INFINITE SUMS OF WEIGHTED FIBONACCI NUMBERS OF ORDER k

SPIROS D. DAFNIS AND ANDREAS N. PHILIPPOU

Abstract. For integers $m \geq 0$ and $k \geq 2$, set $\alpha_{m,k} := \sum_{n=1}^{\infty} \frac{n^m F_n^{(k)}}{2^{n+k-1}}$, where $F_n^{(k)}$ is the Fibonacci sequence of order k or k-generalized Fibonacci sequence. It is shown that $\alpha_{0,k} = 1$, $\alpha_{1,k} = 2^{k+1} - k - 1$, $\alpha_{2,k} = 2^{k+1}(2^{k+2} - 4k - 3) + k^2 + 2k - 1$, and $\alpha_{m,k} = 1 + \sum_{r=0}^{m-1} \binom{m}{r} \sum_{i=1}^{k} 2^{k-i} i^{m-r} \alpha_{r,k}$, which generalize recent results on weighted Fibonacci sums by Benjamin, Neer, Otero, and Sellers.

1. Introduction and Main Results

Benjamin et al. [1] investigated sums of the form

$$\alpha_m := \sum_{n=1}^{\infty} \frac{n^m F_n}{2^{n+1}}, m = 0, 1, 2, \ldots, \tag{1.1}$$

by probabilistic arguments. They found that

$$\alpha_0 = 1, \alpha_1 = 5, \alpha_2 = 47, \tag{1.2}$$

and

$$\alpha_m = 1 + \sum_{r=0}^{m-1} \binom{m}{r} (2 + 2^{m-r}) \alpha_r, \tag{1.3}$$

which implies $\alpha_3 = 665$, $\alpha_4 = 12551$, and so on (see, also, Vajda [12]). Here, and in the sequel, $\sum_{j=l}^{u} f_j = 0$, for $l > u$.

Presently, we examine sums of the form

$$\alpha_{m,k} := \sum_{n=1}^{\infty} \frac{n^m F_n^{(k)}}{2^{n+k-1}}, m = 0, 1, 2, \ldots, k = 2, 3, \ldots, \tag{1.4}$$

where $F_n^{(k)}$ is the Fibonacci sequence of order k [2,7,8,9,11] (or k-generalized Fibonacci sequence [4, 5, 6]).

We note first [6] that for each $k = 2, 3, \ldots$,

$$\lim_{n \to \infty} \frac{F_{n+1}^{(k)}}{F_n^{(k)}} = r_{k,k}$$

for some $r_{k,k}$ in the open interval $(1,2)$, which implies that the series $\sum_{n=1}^{\infty} \frac{n^m F_n^{(k)}}{2^{n+k-1}}$ converges (to $\alpha_{m,k}$), by the ratio test, since $\sum_{n=1}^{\infty} \frac{n^m F_n^{(k)}}{2^{n+k-1}} > 0$ for all n and

$$\frac{\frac{(n+1)^m F_{n+1}^{(k)}}{2^{n+k}}}{\frac{n^m F_n^{(k)}}{2^{n+k-1}}} = \frac{1}{2}\left(\frac{n+1}{n}\right)^m \frac{F_{n+1}^{(k)}}{F_n^{(k)}} \to \frac{r_{k,k}}{2} < 1.$$

Therefore, $\alpha_{m,k}$ is well-defined.

We shall derive the following two propositions.

Proposition 1.1. *Let $\alpha_{m,k}$ be as in* (1.4). *Then, for* $k = 2, 3, \ldots,$

(a) $\alpha_{0,k} = 1,$
(b) $\alpha_{1,k} = 2^{k+1} - k - 1,$
(c) $\alpha_{2,k} = 2^{k+1}(2^{k+2} - 4k - 3) + k^2 + 2k - 1.$

Proposition 1.2. *Let $\alpha_{m,k}$ be as in* (1.4). *Then,*

$$\alpha_{m,k} = 1 + \sum_{r=0}^{m-1} \binom{m}{r} \sum_{i=1}^{k} 2^{k-i} i^{m-r} \alpha_{r,k}.$$

The proofs of the propositions are direct consequences of two well-known results [6, 7, 8, 9, 10], which we state as lemmas for easy reference.

