A CONNECTION BETWEEN π AND ϕ

MICHAEL D. HIRSCHHORN

ABSTRACT. We find an expression for π as a limit involving the golden ratio ϕ.

1. Introduction

We prove that

$$\sum_{k=0}^{n} \binom{n}{k}^2 \binom{n+k}{n}$$
$$\sim \frac{\phi^{5n+\frac{5}{2}}}{2\pi n \sqrt[4]{5}} \left(1 - \frac{5-\sqrt{5}}{10n} + \frac{13-5\sqrt{5}}{50n^2} - \frac{175-83\sqrt{5}}{1250n^3} + \frac{437-205\sqrt{5}}{6250n^4} - \cdots\right)$$

as $n \to \infty$, where ϕ is the golden ratio, and consequently,

$$\frac{1}{\pi} = \lim_{n\to\infty} 2n\sqrt[4]{5} \sum_{k=0}^{n} \binom{n}{k}^2 \binom{n+k}{n} \Big/ \phi^{5n+\frac{5}{2}}.$$

In order to carry out this program, we use the methods developed in two earlier papers on the Apéry numbers [2, 3].

2. The Dominant Term

The first step is to find the value of k for which the term $\binom{n}{k}^2 \binom{n+k}{n}$ is a maximum. We do this by setting

$$\binom{n}{k}^2 \binom{n+k}{n} = \binom{n}{k+1}^2 \binom{n+k+1}{n}.$$

This yields

$$\frac{n!(n+k)!}{k!^3(n-k)!^2} = \frac{n!(n+k+1)!}{(k+1)!^3(n-k-1)!^2},$$

or

$$(k+1)^3 = (n+k+1)(n-k)^2.$$

If we suppose $k = \theta n$, where θ is to be determined, and divide by n^3, we find

$$\left(\theta + \frac{1}{n}\right)^3 = \left(1 + \theta + \frac{1}{n}\right)(1-\theta)^2.$$

If we let $n \to \infty$, this becomes

$$\theta^3 = (1+\theta)(1-\theta)^2,$$

or

$$1 - \theta - \theta^2 = 0.$$

It follows that

$$\theta = \frac{\sqrt{5}-1}{2} = \frac{1}{\phi},$$

where ϕ is the golden ratio.

Thus, the value of k that we seek is given by

$$k \approx \theta n$$

where $\theta = \dfrac{1}{\phi}$.

At $k \approx \theta n$, the value of the term is

$$\begin{aligned}
H &= \binom{n}{\theta n}^2 \binom{n+\theta n}{n} \\
&= \frac{n!(n+\theta n)!}{(\theta n)!^3 (n-\theta n)!^2} \\
&\approx \frac{\sqrt{2\pi n}\left(\frac{n}{e}\right)^n \sqrt{2\pi(1+\theta)n}\left(\frac{(1+\theta)n}{e}\right)^{(1+\theta)n}}{\left(\sqrt{2\pi\theta n}\left(\frac{\theta n}{e}\right)^{\theta n}\right)^3 \left(\sqrt{2\pi(1-\theta)n}\left(\frac{(1-\theta)n}{e}\right)^{(1-\theta)n}\right)^2} \\
&= \left(\frac{1}{\sqrt{2\pi n}}\right)^3 \frac{\sqrt{1+\theta}}{(\sqrt{\theta})^3(\sqrt{1-\theta})^2}\left(\frac{(1+\theta)^{1+\theta}}{\theta^{3\theta}(1-\theta)^{2(1-\theta)}}\right)^n
\end{aligned}$$

after considerable simplification.

