SOME IDENTITIES FOR SEQUENCES OF BINOMIAL SUMS OF GENERALIZED FIBONACCI NUMBERS

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Abstract. In [2] Komatsu obtained several sequences of binomial sums of generalized Fibonacci numbers. Using a technique involving matrices used in [1] by Hoggatt and Bicknell-Johnson a variety of binomial summation identities of these sequences are obtained.

1. Introduction

In [2] some binomial sums of generalized Fibonacci numbers satisfying $u_n = au_{n-1} + bu_{n-2}$ $(n \geq 2)$ with $u_0 = 0$ and $u_1 = 1$, where a and b are nonzero integers, were explored. The author used power series and generating functions to expand the database of identities for the generalized Fibonacci numbers. Not only were many new summation identities obtained but some new recurrence relations arose suggesting that the opportunity for further exploration was indicated. This paper intends to explore some of the new recurrences presented there and using the technique of matrices and the Caley-Hamilton Theorem are able to provide additional generalized Fibonacci number sums.

These sequences and the recurrence relations are

$$\sum_{k=0}^{n} \binom{n}{k} c^k u_k = r_n,$$

where c is a nonzero real number, satisfies the recurrence

$$r_n = (ac+2)r_{n-1} + (bc^2 - ac - 1)r_{n-2} \quad (n \geq 2) \tag{1.1}$$

with initial conditions $r_0 = 0$ and $r_1 = c$;

$$\sum_{k=0}^{n} \binom{n}{k} c^{n-k} d^k u_k = \lambda_n,$$

where d is a nonzero real number, satisfies the recurrence

$$\lambda_n = (ad+2c)\lambda_{n-1} + (bd^2 - acd - c^2)\lambda_{n-2} \quad (n \geq 2) \tag{1.2}$$

with initial conditions $\lambda_0 = 0$ and $\lambda_1 = d$ [2, Theorem 2.2, 2.3].

Similar sequences were obtained for the Tribonacci numbers defined as

$$T_n = T_{n-1} + T_{n-2} + T_{n-3} \quad (n \geq 3)$$

with $T_0 = 0$ and $T_1 = T_2 = 1$. The summation sequences and recurrences analogous to (1.1) and (1.2) are as follows:

$$\sum_{k=0}^{n} \binom{n}{k} c^k T_k = t_n\,,$$

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satisfying

$$t_n = (c+3)t_{n-1} + (c^2 - 2c - 3)t_{n-2} + (c^3 - c^2 + c + 1)t_{n-3} \quad (n \geq 3)$$

and

$$\sum_{k=0}^{n} \binom{n}{k} c^{n-k} d^k T_k = s_n\,,$$

satisfying

$$s_n = (d+3c)s_{n-1} + (d^2 - 2cd - 3c^2)s_{n-2} + (d^3 - cd^2 + c^2 d + c^3)s_{n-3} \quad (n \geq 3)\,,$$

respectively [2, Theorem 3.2, 3.3].

2. Second Order Sequences

Consider the sequence described in (1.1). To generate some summation identities the matrix technique used in [1] and many other papers will be employed. To this end, let

$$A = \begin{bmatrix} ac+2 & bc^2 - ac - 1 \\ 1 & 0 \end{bmatrix},$$

so that

$$A^n \begin{bmatrix} r_1 \\ r_0 \end{bmatrix} = \begin{bmatrix} r_{n+1} \\ r_n \end{bmatrix}. \tag{2.1}$$

While the characteristic equation for A is not found to be helpful, that of A^2 is, with

$$A^2 = \begin{bmatrix} (ac+2)^2 + (bc^2 - ac - 1) & (ac+2)(bc^2 - ac - 1) \\ ac+2 & bc^2 - ac - 1 \end{bmatrix}.$$

The characteristic equation, after simplification, is found to be

$$\big(\lambda - (bc^2 - ac - 1)\big)^2 = (ac+2)^2 \lambda$$

and

$$\big(\lambda + (bc^2 - ac - 1)\big)^2 = (a^2 + 4b)\lambda.$$

So, by the Caley-Hamilton Theorem for A^2

$$\big(A^2 - (bc^2 - ac - 1)E\big)^2 = (ac+2)^2 A^2 \tag{2.2}$$

and

$$\big(A^2 + (bc^2 - ac - 1)E\big)^2 = (a^2 + 4b)A^2. \tag{2.3}$$

Using the binomial expansion on the left side of (2.2) and (2.3), multiplying by the initial column vector in (2.1), from (2.2) and (2.3)

$$\sum_{k=0}^{2n} \binom{2n}{k} (-1)^k (bc^2 - ac - 1)^{2n-k} \begin{bmatrix} r_{2k+1} \\ r_{2k} \end{bmatrix} = (ac+2)^{2n} \begin{bmatrix} r_{2n+1} \\ r_{2n} \end{bmatrix}$$