2. Preliminary Results

Lemma 2.1. [6, 7, 8, 9]. *Let $F_n^{(k)}$ be the Fibonacci sequence of order k. Then, for $n \geq 0$,*

$$F_{n+1}^{(k)} = \sum \binom{n_1 + \cdots + n_k}{n_1, \ldots, n_k},$$

where the sum is taken over all k-tuples of non-negative integers $n_1, n_2, \ldots, n_k$ such that $n_1 + 2n_2 + \ldots + kn_k = n$.

Lemma 2.2. [8, 9, 10]. *Let N_k be the waiting time until the occurrence of the kth consecutive success in independent trials with success probability p $(0 < p < 1)$. Then, for $n \geq k$,*

(a)
$$P(N_k = n) = p^n \sum \binom{n_1 + \cdots + n_k}{n_1, \cdots, n_k} \left(\frac{q}{p}\right)^{n_1 + \cdots + n_k}$$

and 0 if $n < k$, where the summation is taken over all k-tuples of non-negative integers $n_1, n_2, \ldots, n_k$ such that $n_1 + 2n_2 + \cdots + kn_k = n - k$.

(b)
$$\sum_{n=k}^{\infty} P(N_k = n) = 1.$$

(c)
$$\mu_k(p) = E(N_k) = \frac{1-p^k}{qp^k}, \quad \text{and} \quad \sigma_k^2(p) = V(N_k) = \frac{1 - (2k+1)qp^k - p^{2k+1}}{q^2 p^{2k}}.$$

(d)
$$P(N_k = n+k) = \frac{F_{n+1}^{(k)}}{2^{n+k}}, \quad n \geq 0, \quad \text{for} \quad p = \frac{1}{2}.$$

Part (c) was first established by Feller [3].

3. Proof of Main Results

We proceed to show the main results.

Proof of Proposition 1.1. We have

$$\alpha_{0,k} = \sum_{n=1}^{\infty} \frac{F_n^{(k)}}{2^{n+k-1}} = \sum_{n=k}^{\infty} \frac{F_{n-k+1}^{(k)}}{2^n} = \sum_{n=k}^{\infty} \frac{\sum \binom{n_1+\cdots+n_k}{n_1,\ldots,n_k}}{2^n},$$

where the inner sum is taken over all k-tuples of non-negative integers $n_1, n_2, \ldots, n_k$ such that $n_1 + 2n_2 + \cdots + kn_k = n - k$, by Lemma 2.1,

$$= \sum_{n=k}^{\infty} P(N_k = n), \text{with } p = 1/2, \text{ by Lemma 2.2(a)}$$
$$= 1 \text{ by Lemma 2.2(b), and this establishes Proposition 1.1(a).}$$

Next,

$$\alpha_{1,k} = \sum_{n=1}^{\infty} \frac{nF_n^{(k)}}{2^{n+k-1}} = \sum_{n=k}^{\infty} \frac{(n-k+1)F_{n-k+1}^{(k)}}{2^n} = \sum_{n=k}^{\infty} \frac{nF_{n-k+1}^{(k)}}{2^n} - (k-1)\sum_{n=k}^{\infty} \frac{F_{n-k+1}^{(k)}}{2^n}$$
$$= \mu_k(\frac{1}{2}) - (k-1) = 2^{k+1} - 2 - (k-1) = 2^{k+1} - k - 1,$$

by Proposition 1.1(a) and Lemma 2.2(c), which establishes Proposition 1.1 (b).