Now,

$$1-\theta = \theta^2 \quad \text{and} \quad 1+\theta = \frac{1}{\theta}$$

so

$$\frac{\sqrt{1+\theta}}{(\sqrt{\theta})^3(\sqrt{1-\theta})^2} = \frac{1}{1-\theta}\sqrt{\frac{1+\theta}{\theta^3}} = \frac{1}{\theta^2}\sqrt{\frac{1}{\theta^4}} = \frac{1}{\theta^4} = \phi^4$$

and

$$\frac{(1+\theta)^{1+\theta}}{\theta^{3\theta}(1-\theta)^{2(1-\theta)}} = \frac{1}{\theta^{1+\theta}\theta^{3\theta}(\theta^2)^{2(1-\theta)}} = \frac{1}{\theta^5} = \phi^5.$$

So

$$H \approx \frac{\phi^{5n+4}}{(2\pi n)^{\frac{3}{2}}},$$

(see Figure 1.).

At points near θn, the terms of the sum are given by

$$\begin{aligned}
\binom{n}{\theta n+k}^2\binom{n+\theta n+k}{n} &= H \cdot \binom{n}{\theta n+k}^2\binom{n+\theta n+k}{n} \Big/ \binom{n}{\theta n}^2\binom{n+\theta n}{n} \\
&= H \cdot \frac{n!(n+\theta n+k)!}{(\theta n+k)!^3(n-\theta n-k)!^2} \Big/ \frac{n!(n+\theta n)!}{(\theta n)!^3(n-\theta n)!^2} \\
&= H \cdot \frac{(n+\theta n+1)\cdots(n+\theta n+k)(n-\theta n)^2\cdots(n-\theta n-k+1)^2}{(\theta n+1)^3(\theta n+2)^3\cdots(\theta n+k)^3}
\end{aligned}$$

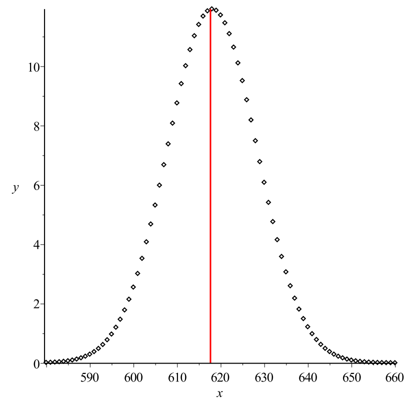

FIGURE 1. The case $n = 1000$, showing the points $(k, \binom{n}{k}^2\binom{n+k}{n})$ for $580 \le k \le 660$, together with the vertical $x = \dfrac{n}{\phi}$, $0 \le y \le \dfrac{\phi^{5n+4}}{(2\pi n)^{\frac{3}{2}}}$.

$$= H \cdot \frac{\left(1+\frac{1}{(1+\theta)n}\right)\cdots\left(1+\frac{k}{(1+\theta)n}\right)\left(1-\frac{1}{(1-\theta)n}\right)^2\cdots\left(1-\frac{k-1}{(1-\theta)n}\right)^2}{\left(1+\frac{1}{\theta n}\right)^3\cdots\left(1+\frac{k}{\theta n}\right)^3}$$
$$\approx H\exp\left\{\frac{1}{(1+\theta)n}(1+\cdots+k) - \frac{2}{(1-\theta)n}\left(1+\cdots+(k-1)\right) - \frac{3}{\theta n}(1+\cdots+k)\right\}$$
$$\approx H\exp\left\{-\frac{k^2}{2n}\left(\frac{2}{1-\theta}+\frac{3}{\theta}-\frac{1}{1+\theta}\right)\right\}$$
$$= H\exp\left\{-\frac{k^2}{2n}\phi^3\sqrt{5}\right\}$$

since

$$\frac{(1+\theta)^k(1-\theta)^{2k}}{\theta^{3k}} = \frac{(\theta^2)^{2k}}{\theta^k\cdot\theta^{3k}} = 1$$

and

$$\begin{aligned}\frac{2}{1-\theta}+\frac{3}{\theta}-\frac{1}{1+\theta} &= \frac{2}{\theta^2}+\frac{3}{\theta}-\theta = \frac{2+3\theta-\theta^3}{\theta^2} = \frac{2+3\theta-\theta(1-\theta)}{\theta^2}\\ &= \frac{2+2\theta+\theta^2}{\theta^2} = (2+2\theta+\theta^2)\phi^2 = 2\phi^2+2\phi+1 = 4\phi+3\\ &= 4\left(\frac{\sqrt{5}+1}{2}\right)+3 = 2\sqrt{5}+5 = (2+\sqrt{5})\sqrt{5} = \phi^3\sqrt{5}.\end{aligned}$$

