FURTHER PHYSICAL DERIVATIONS OF FIBONACCI SUMMATIONS

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Abstract. By interpreting various sums involving Fibonacci and Lucas numbers physically, we show how one can often generate an additional summation with little effort. To illustrate the fruitfulness of the approach, we establish some elegant summations that we believe are new.

1. Introduction

The Fibonacci and Lucas numbers are defined for all integers n by the recurrence relations

$$F_n = F_{n-1} + F_{n-2} \text{ where } F_1 = 1 \text{ and } F_2 = 1;$$
$$L_n = L_{n-1} + L_{n-2} \text{ where } L_1 = 1 \text{ and } L_2 = 3.$$

In articles [12] and [13] we showed that essentially physical arguments can be used to find various summations involving Fibonacci numbers. In this paper we detail a more systematic treatment of this approach. We show how various summation formulas involving Fibonacci and Lucas numbers can be interpreted physically, and how an appealing secondary summation can often be deduced as a natural consequence. For instance, our first example will demonstrate how the first summation in (1.1) implies the second summation,

$$\sum_{j=1}^{n} F_j^2 = F_n F_{n+1} \Longrightarrow \sum_{j=1}^{n} F_j^2 F_{2j} = F_n^2 F_{n+1}^2. \tag{1.1}$$

While both of these are well-known, we believe that the derivation of the second identity as a physical consequence of the first identity to be new. We will also establish some identities that we believe are original. Most notably, in Proposition 3.5 we prove that

$$\sum_{j=1}^{n} F_j^3 F_{j+1}^3 = \left(\sum_{j=1}^{n} F_j^2 F_{j+1} \right)^2. \tag{1.2}$$

The reader will note the resemblance between identity (1.2) and the iconic formula for the sum of cubes,

$$\sum_{j=1}^{n} j^3 = \left(\sum_{j=1}^{n} j \right)^2. \tag{1.3}$$

The similarity is not superficial. We can deduce each of these formulas using the same method, which we will outline in Section 2. In this respect, the work that follows can be regarded as a generalization of (1.3). By looking at the proofs of these identities we will come to see each as a physical imperative rather than a neat algebraic coincidence.

2. The Method and the Main Result

The required theory is elementary. Consider a rectangle $R \subset \mathbb{R}^2$ of uniform density and area A, placed lengthwise along the x-axis. The *balancing point* $\overline{x}$ of this rectangular figure is its midpoint. If R is partitioned into sub-rectangles $(R_j)_{j=1}^{n}$ by a series of vertical lines then the balancing point of R can also be found by finding the midpoint $\overline{x}_j$ and area A_j of each part and then computing a weighted average,

$$\overline{x} = \frac{\sum_{j=1}^{n} A_j \overline{x}_j}{A}. \tag{2.1}$$

This is illustrated below in Figure 1.

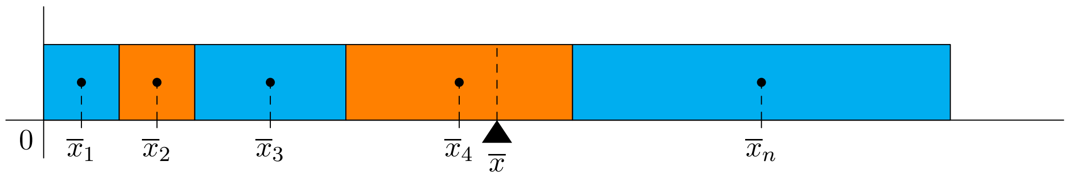

FIGURE 1

We now show how equation (2.1) can be used to deduce each of (1.1), (1.2) and (1.3) in the introduction, amongst other results. To this end, we take any finite sequence of positive numbers $(a_j)_{j=1}^n$ and let $(S_j)_{j=1}^n$ be the sequence of partial sums,

$$S_j = \sum_{k=1}^{j} a_k.$$

We then construct rectangles of dimensions

$$a_1 \times 1, a_2 \times 1, \ldots, a_n \times 1,$$

and then arrange these rectangles along the x-axis as shown below in Figure 2.

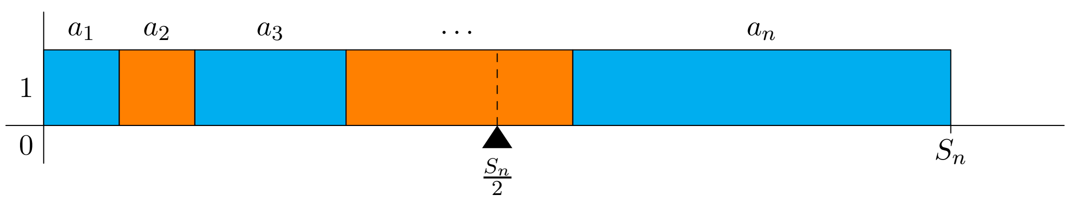

FIGURE 2

For each $1 \leq j \leq n$, the midpoint $\overline{x}_j$ of rectangle j is

$$\overline{x}_j = \frac{1}{2}(a_j + 2S_{j-1}). \tag{2.2}$$

The balancing point $\overline{x}$ of the combined rectangle will be its midpoint,