Finally, we have, by Proposition 1.1(a) and Lemma 2.2(c),

$$\begin{aligned}
\alpha_{2,k} &= \sum_{n=1}^{\infty} \frac{n^2F_n^{(k)}}{2^{n+k-1}} = \sum_{n=k}^{\infty} \frac{(n-k+1)^2F_{n-k+1}^{(k)}}{2^n} \\
&= \sum_{n=k}^{\infty} \frac{n^2F_{n-k+1}^{(k)}}{2^n} - 2(k-1)\sum_{n=k}^{\infty} \frac{nF_{n-k+1}^{(k)}}{2^n} + (k-1)^2 \\
&= E(N_k^2) - 2(k-1)E(N_k) + (k-1)^2, \text{ with } p = \frac{1}{2}, \\
&= \sigma_k^2\left(\frac{1}{2}\right) + \mu_k^2\left(\frac{1}{2}\right) - 2(k-1)\mu_k\left(\frac{1}{2}\right) + (k-1)^2 \\
&= 2^{2k+3} - 2^{k+3} - (2k-1)2^{k+2} - 2^{k+1} + k^2 + 2k - 1 \\
&= 2^{k+1}(2^{k+2} - 4k - 3) + k^2 + 2k - 1,
\end{aligned}$$

and this completes the proof of Proposition 1.1.

For $k = 2$, Proposition 1.1 reduces to relation (1.2).

We proceed now to show our second proposition.

Proof of Proposition 1.2. Let Y_k be the waiting time until the beginning of the occurrence of the kth consecutive success in independent trials with success probability $p = \frac{1}{2}$. Since $Y_k = N_k - (k-1)$ for $p = \frac{1}{2}$, Lemma 2.2(d) gives

$$P(Y_k = n) = P(N_k = n + k - 1) = \frac{F_n^{(k)}}{2^{n+k-1}}, \quad n \geq 1. \tag{3.1}$$

Therefore, by (3.1) and (1.4), the mth moment of Y_k is

$$E(Y_k^m) = \sum_{n=1}^{\infty} n^m P(Y_k = n) = \sum_{n=1}^{\infty} \frac{n^m F_n^{(k)}}{2^{n+k-1}} = \alpha_{m,k}. \tag{3.2}$$

If we denote the trials by T_i, $(i \geq 1)$, success by 1, and failure by 0, it follows that $(Y_k = 1) = (1\ldots1\ (k\ 1\text{'}s))$, $(Y_k = 2) = (01\ldots1\ (k\ 1\text{'}s))$, and for $n \geq 3$ $(Y_k = n) =$ (all outcomes $t_1 \ldots t_{n-2}01\ldots1$ (k 1's), $t_i = 0$ or 1 $(1 \leq i \leq n-2)$ with no k consecutive 1's among the first $n-2$ outcomes).

We define now the events A_0 = no failure occurs in the first k trials, and A_i = the first failure in the first k trials occurs at the ith trial, $1 \leq i \leq k$, i.e.

$$\begin{aligned} A_0 &= \underbrace{1\ldots1}_{k} \\ A_1 &= 0\ldots t_k \\ A_2 &= 10\ldots t_k \\ &\vdots \\ A_k &= \underbrace{1\ldots1}_{k-1}0. \end{aligned}$$

It follows that $(Y_k = n)$ is the union of the mutually exclusive events $(Y_k = n) \cap A_i$ $(0 \leq i \leq k)$, and hence,

$$P(Y_k = n) = \sum_{i=0}^{k} P[(Y_k = n) \cap A_i] = \sum_{i=0}^{k} P[(Y_k = n)|A_i]P(A_i).$$

Therefore,

$$\begin{aligned} E(Y_k^m) &= \sum_{n=1}^{\infty} n^m P(Y_k = n) = \sum_{n=1}^{\infty} n^m \sum_{i=0}^{k} P[(Y_k = n)|A_i]P(A_i) \\ &= \sum_{i=0}^{k}\sum_{n=1}^{\infty} n^m P[(Y_k = n)|A_i]P(A_i) = \sum_{i=0}^{k} E(Y_k^m|A_i)P(A_i). \end{aligned} \tag{3.3}$$