Thus, the terms are essentially distributed normally, with $\sigma^2 = \dfrac{n}{\phi^3\sqrt{5}}$, and the sum is given by

$$\begin{aligned}\sum_{k=0}^{n}\binom{n}{k}^2\binom{n+k}{n} &\approx H\int_{-\infty}^{\infty}\exp\left\{-\frac{x^2}{2\sigma^2}\right\}dx = H\cdot\sigma\sqrt{2\pi}\\ &\approx \frac{\phi^{5n+4}}{(2\pi n)^{\frac{3}{2}}}\cdot\frac{\sqrt{n}}{\phi^{\frac{3}{2}}\sqrt[4]{5}}\sqrt{2\pi}\\ &= \frac{\phi^{5n+\frac{5}{2}}}{2\pi n\sqrt[4]{5}},\end{aligned}$$

as claimed (see Figure 2.).

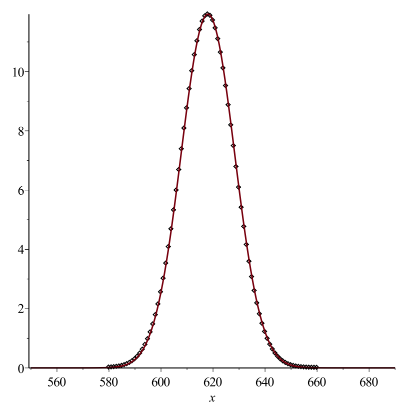

FIGURE 2. The case $n = 1000$, showing the points $(k, \binom{n}{k}^2\binom{n+k}{n})$ for $550 \le k \le 690$, together with the approximating normal, $y = \dfrac{\phi^{5n+4}}{(2\pi n)^{\frac{3}{2}}}\exp\left\{-\dfrac{\phi^3\sqrt{5}}{2n}\left(x-\dfrac{n}{\phi}\right)^2\right\}$.

3. The Correction Term

Let $s_n = \sum_{k=0}^{n}\binom{n}{k}^2\binom{n+k}{n}$. It was stated by Apéry [1] and proved by A. van der Poorten (see Section 4) that s_n satisfies the recurrence

$$(n+1)^2 s_{n+1} - (11n^2+11n+3)s_n - n^2 s_{n-1} = 0,$$

or

$$\left(1+\frac{1}{n}\right)^2 s_{n+1} - \left(11+\frac{11}{n}+\frac{3}{n^2}\right)s_n - s_{n-1} = 0.$$

We now suppose that

$$s_n = Cn^{-1}\Phi^n\left(1+\frac{a_1}{n}+\frac{a_2}{n^2}+\frac{a_3}{n^3}+\cdots\right),$$

where $C = \dfrac{\phi^{\frac{5}{2}}}{2\pi\sqrt[4]{5}}$ and $\Phi = \phi^5$, and substitute into the recurrence, to obtain

$$\begin{aligned}
&\left(1+\frac{1}{n}\right)^2 C(n+1)^{-1}\Phi^{n+1}\left(1+\frac{a_1}{n+1}+\frac{a_2}{(n+1)^2}+\frac{a_3}{(n+1)^3}+\cdots\right)\\
&\quad-\left(11+\frac{11}{n}+\frac{3}{n^2}\right)Cn^{-1}\Phi^n\left(1+\frac{a_1}{n}+\frac{a_2}{n^2}+\frac{a_3}{n^3}+\cdots\right)\\
&\quad-C(n-1)^{-1}\Phi^{n-1}\left(1+\frac{a_1}{n-1}+\frac{a_2}{(n-1)^2}+\frac{a_3}{(n-1)^3}+\cdots\right)=0.
\end{aligned}$$

