ON GENERALIZED MULTI POLY-EULER POLYNOMIALS

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Abstract. In this paper, we establish more identities of generalized multi poly-Euler polynomials with three parameters and obtain a kind of symmetrized generalization of the polynomials.

1. Introduction

Euler numbers and polynomials have a rich literature in the history of mathematics where many identities and properties have been established including their mathematical and physical applications. These numbers and polynomials have close connections with Bernoulli numbers and polynomials, particularly in the structures of their properties and generalizations. In fact, in almost every property and generalization of Bernoulli numbers and polynomials there corresponds property and generalization for Euler numbers and polynomials. For instance, Kaneko [21] introduced the poly-Bernoulli numbers $B_n^{(k)}$ by means of the following exponential generating function

$$\frac{\mathrm{Li}_k(1-e^{-x})}{1-e^{-x}} = \sum_{n=0}^{\infty} B_n^{(k)} \frac{x^n}{n!}$$

where

$$\mathrm{Li}_k(z) = \sum_{n=1}^{\infty} \frac{z^n}{n^k},$$

while Ohno and Sasaki [25] defined poly-Euler numbers as

$$\frac{\mathrm{Li}_k(1-e^{-4t})}{4t\cosh t} = \sum_{n=0}^{\infty} E_n^{(k)} \frac{t^n}{n!}$$

which have been recently extended by H. Jolany et al. [20] in polynomial form as

$$\frac{2\mathrm{Li}_k(1-e^{-t})}{1+e^t} e^{xt} = \sum_{n=0}^{\infty} E_n^{(k)}(x) \frac{t^n}{n!}. \tag{1.1}$$

Arakawa and Kaneko [2] obtained the following explicit formula

$$B_n^{(-k)} = \sum_{m\geq 0} m! \left\{ {n+1 \atop m+1} \right\} m! \left\{ {k+1 \atop m+1} \right\}, \tag{1.2}$$

where $\left\{ {n \atop m} \right\}$ denote Stirling numbers of the second kind. This implies that $B_n^{(-k)}$ are nonnegative integers. In fact, $B_n^{(-k)}$ can be interpreted in terms of the number of partitions as $\left\{ {n \atop m} \right\}$ count the number of partitions of an n-set into m nonempty subsets. Moreover, these numbers can also be interpreted as the number of binary lonesum matrices of size $n \times k$ where

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the binary lonesum matrix is a binary matrix which can be reconstructed from its row and column sums [5].

On the other hand, for any positive integer n, the poly-Euler numbers E_n^k with $k = 0, -1$ can be expressed as (see [25])

$$nE_{n-1}^{(0)} = \frac{3^n - 1}{2}, \quad nE_{n-1}^{(-1)} = \frac{7^n - 5^n}{2}.$$

It has been conjectured by Sasaki and Ohno [25] that the sequence $nE_{n-1}^{(0)} = (3^n - 1)/2$ coincides with the number of non-parallel segments of pairs of distinct vertices in the n-dimensional hypercube, while the sequence $nE_{n-1}^{(-1)} = (7^n - 5^n)/2$ coincides with the number of combinations such that five points exist on the same line inside of the $\underbrace{5 \times \cdots \times 5}_{n}$-dimensional lattice. These interpretations have been verified to be true for $n \leq 9$. We observe that it is quite difficult to deal with combinatorial interpretation of poly-Euler polynomials. However, an explicit formula in [16] for $E_n^{(-k)}$ which is given by

$$E_n^{(-k)} = \sum_{m=0}^{n} \sum_{j=0}^{m} \sum_{i=0}^{j} 2(-1)^{m+n-j+i} j^k \binom{j}{i} (m - j + i + 1)^n \tag{1.3}$$

can be used to give a combinatorial interpretation for poly-Euler numbers. We know that this formula can further be written as

$$E_n^{(-k)} = \sum_{m=0}^{n} 2(-1)^{m+n} \sum_{j=0}^{m} j^k j! \left\{ {n+m-j+1 \atop m+1} \right\}_{m-j+1}$$

where the numbers $\left\{ {n+m-j+1 \atop m+1} \right\}_{m-j+1}$ are called r-Stirling numbers of the second kind (see [7, 24]). The numbers

$$\mathcal{G}(n, k; m) := \sum_{j=0}^{m} j^k j! \left\{ {n+m-j+1 \atop m+1} \right\}_{m-j+1}$$

can be interpreted as the total number of ways to partition an $(n + m - j + 1)$-set into $m + 1$ nonempty subsets such that the first $m - j + 1$ elements are in distinct subsets and that the other j subsets are considered as distinct boxes where additional k distinct elements are distributed, for each j ranges from 0 to m. Hence, the poly-Euler numbers $E_n^{(-k)}$ are equal to the following differences

$$E_n^{(-k)} = \sum_{\substack{n+m \text{ is even} \\ m \in \{0,1,\ldots,n\}}} 2\mathcal{G}(n, k; m) - \sum_{\substack{n+m \text{ is odd} \\ m \in \{0,1,\ldots,n\}}} 2\mathcal{G}(n, k; m).$$