$$\overline{x} = \frac{1}{2}S_n, \tag{2.3}$$

which can also be found using equations (2.1) and (2.2) giving

$$\overline{x} = \frac{\sum_{j=1}^{n} a_j \overline{x}_j}{S_n} = \frac{\sum_{j=1}^{n} \frac{1}{2}\left(a_j^2 + 2a_j S_{j-1}\right)}{S_n}. \tag{2.4}$$

By equating the right-hand sides of (2.4) and (2.3) we prove Theorem 2.1. This is the central result of the paper, and will be deployed extensively throughout Section 3.

Theorem 2.1. *For any finite sequence of positive numbers* $(a_j)_{j=1}^n$ *whose partial sums are* $(S_j)_{j=1}^n$ *we have*

$$\sum_{j=1}^{n} \left(a_j^2 + 2a_j S_{j-1}\right) = S_n^2. \tag{2.5}$$

A given sequence $(a_j)_{j=1}^n$ must satisfy two informal conditions for equation (2.5) to yield an interesting result:

(1) the closed form of S_n must be known and,
(2) $(a_j^2 + 2a_j S_{j-1})_{j=1}^n$ is a sequence that we would like to sum.

As we will now see, such examples are not hard to come by.

3. Applications

For our first example we give a simple deduction of equation (1.3).

Example 3.1. Let $a_j = j$ so that

$$S_n = \sum_{j=1}^{n} j = \frac{n(n+1)}{2}.$$

It follows that

$$a_j^2 + 2a_j S_{j-1} = j^2 + 2j\frac{j(j-1)}{2} = j^3$$

in which case equation (2.5) yields

$$\sum_{j=1}^{n} j^3 = \left(\frac{n(n+1)}{2}\right)^2 = \left(\sum_{j=1}^{n} j\right)^2.$$

Note that we have established the formula for the sum of cubes without assuming the formula for the sum of squares, a prerequisite in many familiar proofs. Of course, the example is elementary. However, it does provide an interesting physical interpretation of a well-known result.

Equation (2.5), while valid for any finite sequence of positive numbers, is often productive when each term of the sequence is some product of Fibonacci or Lucas numbers, as the following two simple examples serve to illustrate.

Example 3.2. We first let $a_j = F_j^2$. It is well-known that

$$S_n = \sum_{j=1}^{n} F_j^2 = F_n F_{n+1}.$$

It follows that

$$\begin{aligned} a_j^2 + 2a_j S_{j-1} &= F_j^4 + 2F_j^3 F_{j-1} \\ &= F_j^3(F_j + 2F_{j-1}) \\ &= F_j^3 L_j && \text{(as } L_j = F_j + 2F_{j-1}) \\ &= F_j^2 F_{2j} && \text{(as } F_{2j} = F_j L_j) \end{aligned}$$

in which case equation (2.5) yields

$$\sum_{j=1}^{n} F_j^2 F_{2j} = F_n^2 F_{n+1}^2,$$

which was also deduced by Melham in [7] using an altogether different approach.

Example 3.3. For our second example we let $a_j = F_{2j}$. We will employ the well-known result that

$$S_n = \sum_{j=1}^{n} F_{2j} = F_{2n+1} - 1.$$

Subsequently,

$$\begin{aligned} a_j^2 + 2a_j S_{j-1} &= F_{2j}^2 + 2F_{2j}(F_{2(j-1)+1} - 1) \\ &= F_{2j}(F_{2j} + 2F_{2j-1}) - 2F_{2j} \\ &= F_{2j}L_{2j} - 2F_{2j} && (\text{as } L_{2j} = F_{2j} + 2F_{2j-1}) \\ &= F_{4j} - 2F_{2j} && (\text{as } F_{4j} = F_{2j}L_{2j}). \end{aligned}$$

Yet another application of equation (2.5) yields

$$\sum_{j=1}^{n} (F_{4j} - 2F_{2j}) = (F_{2n+1} - 1)^2$$

from which we readily obtain

$$\sum_{j=1}^{n} F_{4j} = F_{2n+1}^2 - 1.$$

Of course, the same procedure works for other summations and a sample of eight such results is summarized in Table 1 below. Though each will be familiar, the approach taken to establish these results is original.

TABLE 1. Each of the first summations implies the second summation.

