TWO ALGEBRAIC IDENTITIES AND THE ALTERNATING FIBONACCI SUMS PRODUCED BY THEM

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Abstract. In this paper, we present two algebraic identities that we believe are new. These algebraic identities produce, via the telescoping effect, closed forms for a host of finite sums that involve the Fibonacci/Lucas numbers. All the finite sums contained herein are alternating in nature.

1. Introduction

The Fibonacci and Lucas numbers are defined, respectively, for all integers n, by

$$F_n = F_{n-1} + F_{n-2}, F_0 = 0, F_1 = 1,$$
$$L_n = L_{n-1} + L_{n-2}, L_0 = 2, L_1 = 1.$$

With $\alpha = (1+\sqrt{5})/2$, the Binet (closed) forms for F_n and L_n are

$$F_n = \left(\alpha^n + (-1)^{n+1}\alpha^{-n}\right)/\sqrt{5},$$
$$L_n = \alpha^n + (-1)^n\alpha^{-n},$$

and these closed forms are valid for all integers n.

In this paper, we present closed forms for Fibonacci related sums that alternate with the running variable i. For instance, two summands that we consider are

$$\frac{(-1)^i L_{2ai+b}}{F_{2ai+b}+c} \text{ and } \frac{(-1)^i F_{2ai+b}}{L_{2ai+b}+c},$$

for certain values of the parameters a, b, and c. In each of these two cases, the term in the numerator of the summand, together with its counterpart in the denominator, have subscripts in arithmetic progression. All the finite sums in this paper can be simultaneously written down, and summed, with the use of two elementary algebraic identities. For instance, see Lemma 2.1. Interestingly, the algebraic identities in question have nothing to do with Fibonacci numbers. The link with the Fibonacci numbers is made when the parameter t is replaced by α.

In Section 2, we make use of our first algebraic identity to establish closed forms for four summands. In Section 3, we introduce a second algebraic identity that we use to establish closed forms for a further four summands.

2. Finite Sums I

The lemma that follows is a statement of the first of two algebraic identities that appear in this paper. All the results in this section are consequences of this lemma.

Lemma 2.1. *Let $t > 1$ be a real number, and let n, a, and b be integers. Then*

$$\frac{t^{2an+b} - t^{-2an-b}}{t^{2an+b} + t^{-2an-b} + t^a + t^{-a}} = 1 - \frac{1}{t^{2an+b-a}+1} - \frac{1}{t^{2an+b+a}+1}. \tag{2.1}$$

Proof. Express the right side of (2.1) as

$$\frac{t^{-2an-b}}{t^{-2an-b}}-\frac{1}{t^{2an+b-a}+1}-\frac{1}{t^{2an+b+a}+1},$$

and consider the product $t^{-2an-b}\left(t^{2an+b-a}+1\right)\left(t^{2an+b+a}+1\right)$. The aforementioned product is equal to the denominator of the algebraic fraction on the left side of (2.1). The proof of (2.1) now follows by simple algebra. □

In this paper, $n\geq 1$, a, and b, are always taken to be integers. Many of the results that follow remain valid if a and b are allowed to be negative. However, for simplicity, we always take a and b to be non-negative. With this in mind we define, for $a\geq 1$ and $b\geq 0$,

$$S_1(n,a,b)=\sum_{i=1}^{n}\frac{(-1)^iL_{2ai+b}}{F_{2ai+b}+F_a},$$

$$S_2(n,a,b)=\sum_{i=1}^{n}\frac{(-1)^iF_{2ai+b}}{L_{2ai+b}+L_a},$$

$$S_3(n,a,b)=\sum_{i=1}^{n}\frac{(-1)^iL_{2ai+b}}{\sqrt{5}F_{2ai+b}+L_a},$$

$$S_4(n,a,b)=\sum_{i=1}^{n}\frac{(-1)^iF_{2ai+b}}{L_{2ai+b}+\sqrt{5}F_a}.$$

Let $t=\alpha$, and consider the left side of (2.1). Then the non-alternating part of the summand in each S_i arises from the parities of a and b. Specifically, the non-alternating part of the summand in S_1 arises from the left side of (2.1) when a and b are both assumed to be odd. Furthermore, the non-alternating part of the summand in S_2 arises when a and b are both assumed to be even. For S_3, a is even and b is odd, while for S_4, a is odd and b is even.

