Delta_c^k(1-p^{n-1})\frac{B_n}{n} \equiv 0 \pmod{p^{hk}} \qquad (n \geq 1, k \geq 1, h \geq 1, \phi(p^h)|c),$$

except for the case $n \equiv 0 \pmod{p-1}$ which has to be excluded.

Kummer's congruences for generalized Bernoulli numbers were first proved by L. Carlitz [1]. (In his article the relationship of $B_n(\chi_4)$ to the Euler numbers is incorrect.) There have been many subsequent proofs, one quite recent by P. T. Young [9, 10]. In fact, the results of Young provide an interesting generalization of the original Kummer congruences for $B_n(\chi)$.

The Kummer congruences appear in the theory of p-adic L-functions $L_p(s,\chi)$ of Kubota–Leopoldt in the following way. First, the values of these functions at nonpositive integers $1-n$ (for $n = 1, 2, \ldots$) are equal to $(1-\psi(p)p^{n-1})B_n(\psi)/n$, where ψ depends on χ and on the residue class of n modulo $p-1$. Therefore, the Kummer congruences manifest the continuity (in the p-adic sense, of course) of these functions $L_p(s,\chi)$; one has only to exclude the case $n \equiv 0 \pmod{p-1}$ for $\chi = 1$ corresponding to the pole of $L_p(s,1)$ at $s = 1$. This point of view is discussed, e.g., in [7] and [8]. For more about p-adic L-functions, the reader is referred to [5], for example.

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