(1)	$a_j = F_j$	:	$\sum_{j=1}^{n} F_j = F_{n+2} - 1$	$\Longrightarrow$	$\sum_{j=1}^{n} F_j F_{j+3} = F_{n+2}^2 - 1$
(2)	$a_j = L_j$	:	$\sum_{j=1}^{n} L_j = L_{n+2} - 3$	$\Longrightarrow$	$\sum_{j=1}^{n} L_j L_{j+3} = L_{n+2}^2 - 9$
(3)	$a_j = F_{2j}$	:	$\sum_{j=1}^{n} F_{2j} = F_{2n+1} - 1$	$\Longrightarrow$	$\sum_{j=1}^{n} F_{4j} = F_{2n+1}^2 - 1$
(4)	$a_j = L_{2j}$	:	$\sum_{j=1}^{n} L_{2j} = L_{2n+1} - 1$	$\Longrightarrow$	$\sum_{j=1}^{n} F_{4j} = \frac{1}{5}\left(L_{2n+1}^2 - 1\right)$
(5)	$a_j = F_{2j-1}$	:	$\sum_{j=1}^{n} F_{2j-1} = F_{2n}$	$\Longrightarrow$	$\sum_{j=1}^{n} F_{4j-2} = F_{2n}^2$
(6)	$a_j = L_{2j-1}$	:	$\sum_{j=1}^{n} L_{2j-1} = L_{2n} - 2$	$\Longrightarrow$	$\sum_{j=1}^{n} F_{4j-2} = \frac{1}{5}\left(L_{2n}^2 - 4\right)$
(7)	$a_j = F_j^2$	:	$\sum_{j=1}^{n} F_j^2 = F_n F_{n+1}$	$\Longrightarrow$	$\sum_{j=1}^{n} F_j^2 F_{2j} = F_n^2 F_{n+1}^2$
(8)	$a_j = L_j^2$	:	$\sum_{j=1}^{n} L_j^2 = L_n L_{n+1} - 2$	$\Longrightarrow$	$\sum_{j=1}^{n} L_j^2 F_{2j} = \frac{1}{5}\left(L_n^2 L_{n+1}^2 - 4\right)$

Interestingly, by comparing row (3) with row (4), and row (5) with row (6), we obtain the well-known and so-called *fundamental identity*,

$$L_n^2 - 5F_n^2 = (-1)^n 4, \tag{3.1}$$

the importance of which is detailed by Rabinowitz [9]. By looking at individual rows we can make further observations. For instance, by considering row (5) we see that

$$\sum_{j=1}^{n} F_{4j-2} = \left(\sum_{j=1}^{n} F_{2j-1}\right)^2.$$

Example 3.4. Suppose we now begin with the remarkable (and remarkably imposing) result deduced by Melham [7, equation (1.3)],

$$\sum_{j=1}^{n} F_j F_{j+1} F_{j+2}^2 F_{j+3} F_{j+4} = \frac{1}{4} F_n F_{n+1} F_{n+2} F_{n+3} F_{n+4} F_{n+5}. \tag{3.2}$$

As before, we let $a_j = F_j F_{j+1} F_{j+2}^2 F_{j+3} F_{j+4}$ so that by equation (3.2) we have

$$S_n = \frac{1}{4} F_n F_{n+1} F_{n+2} F_{n+3} F_{n+4} F_{n+5}.$$

It follows that

$$\begin{aligned} a_j^2 + 2a_j S_{j-1} &= F_j^2 F_{j+1}^2 F_{j+2}^4 F_{j+3}^2 F_{j+4}^2 + \frac{1}{2} F_{j-1} F_j^2 F_{j+1}^2 F_{j+2}^3 F_{j+3}^2 F_{j+4}^2 \\ &= \frac{1}{2} F_j^2 F_{j+1}^2 F_{j+2}^3 F_{j+3}^2 F_{j+4}^2 (2F_{j+2} + F_{j-1}) \\ &= \frac{1}{2} F_j^2 F_{j+1}^2 F_{j+2}^3 F_{j+3}^2 F_{j+4}^2 L_{j+2} && (\text{as } L_{j+2} = 2F_{j+2} + F_{j-1}) \\ &= \frac{1}{2} F_j^2 F_{j+1}^2 F_{j+2}^2 F_{j+3}^2 F_{j+4}^2 F_{2j+4} && (\text{as } F_{2j+4} = F_{j+2} L_{j+2}). \end{aligned}$$

Therefore equation (2.5) yields

$$\sum_{j=1}^{n} F_j^2 F_{j+1}^2 F_{j+2}^2 F_{j+3}^2 F_{j+4}^2 F_{2j+4} = \frac{1}{8} F_n^2 F_{n+1}^2 F_{n+2}^2 F_{n+3}^2 F_{n+4}^2 F_{n+5}^2. \tag{3.3}$$

Equation (3.3) is also deduced by Melham [8, equation (2.7)], independently of (3.2) rather than as a consequence of it.