If we now divide by $C\Phi^n$, and multiply by n, we find

$$\begin{aligned}
&\left(1+\frac{1}{n}\right)\Phi\left(1+\frac{a_1}{n+1}+\frac{a_2}{(n+1)^2}+\frac{a_3}{(n+1)^3}+\cdots\right)\\
&\quad-\left(11+\frac{11}{n}+\frac{3}{n^2}\right)\left(1+\frac{a_1}{n}+\frac{a_2}{n^2}+\frac{a_3}{n^3}+\cdots\right)\\
&\quad-\left(1-\frac{1}{n}\right)^{-1}\Phi^{-1}\left(1+\frac{a_1}{n-1}+\frac{a_2}{(n-1)^2}+\frac{a_3}{(n-1)^3}+\cdots\right)=0.
\end{aligned}$$

If we set $\dfrac{1}{n} = u$, $\quad \dfrac{1}{n+1} = \dfrac{u}{1+u}$, $\quad \dfrac{1}{n-1} = \dfrac{u}{1-u}$, and expand in powers of u, we find

$$\begin{aligned}
&\Phi(1+u)(1+a_1u+(a_2-a_1)u^2+(a_3-2a_2+a_1)u^3+\cdots)\\
&\quad-(11+11u+3u^2)(1+a_1u+a_2u^2+a_3u^3+\cdots)\\
&\quad-\Phi^{-1}(1+u+u^2+u^3+\cdots)(1+a_1u+(a_2+a_1)u^2+(a_3+2a_2+a_1)u^3+\cdots)\\
&\quad=0.
\end{aligned}$$

We now set the coefficients of the powers of u equal to zero, and solve for $a_1,\ a_2,\ a_3$ and so on. The constant term and the coefficient of u are automatically zero, because we had Φ correct and the factor n^{-1} correct. The coefficient of u^2 is

$$\Phi a_2 - (11a_2 + 11a_1 + 3) - \Phi^{-1}(a_2 + 2a_1 + 1) = 0,$$

or

$$-(11 + 2\Phi^{-1})a_1 - (3 + \Phi^{-1}) = 0.$$

We find

$$a_1 = -\frac{3+\Phi^{-1}}{11+2\Phi^{-1}} = -\frac{3+\left(\dfrac{5\sqrt{5}-11}{2}\right)}{11+2\left(\dfrac{5\sqrt{5}-11}{2}\right)} = -\frac{5\sqrt{5}-5}{10\sqrt{5}} = -\frac{5-\sqrt{5}}{10}.$$

If we continue in the same way, we find

$$a_2 = \frac{13-5\sqrt{5}}{50}, \quad a_3 = -\frac{175-83\sqrt{5}}{1250}, \quad a_4 = \frac{437-205\sqrt{5}}{6250},$$

and so on.

This completes the proof.

4. The Recurrence

A. van der Poorten's proof [5] goes as follows.

If we define

$$f(k) = \left(k^2 + (6n+3)k - (11n^2 + 9n + 2)\right) \binom{n}{k}^2 \binom{n+k}{n}$$

and

$$g(n) = \binom{n}{k}^2 \binom{n+k}{n}$$

then it is easy to verify that

$$f(k) - f(k-1) = (n+1)^2 g(n+1) - (11n^2 + 11n + 3)g(n) - n^2 g(n-1).$$

The recurrence follows on summing over k from 0 to $n+1$.

Following the work of Sister Celine Fasenmyer and Petrovsek, Wilf and Zeilberger [4], the discovery of such identities is routine.

References

[1] R. Apéry, *Irrationalité de $\zeta(2)$, $\zeta(3)$*, Astérisque, **61** (1979), 11–13.

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School of Mathematics and Statistics, UNSW, Sydney, Australia 2052