and

$$\sum_{k=0}^{2n} \binom{2n}{k} (bc^2 - ac - 1)^{2n-k} \begin{bmatrix} r_{2k+1} \\ r_{2k} \end{bmatrix} = (a^2 + 4b)^{2n} \begin{bmatrix} r_{2n+1} \\ r_{2n} \end{bmatrix},$$

respectively. Several identities follow from these relationships. For example, from the alternating case

$$\sum_{k=0}^{2n} \binom{2n}{k} (-1)^k (bc^2 - ac - 1)^{2n-k} r_{2k+1} = (ac+2)^{2n} r_{2n+1}, \tag{2.4}$$

$$\sum_{k=0}^{2n} \binom{2n}{k} (-1)^k (bc^2 - ac - 1)^{2n-k} r_{2k} = (ac+2)^{2n} r_{2n}. \tag{2.5}$$

Multiplying (2.4) by $(ac+2)$ and (2.5) by $(bc^2 - ac - 1)$ the adding yields

$$\sum_{k=0}^{2n} \binom{2n}{k} (-1)^k (bc^2 - ac - 1)^{2n-k} r_{2k+2} = (ac+2)^{2n} r_{2n+2}\,. \tag{2.6}$$

Similar identities can be obtained for the second case.

By raising A^2 to higher powers, matrix equations analogous to (2.2) and (2.3) yield a plethora of binomial identities for the $\{r_n\}$ sequence. For example, A^4, A^6 and A^8 yield the following:

$$\left(A^4 - (bc^2 - ac - 1)^2 E\right)^2 = c^2 (ac+2)^2 (a^2 + 4b) A^4.$$

It follows that

$$\sum_{k=0}^{2n} \binom{2n}{k} (-1)^k (bc^2 - ac - 1)^{2(2n-k)} r_{4k+1} = c^{2n} (ac+2)^{2n} (a^2+4b)^n r_{4n+1},$$

$$\sum_{k=0}^{2n} \binom{2n}{k} (-1)^k (bc^2 - ac - 1)^{2(2n-k)} r_{4k} = c^{2n} (ac+2)^{2n} (a^2+4b)^n r_{4n},$$

$$\sum_{k=0}^{2n} \binom{2n}{k} (-1)^k (bc^2 - ac - 1)^{2(2n-k)} r_{4k+2} = c^{2n} (ac+2)^{2n} (a^2+4b)^n r_{4n+2}.$$

If $a = b = 1$ and $c = 2$ in the first two of these identities, by $r_n = F_{3n}$, we obtain

$$\sum_{k=0}^{2n} \binom{2n}{k} (-1)^k F_{12k+3} = (2^6 \cdot 5)^n F_{12n+3},$$

$$\sum_{k=0}^{2n} \binom{2n}{k} (-1)^k F_{12k} = (2^6 \cdot 5)^n F_{12n}.$$

Similar binomial sums are obtained for each of the following cases:

$$\begin{aligned}
\left(A^4 + (bc^2 - ac - 1)^2 E\right)^2 &= \left((ac+2)^2 + 2(bc^2 - ac - 1)\right)^2 A^4, \\
\left(A^6 - (bc^2 - ac - 1)^3 E\right)^2 &= (ac+2)^2 (a^2c^2 + 3bc^2 + ac + 1)^2 A^6, \\
\left(A^6 + (bc^2 - ac - 1)^3 E\right)^2 &= \left((ac+2)^2 (a^2c^2 + 3bc^2 + ac + 1)^2 + 4(bc^2 - ac - 1)^3\right)^2 A^6, \\
\left(A^8 - (bc^2 - ac - 1)^4 E\right)^2 &= c^2 (ac+2)^2 (a^2+4b)(a^2c^2 + 2bc^2 + 2ac + 2)^2 A^8, \\
\left(A^8 + (bc^2 - ac - 1)^4 E\right)^2 &= \left(c^2 (ac+2)^2 (a^2+4b)(a^2c^2 + 2bc^2 + 2ac + 2)^2 \right. \\
&\qquad \left. + 4(bc^2 - ac - 1)^2\right)^2 A^8.
\end{aligned}$$

No particular pattern emerges for generalization, but it is interesting to note that if

$$P = ac + 2 \quad \text{and} \quad Q = bc^2 - ac - 1$$

the coefficients of A^4 on the right-hand side of the identities are respectively the minus and plus cases $P^2(P^2+4Q)$ and $(P^2+2Q)^2$; those of A^6 are $P^2(P^2+3Q)^2$ and $P^2(P^2+3Q^2)+4Q^3$; and those of A^8 are $P^2(P^2+4Q)(P^2+2Q)^2$ and $P^2(P^2+4Q)(P^2+2Q)^2+4Q^2$.