The multiple polylogarithms are usually defined as

$$\mathrm{Li}_{(k_1,k_2,\ldots,k_r)}(z_1, z_2, \ldots, z_r) = \sum_{0<m_1<m_2<\ldots<m_r} \prod_{j=1}^{r} m_j^{-k_j} z_j^{n_j}, \tag{1.4}$$

where k_1, k_2, ..., k_r and z_1, z_2, ..., z_r are complex numbers suitably restricted so that the sum (1.4) converges. These polynomials are certain generalization of the nested harmonic sums as well as the Riemann zeta function and the ordinary polylogarithm, which preserve many interesting properties. They occur in various fields like combinatorics, knot theory, quantum field theory and mirror symmetry. In fact, there are several sophisticated studies that relate multiple polylogarithms to arithmetic and algebraic geometry and to algebraic K-theory.

The idea of poly-Bernoulli and poly-Euler numbers and polynomials can be extended to a multiple parameter case. In particular, some studies consider a special case of (1.4) in which

$$z_1 = z_3 = \ldots = z_{r-1} = 1, z_r = z,$$

that is, the studies that deal with the multiple polylogarithm of the form

$$\mathrm{Li}_{(k_1,k_2,\ldots,k_r)}(z) = \sum_{0<m_1<m_2<\ldots<m_r} \frac{z^{m_r}}{m_1^{k_1} m_2^{k_2} \ldots m_r^{k_r}}. \tag{1.5}$$

For instance, Imatomi et al. [14] have defined a certain generalization of Bernoulli numbers in terms of these multiple logarithms as follows

$$\frac{\mathrm{Li}_{(k_1,k_2,\ldots,k_r)}(1-e^{-t})}{1-e^{-t}} = \sum_{n=0}^{\infty} B_n^{(k_1,k_2,\ldots,k_r)} \frac{t^n}{n!}. \tag{1.6}$$

These numbers possess, respectively, the following recurrence relation and explicit formula

$$B_n^{(k_1,k_2,\ldots,k_r)} = \frac{1}{n+1}\left(B_n^{(k_1-1,k_2,\ldots,k_r)} - \sum_{m=1}^{n-1}\binom{n}{m-1} B_m^{(k_1,k_2,\ldots,k_r)}\right) \tag{1.7}$$

$$B_n^{(k_1,k_2,\ldots,k_r)} = (-1)^n \sum_{n+1\geq m_1>m_2>\ldots>m_r>0} \frac{(-1)^{m_1-1}(m_1-1)!S(n,m_1-1)}{m_1^{k_1} m_2^{k_2} \ldots m_r^{k_r}}. \tag{1.8}$$

Parallel to the above generalization is the generalized multi poly-Euler polynomials which are denoted by $E_n^{(k_1,k_2,\ldots,k_r)}(x;a,b,c)$. These polynomials have been introduced in [19] by means of the above multiple poly-logarithm, also known as multiple zeta values. More precisely, we have

$$\frac{2\mathrm{Li}_{(k_1,k_2,\ldots,k_r)}(1-(ab)^{-t})}{(a^{-t}+b^t)^r} c^{rxt} = \sum_{n=0}^{\infty} E_n^{(k_1,k_2,\ldots,k_r)}(x;a,b,c)\frac{t^n}{n!}. \tag{1.9}$$

When $r=1$, (1.9) boils down to the generalized poly-Euler polynomials with three parameters a, b, c. Moreover, when $c = e$, (1.9) reduces to the multi poly-Euler polynomials with two parameters a, b. These special cases have been discussed intensively in [19, 12].

Some properties of generalized multi poly-Euler polynomials $E_n^{(k_1,k_2,\ldots,k_r)}(x;a,b,c)$ are established in [19] which include the following identities

$$E_n^{(k_1,k_2,\ldots,k_r)}(x;a,b,c) = \sum_{i=0}^{n}\binom{n}{i}(r\ln c)^{n-i} E_i^{(k_1,k_2,\ldots,k_r)}(a,b)x^{n-i} \tag{1.10}$$

$$E_n^{(k_1,k_2,\ldots,k_r)}(x;a,b,c) = (\ln a + \ln b)^n E_n^{(k_1,k_2,\ldots,k_r)}\left(\frac{rx\ln c + \ln a}{\ln a + \ln b}\right) \tag{1.11}$$

$$\frac{d}{dx}E_{n+1}^{(k_1,k_2,\ldots,k_r)}(x;a,b,c) = (n+1)(r\ln c)E_n^{(k_1,k_2,\ldots,k_r)}(x;a,b,c)$$

$$E_n^{(k_1,k_2,\ldots,k_r)}(x+y;a,b,c) = \sum_{i=0}^{n}\binom{n}{i}(r\ln c)^{n-i}E_i^{(k_1,k_2,\ldots,k_r)}(x;a,b,c)y^{n-i}.$$