For our first theorem, we let $t=\alpha$ in (2.1), and take a and b to be odd. Thus, we establish the closed form for S_1.

Theorem 2.2. *Let $n\geq 1$. Let $a\geq 1$ and $b\geq 1$ both be odd. Then*

$$S_1(n,a,b)=\begin{cases}\frac{1}{2}\left(\frac{L_{an+(a+b)/2}}{F_{an+(a+b)/2}}-\frac{L_{(a+b)/2}}{F_{(a+b)/2}}\right), & \text{if } n \text{ is even and } (a+b)/2 \text{ is odd;}\\ \frac{5}{2}\left(\frac{F_{an+(a+b)/2}}{L_{an+(a+b)/2}}-\frac{F_{(a+b)/2}}{L_{(a+b)/2}}\right), & \text{if } n \text{ is even and } (a+b)/2 \text{ is even;}\\ -\frac{1}{2}\left(\frac{5F_{an+(a+b)/2}}{L_{an+(a+b)/2}}+\frac{L_{(a+b)/2}}{F_{(a+b)/2}}\right), & \text{if } n \text{ is odd and } (a+b)/2 \text{ is odd;}\\ -\frac{1}{2}\left(\frac{L_{an+(a+b)/2}}{F_{an+(a+b)/2}}+\frac{5F_{(a+b)/2}}{L_{(a+b)/2}}\right), & \text{if } n \text{ is odd and } (a+b)/2 \text{ is even.}\end{cases}$$

Proof. For (2.1) to telescope when summed, an alternating sum is required. Accordingly, when terms are written vertically, it is seen that

$$\begin{aligned}&\sum_{i=1}^{n}\frac{(-1)^i\left(t^{2ai+b}-t^{-2ai-b}\right)}{t^{2ai+b}+t^{-2ai-b}+t^a+t^{-a}}\\ &=\sum_{i=1}^{n}(-1)^i+\frac{1}{t^{a+b}+1}+\frac{(-1)^{n+1}}{t^{2an+a+b}+1} \qquad (2.2)\\ &=\sum_{i=1}^{n}(-1)^i+\frac{t^{-(a+b)/2}}{t^{(a+b)/2}+t^{-(a+b)/2}}+\frac{(-1)^{n+1}t^{-an-(a+b)/2}}{t^{an+(a+b)/2}+t^{-an-(a+b)/2}}.\end{aligned}$$

We require two identities that appear in [4]. These are

$$\frac{\alpha^q}{F_q} = \frac{L_q}{2F_q} + \frac{\sqrt{5}}{2}, \text{for all integers } q \neq 0, \tag{2.3}$$

and

$$\frac{\alpha^q}{L_q} = \frac{\sqrt{5}F_q}{2L_q} + \frac{1}{2}, \text{for all integers } q. \tag{2.4}$$

We also recall that $F_{-q} = (-1)^{q+1}F_q$, and $L_{-q} = (-1)^q L_q$. Now let $t = \alpha$ in (2.2), and assume that n is even, and $(a+b)/2$ is odd. With the use of the Binet forms, together with (2.3), we then transform the first and third lines of (2.2) into Fibonacci/Lucas numbers. After some simplification, we obtain the first case of Theorem 2.2. Likewise, to establish each of the remaining three cases of Theorem 2.2, we transform the first and third lines of (2.2) into Fibonacci/Lucas numbers, making use of (2.3) or (2.4), as appropriate. We leave the details to the interested reader. $\square$

Our next theorem follows, and this is proved in precisely the same manner as Theorem 2.2. That is, we transform the first and third lines of (2.2) into Fibonacci/Lucas numbers, making use of (2.3) or (2.4), as appropriate.