Next, if these sequence are extended to the left, the inverse of A,

$$A^{-1} = \frac{1}{bc^2 - ac - 1}\begin{bmatrix} 0 & bc^2 - ac - 1 \\ 1 & -(ac+2) \end{bmatrix}$$

yields

$$(A^{-1})^n \begin{bmatrix} r_1 \\ r_0 \end{bmatrix} = \begin{bmatrix} r_{1-n} \\ r_{-n} \end{bmatrix}.$$

Recalling that the eigenvalues of the inverse matrix are the inverses of the eigenvalues of the given matrix, it follows, again using the Cayley-Hamilton Theorem, that the inverse of A^2 yields

$$\left((bc^2 - ac - 1)(A^{-1})^2 - E\right)^{2n} = (ac+2)^{2n}(A^{-1})^{2n}, \tag{2.7}$$

$$\left((bc^2 - ac - 1)(A^{-1})^2 + E\right)^{2n} = (a^2+4b)^{2n}(A^{-1})^{2n}. \tag{2.8}$$

Again by the binomial expansion and applying (2.7) and (2.8), it follows that

$$\sum_{k=0}^{2n}\binom{2n}{k}(-1)^k(bc^2 - ac - 1)^k \begin{bmatrix} r_{1-2k} \\ r_{-2k} \end{bmatrix} = (ac+2)^{2n}\begin{bmatrix} r_{1-2n} \\ r_{-2n} \end{bmatrix},$$

$$\sum_{k=0}^{2n}\binom{2n}{k}(bc^2 - ac - 1)^k \begin{bmatrix} r_{1-2k} \\ r_{-2k} \end{bmatrix} = (a^2+4b)^{2n}\begin{bmatrix} r_{1-2n} \\ r_{-2n} \end{bmatrix}.$$

So for examples analogous to (2.4), (2.5), and (2.6), the following are obtained:

$$\sum_{k=0}^{2n}(-1)^k(bc^2 - ac - 1)^k r_{1-2k} = (a^2+4b)^{2n} r_{1-2n},$$

$$\sum_{k=0}^{2n}(-1)^k(bc^2 - ac - 1)^k r_{-2k} = (a^2+4b)^{2n} r_{-2n},$$

$$\sum_{k=0}^{2n}(-1)^k(bc^2 - ac - 1)^{2n-k} r_{2-2k} = (a^2+4b)^{2n} r_{2-2n}.$$

As above, if $a = b = 1$ and $c = 2$ in the first two of these identities and $r_n = F_{3n}$, then

$$\sum_{k=0}^{2n}\binom{2n}{k}(-1)^k F_{1-6k} = 5^n F_{1-6n},$$

$$\sum_{k=0}^{2n}\binom{2n}{k}(-1)^k F_{-6k} = 5^n F_{-6n}.$$

The inverses for A^4, A^6, and A^8 cases are as follows:

$$\begin{aligned}
\big((bc^2-ac-1)^2(A^{-1})^4-E\big)^2 &= c^2(ac+2)^2(a^2+4b)(A^{-1})^4,\\
\big((bc^2-ac-1)^2(A^{-1})^4+E\big)^2 &= \big(c^2(ac+2)^2(a^2+4b)+4(bc^2-ac-1)^2\big)(A^{-1})^4,\\
\big((bc^2-ac-1)^3(A^{-1})^6-E\big)^2 &= (ac+2)^2(a^2c^2+3bc^2+ac+1)^2(A^{-1})^6,\\
\big((bc^2-ac-1)^3(A^{-1})^6+E\big)^2 &= \big((ac+2)^2(a^2c^2+3bc^2+ac+1)^2+4(bc^2-ac-1)^3\big)(A^{-1})^6,\\
\big((bc^2-ac-1)^4(A^{-1})^8-E\big)^2 &= c^2(ac+2)^2(a^2+4b)(a^2c^2+2bc^2+2ac+2)^2(A^{-1})^8,\\
\big((bc^2-ac-1)^4(A^{-1})^8+E\big)^2 &= \big(c^2(ac+2)^2(a^2+4b)(a^2c^2+2bc^2+2ac+2)^2\\
&\qquad +4(bc^2-ac-1)^4\big)(A^{-1})^8.
\end{aligned}$$

From these several binomial identities can be constructed. All of the identities obtained for the sequence described in (1.1) can be reproduced for the sequence of (1.2) by replacing $(ac+2)$ with $(ad+2c)$ and (bc^2-ac-1) with $(bd^2-acd-c^2)$.