When $r = 1$, these identities reduce to

$$E_n^{(k)}(x;a,b,c) = \sum_{i=0}^{n}\binom{n}{i}(\ln c)^{n-i}E_i^{(k)}(a,b)x^{n-i}$$
$$E_n^{(k)}(x;a,b,c) = (\ln a+\ln b)^n E_n^{(k)}\left(\frac{x\ln c+\ln a}{\ln a+\ln b}\right)$$
$$\frac{d}{dx}E_{n+1}^{(k)}(x;a,b,c) = (n+1)(\ln c)E_n^{(k)}(x;a,b,c)$$
$$E_n^{(k)}(x+y;a,b,c) = \sum_{i=0}^{n}\binom{n}{i}(\ln c)^{n-i}E_i^{(k)}(x;a,b,c)y^{n-i}$$

which are properties of generalized poly-Euler polynomials (see [19]).

Furthermore, an explicit formula for $E_n^{(k_1,k_2,\ldots,k_r)}(x;a,b,c)$ is given by

$$E_n^{(k_1,k_2,\ldots,k_r)}(x;a,b,c) = \sum_{i=0}^{n}\sum_{\substack{0<m_1<m_2<\ldots<m_r\\ c_1+c_2+\ldots=r}}\mathcal{J}(m_1,m_2,\ldots,m_r) \tag{1.12}$$

where

$$\mathcal{J}(m_1,m_2,\ldots,m_r) = \sum_{j=0}^{m_r}\frac{2(rx\ln c-j\ln ab)^{n-i}r!(-1)^{j+s}(s\ln ab+r\ln a)^i\binom{m_r}{j}\binom{n}{i}}{(c_1!c_2!\ldots)(m_1^{k_1}m_2^{k_2}\ldots m_r^{k_r})}$$

with $s = c_1 + 2c_2 + \cdots$. Clearly, for appropriate restriction of the parameters involved, the polynomials $E_n^{(-k_1,-k_2,\ldots,-k_r)}(x;a,b,c)$ would give nonnegative integer values and may be given combinatorial interpretation using (1.12) parallel to the interpretation constructed for the explicit formula in (1.3).

In this paper, some identities of $E_n^{(k_1,k_2,\ldots,k_r)}(x;a,b,c)$ related to Stirling numbers of the second kind are established and a certain symmetrized generalization for $E_n^{(k_1,k_2,\ldots,k_r)}(x;a,b,c)$ is obtained.

2. Generalized Multi Poly-Euler Polynomials and Stirling Numbers

The generalized poly-Euler polynomials with three parameters a, b, and c are defined in [19] as follows

$$\frac{2\mathrm{Li}_k(1-(ab)^{-t})}{a^{-t}+b^t}c^{xt} = \sum_{n=0}^{\infty}E_n^{(k)}(x;a,b,c)\frac{t^n}{n!}. \tag{2.1}$$

Some identities on generalized poly-Euler polynomials are expressed in terms of Stirling numbers of the second kind. Such identities have appeared in Theorem 2.6 of [19] but with $c = e$. More precisely, we have the following theorem.

Theorem 2.1. [19] *The generalized poly-Euler polynomials $E_n^{(k)}(x;a,b)$ satisfy the following explicit formulas*

$$E_n^{(k)}(x;a,b)=\sum_{m=0}^{\infty}\sum_{l=m}^{n}\left\{{l \atop m}\right\}\binom{n}{l}E_{n-l}^{(k)}(-m;a,b)(x)^{(m)} \tag{2.2}$$

$$E_n^{(k)}(x;a,b)=\sum_{m=0}^{\infty}\sum_{l=m}^{n}\left\{{l \atop m}\right\}\binom{n}{l}E_{n-l}^{(k)}(0;a,b)(x)_m \tag{2.3}$$

$$E_n^{(k)}(x;a,b)=\sum_{m=0}^{\infty}\binom{n}{m}\sum_{l=0}^{n-m}\frac{\binom{n-m}{l}}{\binom{l+s}{l}}\left\{{l+s \atop s}\right\}E_{n-m-l}^{(k)}(0;a,b)B_m^{(s)}(x) \tag{2.4}$$

$$E_n^{(k)}(x;a,b)=\sum_{m=0}^{\infty}\frac{\binom{n}{m}}{(1-\lambda)^s}\sum_{j=0}^{s}\binom{s}{j}(-\lambda)^{s-j}E_{n-m}^{(k)}(j;a,b)H_m^{(s)}(x;\lambda), \tag{2.5}$$

where $(x)^{(n)}=x(x+1)\ldots(x+n-1)$, $(x)_n=x(x-1)\ldots(x-n+1)$,

$$\left(\frac{t}{e^t-1}\right)^s e^{xt}=\sum_{n=0}^{\infty}B_n^{(s)}(x)\frac{t^n}{n!} \text{ and } \left(\frac{1-\lambda}{e^t-\lambda}\right)^s e^{xt}=\sum_{n=0}^{\infty}H_n^{(s)}(x;\lambda)\frac{t^n}{n!}.$$