Theorem 2.3. *Let $n \geq 1$. Let $a \geq 2$, and $b \geq 0$ both be even. Then*

$$S_2(n,a,b) = \begin{cases} \frac{1}{10}\left(\frac{L_{an+(a+b)/2}}{F_{an+(a+b)/2}} - \frac{L_{(a+b)/2}}{F_{(a+b)/2}}\right), & \text{if } n \text{ is even and } (a+b)/2 \text{ is odd;} \\ \frac{1}{2}\left(\frac{F_{an+(a+b)/2}}{L_{an+(a+b)/2}} - \frac{F_{(a+b)/2}}{L_{(a+b)/2}}\right), & \text{if } n \text{ is even and } (a+b)/2 \text{ is even;} \\ -\frac{1}{10}\left(\frac{L_{an+(a+b)/2}}{F_{an+(a+b)/2}} + \frac{L_{(a+b)/2}}{F_{(a+b)/2}}\right), & \text{if } n \text{ is odd and } (a+b)/2 \text{ is odd;} \\ -\frac{1}{2}\left(\frac{F_{an+(a+b)/2}}{L_{an+(a+b)/2}} + \frac{F_{(a+b)/2}}{L_{(a+b)/2}}\right), & \text{if } n \text{ is odd and } (a+b)/2 \text{ is even.} \end{cases}$$

The last theorem in this section gives the closed forms for S_3 and S_4, where the assumption is that a and b have different parities. Here, we transform the first and second lines of (2.2) into Fibonacci/Lucas numbers.

Theorem 2.4. *Let $n \geq 1$. Let $a \geq 1$, $b \geq 0$. Then*

$$S_3(n,a,b) = \frac{1}{2}\left(\frac{(-1)^n\left(\sqrt{5}F_{2an+a+b} - 2\right)}{L_{2an+a+b}} - \frac{\sqrt{5}F_{a+b} - 2}{L_{a+b}}\right), a \text{ even and } b \text{ odd,}$$

$$\sqrt{5}S_4(n,a,b) = \frac{1}{2}\left(\frac{(-1)^n\left(\sqrt{5}F_{2an+a+b} - 2\right)}{L_{2an+a+b}} - \frac{\sqrt{5}F_{a+b} - 2}{L_{a+b}}\right), a \text{ odd and } b \text{ even.}$$

Proof. Observe that, for q an odd integer,

$$\begin{aligned} \frac{1}{\alpha^q + 1} &= \frac{\alpha^q - 1}{\alpha^{2q} - 1} = \frac{\alpha^q - 1}{\alpha^{2q} - \alpha^q\alpha^{-q}} = \frac{\alpha^q - 1}{\alpha^q\left(\alpha^q - \alpha^{-q}\right)} = \frac{1 - \alpha^{-q}}{\left(\alpha^q - \alpha^{-q}\right)} \\ &= \frac{F_{q+1} - \alpha F_q + 1}{L_q} = \frac{1}{L_q} + \frac{1}{2} - \frac{\sqrt{5}F_q}{2L_q}. \end{aligned} \tag{2.5}$$

Here, in addition to $F_{-q} = (-1)^{q+1}F_q$, mentioned earlier, we have used the well-known identity $\alpha^q = \alpha F_q + F_{q-1}$, which is valid for all integers q.

If a and b have different parities, then (2.5) shows that

$$\frac{1}{\alpha^{a+b} + 1} = \frac{1}{L_{a+b}} + \frac{1}{2} - \frac{\sqrt{5}F_{a+b}}{2L_{a+b}}. \tag{2.6}$$

Let $t = \alpha$, and assume that a is even, and b is odd. Then, with the help of (2.6), we transform the first and second lines of (2.2) into Fibonacci and Lucas numbers under the assumption that n is even. We then do likewise under the assumption that n is odd. The closed form for S_3, as stated in Theorem 2.4, brings these two cases together, so that the closed form for S_3 is independent of the parity of n. The closed form for S_4 is similarly obtained. $\square$

We have chosen to present the closed form for S_4 in the manner above (with $\sqrt{5}$ transposed to the left side) in order to highlight its relationship with the closed form for S_3.