The method that we have discovered can also be applied sequentially. For instance, if we begin with an elegant result first discovered by Block in [2],

$$\sum_{j=1}^{n} F_j^2 F_{j+1} = \frac{1}{2} F_n F_{n+1} F_{n+2}, \tag{3.4}$$

then we can establish equations (3.5) and (3.6) below. We believe that these are original. Note that (3.4) is easily proved; although induction will suffice, Clary and Hemenway [3, p.135] demonstrate a particularly clean approach.

Proposition 3.5. *Equation* (3.4) *implies* (3.5) *which in turn implies* (3.6) *where*

$$\sum_{j=1}^{n} F_j^3 F_{j+1}^3 = \frac{1}{4} F_n^2 F_{n+1}^2 F_{n+2}^2, \tag{3.5}$$

$$\sum_{j=1}^{n} F_j^5 F_{j+1}^5 F_{2j+1} = \frac{1}{8} F_n^4 F_{n+1}^4 F_{n+2}^4. \tag{3.6}$$

Proof. We prove that equation (3.5) implies (3.6). Let $a_j = F_j^3 F_{j+1}^3$ so that by equation (3.5) we have

$$S_n = \frac{1}{4} F_n^2 F_{n+1}^2 F_{n+2}^2.$$

Therefore,

$$\begin{aligned} a_j^2 + 2a_j S_{j-1} &= F_j^6 F_{j+1}^6 + \frac{1}{2} F_{j-1}^2 F_j^5 F_{j+1}^5 \\ &= \frac{1}{2} F_j^5 F_{j+1}^5 (2F_j F_{j+1} + F_{j-1}^2) \\ &= \frac{1}{2} F_j^5 F_{j+1}^5 F_{2j+1} \end{aligned} \tag{3.7}$$

where line (3.7) is deduced from the fact that

$$\begin{aligned} 2F_j F_{j+1} + F_{j-1}^2 &= 2(F_j + F_{j-1}) F_j + F_{j-1}^2 \\ &= F_j^2 + F_j^2 + 2F_j F_{j-1} + F_{j-1}^2 \\ &= F_j^2 + (F_j + F_{j-1})^2 \\ &= F_j^2 + F_{j+1}^2 \\ &= F_{2j+1}. \end{aligned}$$

Finally, we obtain equation (3.6) by making use of equation (3.7) in (2.5). The proof that equation (3.4) implies (3.5) is similar and is therefore omitted. □

A couple remarks. First, note that equations (3.4) and (3.5) imply (1.2) that is given in the introduction. And secondly, we do not obtain a formula of any particular interest if we apply the method once more to (3.6).

The method of discovery of equations (3.5) and (3.6) does not work quite as gracefully for Lucas numbers. Nonetheless, each *does* have a corresponding counterpart. In this case, we begin with the Lucas counterpart to equation (3.4) [7, equation (3.2)],

$$\sum_{j=1}^{n} L_j^2 L_{j+1} = \frac{[L_j L_{j+1} L_{j+2}]_0^n}{2}. \tag{3.8}$$

Here, and later, we follow the convention that j serves as a dummy variable on the right-hand side of (3.8),

$$[L_j L_{j+1} L_{j+2}]_0^n = L_n L_{n+1} L_{n+2} - L_0 L_1 L_2.$$

Assuming (3.8) we easily obtain Proposition 3.6, which we also believe to be original. We have omitted the proof as it is no more illuminating than the proof of Proposition 3.5.

Proposition 3.6. *Given equation* (3.8) *we have*

$$\sum_{j=1}^{n} L_j^3 L_{j+1}^3 = \frac{[L_j^2 L_{j+1}^2 L_{j+2}^2]_0^n}{4}, \tag{3.9}$$

$$\sum_{j=1}^{n} L_j^5 L_{j+1}^5 F_{2j+1} = \frac{[L_j^4 L_{j+1}^4 L_{j+2}^4]_0^n}{40}. \tag{3.10}$$

We conclude with one final result, the proof of which depends on an identity that has recently been considered by Sury [11], and which has received further attention in this *Fibonacci Quarterly* [4] and elsewhere (see, e.g., [5] and [6]),

$$\sum_{j=0}^{n} 2^j L_j = 2^{n+1} F_{n+1}. \tag{3.11}$$

Using the method outlined in this paper, the reader might like to show that equation (3.11) yields what we believe is a further original identity,

$$\sum_{j=0}^{n} 2^{2j} L_j F_{j+3} = 2^{2n+2} F_{n+1}^2. \tag{3.12}$$

4. Closing Remarks

This paper could not possibly provide a complete account of the Fibonacci and Lucas summations discoverable by the method that we have detailed. Therefore the reader has ample opportunity to discover additional results using this method. Moreover, combinatorists might also like to ponder if the new identities presented here have proofs, similar to those detailed in Benjamin and Quinn [1].

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