We notice that the generalized poly-Euler polynomials $E_n^{(k)}(x;a,b)$ are expressed not only in terms of Stirling numbers but also using certain generalizations of Bernoulli numbers. This will help us further understand the structure of $E_n^{(k)}(x;a,b)$. This idea can also be done to further understand the structure of $E_n^{(k_1,k_2,\ldots,k_r)}(x;a,b,c)$. Here, we derive some identities for $E_n^{(k_1,k_2,\ldots,k_r)}(x;a,b,c)$ which are parallel to those in Theorem 2.1. The first such identity is given in the following theorem.

Theorem 2.2.

$$E_n^{(k_1,k_2,\ldots,k_r)}(x;a,b,c)=\sum_{m=0}^{\infty}\sum_{l=m}^{n}(r\log c)^l\left\{{l \atop m}\right\}\binom{n}{l}E_{n-l}^{(k_1,k_2,\ldots,k_r)}(-m\log c;a,b)(x)^{(m)}. \tag{2.6}$$

Proof. Note that (1.9) can be written as

$$\sum_{n=0}^{\infty}E_n^{(k_1,k_2,\ldots,k_r)}(x;a,b,c)\frac{t^n}{n!}=\frac{2\mathrm{Li}_{(k_1,k_2,\ldots,k_r)}(1-(ab)^{-t})}{(a^{-t}+b^t)^r}(1-(1-e^{-rt\log c}))^{-x}.$$

Using Newton's Binomial Theorem, we have

$$\sum_{n=0}^{\infty}E_n^{(k_1,k_2,\ldots,k_r)}(x;a,b,c)\frac{t^n}{n!}=\frac{2\mathrm{Li}_{(k_1,k_2,\ldots,k_r)}(1-(ab)^{-t})}{(a^{-t}+b^t)^r}\sum_{m=0}^{\infty}\binom{x+m-1}{m}(1-e^{-rt\log c})^m$$

$$=\sum_{m=0}^{\infty}(x)^{(m)}\frac{(e^{rt\log c}-1)^m}{m!}\frac{2\mathrm{Li}_{(k_1,k_2,\ldots,k_r)}(1-(ab)^{-t})}{(a^{-t}+b^t)^r}e^{-mrt\log c}$$

$$=\sum_{m=0}^{\infty}(x)^{(m)}\left(\sum_{n=0}^{\infty}\left\{{n \atop m}\right\}\frac{(rt\log c)^n}{n!}\right)\left(\sum_{n=0}^{\infty}E_n^{(k_1,k_2,\ldots,k_r)}(-mr\log c;a,b)\frac{t^n}{n!}\right)$$

$$=\sum_{n=0}^{\infty}\left\{\sum_{m=0}^{\infty}\sum_{l=m}^{n}(r\log c)^l\left\{{l \atop m}\right\}\binom{n}{l}E_{n-l}^{(k_1,k_2,\ldots,k_r)}(-mr\log c;a,b)(x)^{(m)}\right\}\frac{t^n}{n!}$$

Comparing coefficients completes the proof of (2.6). □

In particular, when $c = e$ and $r = 1$, (2.6) yields (2.2). For the generalization of (2.3), we have the following theorem.

Theorem 2.3.

$$E_n^{(k_1,k_2,\ldots,k_r)}(x;a,b,c) = \sum_{m=0}^{\infty}\sum_{l=m}^{n}(r\log c)^l \left\{ {l \atop m} \right\} \binom{n}{l} E_{n-l}^{(k_1,k_2,\ldots,k_r)}(0;a,b)(x)_m. \tag{2.7}$$

Proof. The proof also makes use of (1.9) and Newton's Binomial Theorem. □

Theorem 2.4.

$$E_n^{(k_1,k_2,\ldots,k_r)}(x;a,b,c) = \sum_{m=0}^{\infty}\binom{n}{m}\sum_{l=0}^{n-m}\frac{\binom{n-m}{l}}{\binom{l+s}{l}}\left\{ {l+s \atop s} \right\} E_{n-m-l}^{(k_1,k_2,\ldots,k_r)}(0;a,b)B_m^{(s)}(xr\log c). \tag{2.8}$$