We conclude this section with an example. In Theorem 2.2, let n be even, and take $a = b = 1$. We then see that

$$\sum_{i=1}^{n} \frac{(-1)^i L_{2i+1}}{F_{2i+1}+1} = \frac{F_n}{F_{n+1}}, \text{ for } n \text{ even.}$$

Here, we simplify the right side with the use of the well-known identity $L_n = F_{n+1} + F_{n-1}$.

3. Finite Sums II

In this section, we present results that are parallel to those in Section 2. Since the methods used in this section are similar to those used in Section 2, we state our results with minimal commentary. The lemma that follows is a statement of our second algebraic identity, and it is from this algebraic identity that all the results in this section flow.

Lemma 3.1. *Let $t > 1$ be a real number and let $n \geq 1$, $a \geq 1$, and $b \geq 0$ be integers. Then*

$$\frac{t^{2an+b} - t^{-2an-b}}{t^{2an+b} + t^{-2an-b} - t^a - t^{-a}} = 1 + \frac{1}{t^{2an+b-a} - 1} + \frac{1}{t^{2an+b+a} - 1}. \tag{3.1}$$

It is a simple matter to verify that, with the constraints placed upon the parameters, no denominator in identity (3.1) ever vanishes. The next identity arises from (3.1), and is a parallel result to (2.2). We have

$$\begin{aligned} &\sum_{i=1}^{n} \frac{(-1)^i \left(t^{2ai+b} - t^{-2ai-b}\right)}{t^{2ai+b} + t^{-2ai-b} - t^a - t^{-a}} \\ &= \sum_{i=1}^{n} (-1)^i - \frac{1}{t^{a+b} - 1} + \frac{(-1)^n}{t^{2an+a+b} - 1} \\ &= \sum_{i=1}^{n} (-1)^i - \frac{t^{-(a+b)/2}}{t^{(a+b)/2} - t^{-(a+b)/2}} + \frac{(-1)^n t^{-an-(a+b)/2}}{t^{an+(a+b)/2} - t^{-an-(a+b)/2}}. \end{aligned} \tag{3.2}$$

We now define the finite sums whose closed forms are given in this section. For $a \geq 1$ and $b \geq 0$, these sums are

$$\begin{aligned} S_5(n,a,b) &= \sum_{i=1}^{n} \frac{(-1)^i L_{2ai+b}}{F_{2ai+b} - F_a}, \\ S_6(n,a,b) &= \sum_{i=1}^{n} \frac{(-1)^i F_{2ai+b}}{L_{2ai+b} - L_a}, \\ S_7(n,a,b) &= \sum_{i=1}^{n} \frac{(-1)^i L_{2ai+b}}{\sqrt{5}F_{2ai+b} - L_a}, \\ S_8(n,a,b) &= \sum_{i=1}^{n} \frac{(-1)^i F_{2ai+b}}{L_{2ai+b} - \sqrt{5}F_a}. \end{aligned}$$

The first theorem in this section gives the closed form for S_5. The proof makes use of (3.2), and follows the same lines as the proof of Theorem 2.2.

Theorem 3.2. *Let $n \geq 1$. Let $a \geq 1$ and $b \geq 1$ both be odd. Then*

$$S_5(n,a,b) = \begin{cases} \frac{5}{2}\left(\frac{F_{an+(a+b)/2}}{L_{an+(a+b)/2}} - \frac{F_{(a+b)/2}}{L_{(a+b)/2}}\right), & \text{if } n \text{ is even and } (a+b)/2 \text{ is odd;} \\ \frac{1}{2}\left(\frac{L_{an+(a+b)/2}}{F_{an+(a+b)/2}} - \frac{L_{(a+b)/2}}{F_{(a+b)/2}}\right), & \text{if } n \text{ is even and } (a+b)/2 \text{ is even;} \\ -\frac{1}{2}\left(\frac{L_{an+(a+b)/2}}{F_{an+(a+b)/2}} + \frac{5F_{(a+b)/2}}{L_{(a+b)/2}}\right), & \text{if } n \text{ is odd and } (a+b)/2 \text{ is odd;} \\ -\frac{1}{2}\left(\frac{5F_{an+(a+b)/2}}{L_{an+(a+b)/2}} + \frac{L_{(a+b)/2}}{F_{(a+b)/2}}\right), & \text{if } n \text{ is odd and } (a+b)/2 \text{ is even.} \end{cases}$$

Next, we give the closed form for S_6.