Now, given that the event A_i $(1 \leq i \leq k)$ has occurred, the beginning of the kth consecutive success may start at the $i+1$ trial. Thus, $E(Y_k^m|A_i) = E((Y_k+i)^m)$ $(1 \leq i \leq k)$. Furthermore, $Y_k^m|A_0 = 1$, $P(A_0) = (\frac{1}{2})^k$ and $P(A_i) = (\frac{1}{2})^i$ $(1 \leq i \leq k)$. It follows, by (3.3),

$$\begin{aligned} E(Y_k^m) &= \left(\frac{1}{2}\right)^k + \sum_{i=1}^{k}\left(\frac{1}{2}\right)^i E((Y_k+i)^m) \\ &= \left(\frac{1}{2}\right)^k + \sum_{i=1}^{k}\left(\frac{1}{2}\right)^i E\left(\sum_{r=0}^{m}\binom{m}{r} i^{m-r} Y_k^r\right) \\ &= \left(\frac{1}{2}\right)^k + \left(1 - \left(\frac{1}{2}\right)^k\right) E(Y_k^m) + \sum_{i=1}^{k}\left(\frac{1}{2}\right)^i E\left(\sum_{r=0}^{m-1}\binom{m}{r} i^{m-r} Y_k^r\right). \end{aligned}$$

Solving for $E(Y_k^m)$, we get

$$E(Y_k^m) = 1 + \sum_{i=1}^{k} 2^{k-i} E\left(\sum_{r=0}^{m-1} \binom{m}{r} i^{m-r} Y_k^r\right)$$
$$= 1 + \sum_{r=0}^{m-1} \binom{m}{r} \sum_{i=1}^{k} 2^{k-i} i^{m-r} E(Y_k^r),$$

which, by means of (3.2), completes the proof of Proposition 1.2.

For $k = 2$, Proposition 1.2 reduces to relation (1.3).

References

[1] A. T. Benjamin, J. D. Neer, D. E. Otero, and J. A. Sellers, *A probabilistic view of certain weighted Fibonacci sums*, The Fibonacci Quarterly, **41.4** (2003), 360–364.

[2] R. C. Bollinger, *Fibonacci k-sequences, Pascal-T triangles and k-in-a-row problems*, The Fibonacci Quarterly, **22.2** (1984), 146–151.

[3] W. Feller, *An Introduction to Probability Theory and Its Applications*, third ed., Wiley, 1968.

[4] F. T. Howard and C. Cooper, *Some identities for the r-Fibonacci numbers*, The Fibonacci Quarterly, **49.3** (2011), 231–243.

[5] D. Kessler and J. Schiff, *A combinatoric proof of generalization of Ferguson's formula for k-generalized Fibonacci numbers*, The Fibonacci Quarterly, **42.3** (2004), 266–273.

[6] E. P. Miles, Jr., *Generalized Fibonacci numbers and associated matrices*, American Mathematical Monthly, **67.8** (1960), 745–752.

[7] A. N. Philippou, *A note on the Fibonacci sequence of order k and multinomial coefficients*, The Fibonacci Quarterly, **24.2** (1983), 82–86.

[8] A. N. Philippou, *Distributions and Fibonacci polynomials of order k, longest runs, and reliability of consecutive-k-out-of-n: F systems*, Fibonacci Numbers and Their Applications, A. N. Philippou, G. E. Bergum, and A. F. Horadam, eds., 203–227, Reidel, Dordrecht, 1986.

[9] A. N. Philippou and A. A. Muwafi, *Waiting for the kth consecutive success and the Fibonacci sequence of order k*, The Fibonacci Quarterly, **20.1** (1982), 28–32.

[10] A. N. Philippou, C. Georghiou, and G. N. Philippou, *A generalized geometric distribution and some of its properties*, Statistics and Probability Letters, **1** (1983), 171–175.

[11] H. Prodinger, *Two families of series of the generalized golden ratio*, The Fibonacci Quarterly, **53.1** (2015), 74–77.

[12] S. Vajda, *Fibonacci and Lucas Numbers and the Golden Section: Theory and Applications*, John Wiley and Sons, New York, 1989.

MSC2010: 11B39, 40A05

Department of Mathematics, University of Patras, 265-00 Patras, Greece
E-mail address: dafnisspyros@gmail.com

Department of Mathematics, University of Patras, 265-00 Patras, Greece
E-mail address: anphilip@math.upatras.gr