Proof. Note that (1.9) can be written as

$$G(t) = \sum_{n=0}^{\infty} E_n^{(k_1,k_2,\ldots,k_r)}(x;a,b,c)\frac{t^n}{n!}$$

$$= \left(\frac{(e^t-1)^s}{s!}\right)\left(\frac{t^s e^{xrt\log c}}{(e^t-1)^s}\right)\left(\frac{2\mathrm{Li}_{(k_1,k_2,\ldots,k_r)}(1-(ab)^{-t})}{(a^{-t}+b^t)^r}\right)\frac{s!}{t^s}$$

$$= \left(\sum_{n=0}^{\infty}\left\{ {n+s \atop s} \right\}\frac{t^{n+s}}{(n+s)!}\right)\left(\sum_{m=0}^{\infty}B_m^{(s)}(xr\log c)\frac{t^m}{m!}\right)\left(\sum_{n=0}^{\infty}E_n^{(k_1,k_2,\ldots,k_r)}(0;a,b)\frac{t^n}{n!}\right)\frac{s!}{t^s}$$

$$= \sum_{n=0}^{\infty}\left\{\sum_{m=0}^{n}\binom{n}{m}\sum_{l=0}^{n-m}\frac{\binom{n-m}{l}}{\binom{l+s}{s!}}\left\{ {l+s \atop s} \right\}E_{n-m-l}^{(k_1,k_2,\ldots,k_r)}(0;a,b)B_m^{(s)}(xr\log c)\right\}\frac{t^n}{n!}.$$

Comparing coefficients completes the proof of (2.8). □

Theorem 2.5.

$$E_n^{(k_1,k_2,\ldots,k_r)}(x;a,b,c) = \sum_{m=0}^{n}\frac{\binom{n}{m}}{(1-\lambda)^s}\sum_{j=0}^{s}\binom{s}{j}(-\lambda)^{s-j}E_{n-m}^{(k_1,k_2,\ldots,k_r)}(j;a,b)H_m^{(s)}(xr\log c;\lambda). \tag{2.9}$$

Proof. Using (1.9), we have

$$\sum_{n=0}^{\infty}E_n^{(k_1,k_2,\ldots,k_r)}(x;a,b,c)\frac{t^n}{n!} = \left(\frac{(1-\lambda)^s}{(e^t-\lambda)^s}e^{xrt\log c}\right)\left(\frac{(e^t-\lambda)^s}{(1-\lambda)^s}\right)\left(\frac{2\mathrm{Li}_{(k_1,k_2,\ldots,k_r)}(1-(ab)^{-t})}{(a^{-t}+b^t)^r}\right)$$

$$= \left(\sum_{n=0}^{\infty}H_n^{(s)}(xr\log c;\lambda)\frac{t^n}{n!}\right)\left(\sum_{j=0}^{s}\binom{s}{j}(-\lambda)^{s-j}\frac{2\mathrm{Li}_{(k_1,k_2,\ldots,k_r)}(1-(ab)^{-t})}{(a^{-t}+b^t)^r}e^{jt}\right)$$

$$= \sum_{n=0}^{\infty}\left(\sum_{m=0}^{n}\frac{\binom{n}{m}}{(1-\lambda)^s}\sum_{j=0}^{s}\binom{s}{j}(-\lambda)^{s-j}E_{n-m}^{(k_1,k_2,\ldots,k_r)}(j;a,b)H_m^{(s)}(xr\log c;\lambda)\right)\frac{t^n}{n!}.$$

Comparing coefficients completes the proof of (2.9). □

The next theorem contains an identity which is obtained by making use of the following differential formula for the generalized poly-logarithm (Hamahata and Masubuchi, Integers)

$$\frac{d}{dz}\mathrm{Li}_{k_1,\ldots,k_r}(z) = \begin{cases} \frac{1}{z}\mathrm{Li}_{k_1,\ldots,k_{r-1},k_r-1}(z) & \text{if } k_r > 1; \\ \frac{1}{1-z}\mathrm{Li}_{k_1,\ldots,k_{r-1}}(z) & \text{if } k_r = 1. \end{cases} \tag{2.10}$$

Theorem 2.6. *If $k_r > 1$, then*

$$\begin{aligned} &E_{n+1}^{(k_1,\ldots,k_r)}(x) - r\log(ac^x)E_n^{(k_1,\ldots,k_r)}(x) \\ &\quad = r\sum_{\nu=0}^{\infty}\binom{n}{\nu}\sum_{j=0}^{\infty}\big(-j\log(ab)\big)^{n-\nu}E_\nu^{(k_1,\ldots,k_r)}(x) \\ &\qquad - \log(ab)\sum_{\nu=0}^{\infty}\binom{n}{\nu}\sum_{j=0}^{\infty}\big(j\log(ab)\big)^{n-\nu}E_\nu^{(k_1,\ldots,k_{r-1},k_r-1)}(x). \end{aligned}$$

If $k_r = 1$, then

$$\begin{aligned} &E_{n+1}^{(k_1,\ldots,k_r)}(x) - r\log(ac^x)E_n^{(k_1,\ldots,k_r)}(x) \\ &\quad = r\sum_{\nu=0}^{\infty}\binom{n}{\nu}\sum_{j=0}^{\infty}\big(-j\log(ab)\big)^{n-\nu}E_\nu^{(k_1,\ldots,k_r)}(x) + \log(ab)E_n^{(k_1,\ldots,k_{r-1})}(x). \end{aligned}$$