Theorem 3.3. *Let $n \geq 1$. Let $a \geq 2$ and $b \geq 0$ both be even. Then*

$$S_6(n,a,b) = \begin{cases} \frac{1}{2}\left(\frac{F_{an+(a+b)/2}}{L_{an+(a+b)/2}} - \frac{F_{(a+b)/2}}{L_{(a+b)/2}}\right), & \text{if } n \text{ is even and } (a+b)/2 \text{ is odd;} \\ \frac{1}{10}\left(\frac{L_{an+(a+b)/2}}{F_{an+(a+b)/2}} - \frac{L_{(a+b)/2}}{F_{(a+b)/2}}\right), & \text{if } n \text{ is even and } (a+b)/2 \text{ is even;} \\ -\frac{1}{2}\left(\frac{F_{an+(a+b)/2}}{L_{an+(a+b)/2}} + \frac{F_{(a+b)/2}}{L_{(a+b)/2}}\right), & \text{if } n \text{ is odd and } (a+b)/2 \text{ is odd;} \\ -\frac{1}{10}\left(\frac{L_{an+(a+b)/2}}{F_{an+(a+b)/2}} + \frac{L_{(a+b)/2}}{F_{(a+b)/2}}\right), & \text{if } n \text{ is odd and } (a+b)/2 \text{ is even.} \end{cases}$$

Our final theorem gives the closed forms for S_7 and S_8, where the assumption is that a and b have different parities. The proof is similar to the proof of Theorem 2.4, and makes use of an identity that is analogous to (2.5). This identity is

$$\frac{1}{\alpha^q - 1} = \frac{1}{L_q} - \frac{1}{2} + \frac{\sqrt{5}F_q}{2L_q}, \tag{3.3}$$

in which q is odd.

Theorem 3.4. *Let $n \geq 1$. Let $a \geq 1$, $b \geq 0$. Then*

$$S_7(n,a,b) = \frac{1}{2}\left(\frac{(-1)^n\left(\sqrt{5}F_{2an+a+b}+2\right)}{L_{2an+a+b}} - \frac{\sqrt{5}F_{a+b}+2}{L_{a+b}}\right), a \text{ even and } b \text{ odd,}$$

$$\sqrt{5}S_8(n,a,b) = \frac{1}{2}\left(\frac{(-1)^n\left(\sqrt{5}F_{2an+a+b}+2\right)}{L_{2an+a+b}} - \frac{\sqrt{5}F_{a+b}+2}{L_{a+b}}\right), a \text{ odd and } b \text{ even.}$$

Finally for this section, consider Theorem 3.2 with n even, and $a = b = 1$. We then see that

$$\sum_{i=1}^{n} \frac{(-1)^i L_{2i+1}}{F_{2i+1} - 1} = \frac{-5F_n}{L_{n+1}}, \text{ for } n \text{ even.}$$

4. Concluding Comments

Our search for algebraic identities that produce Fibonacci related sums (see also [3] and [4]) was initiated by the response of Almkvist [1] to the paper of Backstrom [2], who studies both finite and infinite sums associated with summands of the form

$$\frac{1}{F_{2ai+b} + c} \text{ and } \frac{1}{L_{2ai+b} + c},$$

for certain constants a, b, and c. While Backstrom's paper generated much interest his proofs are lengthy. Almkvist's contribution is to demonstrate that all of Backstrom's sums can be established, and proved quickly with the use of an algebraic identity. For more details and references, see [3, 4].

Algebraic identities like those employed here, and in [3] and [4], seem rare. However, as the reader can now appreciate, each such algebraic identity allows its discoverer to write down closed forms for a host of Fibonacci related sums and to prove them without fuss.

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