3. Third Order Sequences

The matrix procedure for determining binomial sums for second order recurrences used in [1] does not appear to lead to useful Cayley-Hamilton patterns, but some special cases of third order sequences have been considered in [3] and [4] for some special cases and will be used here.

Let

$$B=\begin{bmatrix} c+3 & c^2-2c-3 & c^3-c^2+c+1\\ 1 & 0 & 0\\ 0 & 1 & 0\end{bmatrix}.$$

Then for $n\ge 0$ we have

$$B^n\begin{bmatrix} t_2\\ t_1\\ t_0\end{bmatrix}=\begin{bmatrix} t_{n+2}\\ t_{n+1}\\ t_n\end{bmatrix}.$$

Since the characteristic equation of B^3 is given by

$$(c^3-c^2+c+1-x)^3=2c^2(c+1)(c-3)(c^2+5c-2)x-4c^2(c+3)x^2,$$

we have

$$\begin{aligned}
&\sum_{k=0}^{3n}\binom{3n}{k}(-1)^k(c^3-c^2+c+1)^{3n-k}B^{3k}\\
&=\sum_{k=0}\binom{n}{k}\big(2c^2(c+1)(c-3)(c^2+5c-2)\big)^kB^{3k}(-4)^{n-k}\big(c^2(c+3)\big)^{n-k}B^{6(n-k)}\\
&=\sum_{k=0}^{n}\binom{n}{k}(-1)^{n-k}2^{2n-k}c^{2n}(c+1)^k(c-3)^k(c^2+5c-2)^k(c+3)^{n-k}B^{6n-3k}. \qquad (3.1)
\end{aligned}$$

If we let $c=-1$ in (3.1), we obtain

$$\sum_{k=0}^{3n}(-1)^k(-2)^{3n-k}B^{3k}=(-2)^{3n}B^{6n}.$$

Thus, we have for $i = 0, 1, 2$

$$(-1)^n \sum_{k=0}^{3n} 2^{3n-k} t_{3k+i} = (-2)^{3n} t_{6n+i},$$

where $t_n = 2t_{n-1} - 2t_{n-3}$ $(n \geq 3)$ with $t_0 = 0$ and $t_1 = t_2 = -1$. Since

$$t_n = \sum_{j=0}^{n} \binom{n}{j} (-1)^j T_j,$$

where T_j is the jth Tribonacci number in this case, we get

$$\sum_{k=0}^{3n} 2^{3n-k} \sum_{j=0}^{3k+i} \binom{3k+i}{j} (-1)^j T_j = 2^{3n} \sum_{j=0}^{6n+i} \binom{6n+i}{j} (-1)^j T_j. \tag{3.2}$$

If we let $c = 3$ in (3.1), we obtain

$$\sum_{k=0}^{3n} (-1)^k 22^{3n-k} B^{3k} = (-6)^{3n} B^{6n},$$

where $t_n = 6t_{n-1} + 22t_{n-3}$ $(n \geq 3)$ with $t_0 = 0$, $t_1 = 3$ and $t_2 = 15$. Since

$$t_n = \sum_{j=0}^{n} \binom{n}{j} 3^j T_j$$

in this case, we get for $i = 0, 1, 2$

$$\sum_{k=0}^{3n} (-1)^k 22^{3n-k} \sum_{j=0}^{3k+i} \binom{3k+i}{j} 3^j T_j = (-6)^{3n} \sum_{j=0}^{6n+i} \binom{6n+i}{j} 3^j T_j. \tag{3.3}$$

If we let $c = -3$ in (3.1), we obtain

$$\sum_{k=0}^{3n} (-1)^k (-38)^{3n-k} B^{3k} = (-2^6 \cdot 3^3)^{3n} B^{3n},$$

where $t_n = -6t_{n-2} - 38t_{n-3}$ $(n \geq 3)$ with $t_0 = 0$, $t_1 = -3$ and $t_2 = 3$. Since

$$t_n = \sum_{j=0}^{n} \binom{n}{j} (-3)^j T_j$$

in this case, we get for $i = 0, 1, 2$

$$\sum_{k=0}^{3n} (-1)^k (-38)^{3n-k} \sum_{j=0}^{3k+i} \binom{3k+i}{j} (-3)^j T_j = (-2^6 \cdot 3^3)^{3n} \sum_{j=0}^{3n+i} \binom{3n+i}{j} (-3)^j T_j. \tag{3.4}$$

4. Concluding Remarks

It is hoped that the new identities presented in this paper, like so many identities obtained historically, will be of use to Fibonacci researchers investigating both physical and mathematical applications. The techniques in this paper and those found in [2] should also be useful in exploring sequences that do not fit the generalized Fibonacci numbers model.

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