Proof. We differentiate both sides of

$$2(ac^x)^{rt}\mathrm{Li}_{k_1,\ldots,k_r}\big(1-(ab)^{-t}\big) = \big(1+(ab)^t\big)^r\sum_{n=0}^{\infty}E_n^{(k_1,\ldots,k_r)}(x)\frac{t^n}{n!}$$

with respect to t using (2.10). If $k_1 \neq 1$, then

$$\begin{aligned} &2r(ac^x)^{rt}\log(ac^x)\mathrm{Li}_{k_1,\ldots,k_r}\big(1-(ab)^{-t}\big) \\ &\quad +2(ac^x)^{rt}\frac{-(ab)^{-t}\log(ab)}{1-(ab)^{-t}}\mathrm{Li}_{k_1,\ldots,k_{r-1},k_r-1}\big(1-(ab)^{-t}\big) \\ &\qquad = r\big(1+(ab)^t\big)^{r-1}(ab)^t\log(ab)\sum_{n=0}^{\infty}E_n^{(k_1,\ldots,k_r)}(x)\frac{t^n}{n!} \\ &\qquad\quad + \big(1+(ab)^t\big)^r\sum_{n=1}^{\infty}E_n^{(k_1,\ldots,k_r)}(x)\frac{t^{n-1}}{(n-1)!}. \end{aligned}$$

Dividing both sides by $\big(1+(ab)^t\big)^r$, we have

$$\begin{aligned} &r\log(ac^x)\frac{2(ac^x)^{rt}}{\big(1+(ab)^t\big)^r}\mathrm{Li}_{k_1,\ldots,k_r}\big(1-(ab)^{-t}\big) \\ &\quad + \frac{\log(ab)}{1-(ab)^t}\frac{2(ac^x)^{rt}}{\big(1+(ab)^t\big)^r}\mathrm{Li}_{k_1,\ldots,k_{r-1},k_r-1}\big(1-(ab)^{-t}\big) \\ &= \frac{r(ab)^t}{1+(ab)^t}\log(ab)\sum_{n=0}^{\infty}E_n^{(k_1,\ldots,k_r)}(x)\frac{t^n}{n!} + \sum_{n=0}^{\infty}E_{n+1}\frac{t^n}{n!}. \end{aligned}$$

Since

$$\begin{aligned}\frac{\log(ab)}{1-(ab)^t} &= \log(ab)\sum_{j=0}^{\infty}(ab)^{jt} = \log(ab)\sum_{j=0}^{\infty}e^{jt\log(ab)}\\ &= \log(ab)\sum_{j=0}^{\infty}\sum_{\nu=0}^{\infty}\frac{\big(j\log(ab)\big)^{\nu}}{\nu!}t^{\nu}\end{aligned}$$

and

$$\begin{aligned}\frac{r(ab)^t}{1+(ab)^t} &= r\sum_{j=0}^{\infty}(ab)^{-jt}\\ &= r\sum_{j=0}^{\infty}\sum_{\nu=0}^{\infty}\frac{\big(-j\log(ab)\big)^{\nu}}{\nu!}t^{\nu},\end{aligned}$$

we obtain

$$\begin{aligned}&r\log(ac^x)\sum_{n=0}^{\infty}E_n^{(k_1,\dots,k_r)}(x)\frac{t^n}{n!}\\ &\qquad+\log(ab)\sum_{j=0}^{\infty}\sum_{\nu=0}^{\infty}\big(j\log(ab)\big)^{\nu}\frac{t^{\nu}}{\nu!}\sum_{\mu=0}^{\infty}E_{\mu}^{(k_1,\dots,k_{r-1},k_r-1)}(x)\frac{t^{\mu}}{\mu!}\\ &=r\sum_{j=0}^{\infty}\sum_{\nu=0}^{\infty}\big(-j\log(ab)\big)^{\nu}\frac{t^{\nu}}{\nu!}\sum_{\mu=0}^{\infty}E_{\mu}\frac{t^{\mu}}{\mu!}+\sum_{n=0}^{\infty}E_{n+1}^{(k_1,\dots,k_r)}\frac{t^n}{n!}.\end{aligned}$$

Hence,

$$\begin{aligned}&-r\log(ac^x)\sum_{n=0}^{\infty}E_n^{(k_1,\dots,k_r)}(x)\frac{t^n}{n!}+\sum_{n=0}^{\infty}E_{n+1}^{(k_1,\dots,k_r)}(x)\frac{t^n}{n!}\\ &=r\sum_{n=0}^{\infty}\left(\sum_{\nu=0}^{\infty}\binom{n}{\nu}\sum_{j=0}^{\infty}\big(-j\log(ab)\big)^{n-\nu}E_{\nu}^{(k_1,\dots,k_r)}(x)\right)\frac{t^n}{n!}\\ &\quad-\log(ab)\sum_{n=0}^{\infty}\left(\sum_{\nu=0}^{\infty}\binom{n}{\nu}\sum_{j=0}^{\infty}\big(j\log(ab)\big)^{n-\nu}E_{\nu}^{(k_1,\dots,k_{r-1},k_r-1)}(x)\right)\frac{t^n}{n!}.\end{aligned}$$

Comparing the coefficients on both sides, we have

$$\begin{aligned}&E_{n+1}^{(k_1,\dots,k_r)}(x)-r\log(ac^x)E_n^{(k_1,\dots,k_r)}(x)\\ &=r\sum_{\nu=0}^{\infty}\binom{n}{\nu}\sum_{j=0}^{\infty}\big(-j\log(ab)\big)^{n-\nu}E_{\nu}^{(k_1,\dots,k_r)}(x)\\ &\quad-\log(ab)\sum_{\nu=0}^{\infty}\binom{n}{\nu}\sum_{j=0}^{\infty}\big(j\log(ab)\big)^{n-\nu}E_{\nu}^{(k_1,\dots,k_{r-1},k_r-1)}(x).\end{aligned}$$

If $k_r=1$, then the second term on the left-hand side becomes

$$2(ac^x)^{rt}\frac{-(ab)^{-t}\log(ab)}{(ab)^{-t}}\mathrm{Li}_{k_1,\dots,k_{r-1}}.$$

After dividing of $\big(1+(ab)^t\big)^r$, this second term becomes

$$\begin{aligned}&-\log(ab)\frac{2(ac^x)^{rt}}{\big(1+(ab)^t\big)^r}\mathrm{Li}_{k_1,\ldots,k_{r-1}}\\&=-\log(ab)\sum_{n=0}^{\infty}E_n^{(k_1,\ldots,k_{r-1})}(x)\frac{t^n}{n!}.\end{aligned}$$

Finally, we get

$$\begin{aligned}&E_{n+1}^{(k_1,\ldots,k_r)}(x)-r\log(ac^x)E_n^{(k_1,\ldots,k_r)}(x)\\&=r\sum_{\nu=0}^{\infty}\binom{n}{\nu}\sum_{j=0}^{\infty}\big(-j\log(ab)\big)^{n-\nu}E_\nu^{(k_1,\ldots,k_r)}(x)+\log(ab)E_n^{(k_1,\ldots,k_{r-1})}(x).\end{aligned}$$

□

3. Symmetrized Generalization of $E_n^{(k_1,k_2,\ldots,k_r)}(x;a,b,c)$

The idea of constructing symmetrized generalization is originated in the work of Kaneko [21] in constructing symmetrized poly-Bernoulli numbers that satisfy certain duality relation. This idea is being used to establish the symmetrized generalization of poly-Bernoulli polynomials with parameters a and b [18] which is given by

$$C_n^{(-m)}(x,y;a,b)=\frac{1}{(\ln a+\ln b)^n}\sum_{k=0}^{m}\binom{m}{k}B_n^{(-k)}(x;a,b)\left(y-\frac{\ln b}{\ln a+\ln b}\right)^{m-k} \tag{3.1}$$

where the following duality relation holds

$$C_n^{(-m)}(x,y;a,b)=C_n^{(-m)}(y,x;b,a). \tag{3.2}$$

Parallel to this, the poly-Euler polynomials $E_n^{(-k)}(x;a,b,c)$ with parameters a, b and c have been given symmetrized generalization [19] of the form

$$D_n^{(m)}(x,y;a,b,c)=\frac{1}{(\ln a+\ln b)^n}\sum_{k=0}^{m}\binom{m}{k}E_n^{(-k)}(x;a,b,c)\left(\frac{y\ln c+\ln a}{\ln a+\ln b}\right)^{m-k}.$$

This symmetrized generalization possesses the following double generating function

$$\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}D_n^{(m)}(x,y;a,b,c)\frac{t^n}{n!}\frac{u^m}{m!}=\frac{2e^{\left(\frac{y\ln c+\ln a}{\ln a+\ln b}\right)u}e^{\left(\frac{x\ln c+\ln a}{\ln a+\ln b}\right)t}e^{t+u}\left(1-e^{-t}\right)}{(e^t+1)(e^t+e^u-e^{t+u})} \tag{3.3}$$

and explicit formula

$$\begin{aligned}D_n^{(m)}(x,y;a,b,c)=2\sum_{j=0}^{\infty}(j!)^2&\left(\sum_{l=0}^{n}\sum_{i=0}^{\infty}(-1)^i\frac{\big(\ln c^xa^{i+2}b^{i+1}\big)^{n-l}-\big(\ln c^xa^{i+1}b^i\big)^{n-l}}{(\ln a+\ln b)^{n-l}}\binom{n}{l}\left\{{l \atop j}\right\}\right)\\&\times\left(\sum_{r=0}^{m}\left(\frac{y\ln c+2\ln a+\ln b}{\ln a+\ln b}\right)^{m-r}\binom{m}{r}\left\{{r \atop j}\right\}\right).\end{aligned} \tag{3.4}$$

For multi poly-Euler polynomials with parameters a, b, and c, let us consider the following symmetrized generalization.

Definition 3.1. *For $m,n \geq 0$, we define the symmetrized generalization of multi poly-Euler polynomials with parameters a, b, and c as follows*

$$\begin{aligned}&\mathcal{D}_n^{(m)}(x,y;a,b,c)\\&=\sum_{k_1+k_2+\ldots+k_r=m}\binom{m}{k_1,k_2,\ldots k_r}\frac{E_n^{(-k_1,-k_2,\ldots -k_{r-1})}(x;a,b,c)}{(\ln a+\ln b)^n}\left(\frac{(r-1)y\ln c+\ln a}{\ln a+\ln b}\right)^{k_r}.\end{aligned} \tag{3.5}$$

The following theorem contains the double generating function for $D_n^{(m)}(x,y;a,b,c)$.

Theorem 3.2. *For $n,m \geq 0$, we have*

$$\begin{aligned}&\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\mathcal{D}_n^{(m)}(x,y;a,b,c)\frac{t^n}{n!}\frac{u^m}{m!}\\&=\frac{2e^{\left(\frac{(r-1)y\ln c+\ln a}{\ln a+\ln b}\right)u}e^{(r-1)\left(\frac{(r-1)x\ln c+\ln a}{\ln a+\ln b}\right)t}e^{\binom{r}{2}u+(r-1)t}(1-e^{-t})^{r-1}}{(1+e^t)^{r-1}\prod_{i=1}^{r-1}(e^t+e^{iu}-e^{t+iu})}.\end{aligned} \tag{3.6}$$

Proof.

$$\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\mathcal{D}_n^{(m)}(x,y;a,b,c)\frac{t^n}{n!}\frac{u^m}{m!}$$

$$\begin{aligned}&=\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\sum_{k_1+k_2+\ldots+k_r=m}\frac{E_n^{(-k_1,-k_2,\ldots -k_{r-1})}(x;a,b,c)}{(\ln a+\ln b)^n}\left(\frac{(r-1)y\ln c+\ln a}{\ln a+\ln b}\right)^{k_r}\frac{t^n}{n!}\\&\quad\times\frac{u^m}{k_1!k_2!\ldots k_r!}\\&=\sum_{n=0}^{\infty}\sum_{k_1+k_2+\ldots+k_r\geq 0}\frac{E_n^{(-k_1,-k_2,\ldots -k_{r-1})}(x;a,b,c)}{(\ln a+\ln b)^n}\left(\frac{(r-1)y\ln c+\ln a}{\ln a+\ln b}\right)^{k_r}\frac{t^n}{n!}\\&\quad\times\frac{u^{k_1+k_2+\ldots+k_r}}{k_1!k_2!\ldots k_r!}\end{aligned}$$

$$\begin{aligned}&=\sum_{n=0}^{\infty}\sum_{k_1+k_2+\ldots+k_{r-1}\geq 0}\frac{E_n^{(-k_1,-k_2,\ldots -k_{r-1})}(x;a,b,c)}{(\ln a+\ln b)^n}\sum_{k_r\geq 0}\left(\frac{(r-1)y\ln c+\ln a}{\ln a+\ln b}\right)^{k_r}\frac{u^{k_r}}{k_r!}\\&\quad\times\frac{t^n}{n!}\frac{u^{k_1+k_2+\ldots+k_{r-1}}}{k_1!k_2!\ldots k_{r-1}!}\\&=e^{\left(\frac{(r-1)y\ln c+\ln a}{\ln a+\ln b}\right)u}\sum_{n=0}^{\infty}\sum_{k_1+k_2+\ldots+k_{r-1}\geq 0}\frac{E_n^{(-k_1,-k_2,\ldots -k_{r-1})}(x;a,b,c)}{(\ln a+\ln b)^n}\frac{t^n}{n!}\frac{u^{k_1+k_2+\ldots+k_{r-1}}}{k_1!k_2!\ldots k_{r-1}!}\end{aligned}$$

Using identity (1.11), we obtain

$$\begin{aligned}
&\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\mathcal{D}_n^{(m)}(x,y;a,b,c)\frac{t^n}{n!}\frac{u^m}{m!}\\
&\quad= e^{\left(\frac{(r-1)y\ln c+\ln a}{\ln a+\ln b}\right)u}\sum_{k_1+k_2+\ldots+k_{r-1}\geq 0}\sum_{n=0}^{\infty}E_n^{(-k_1,-k_2,\ldots -k_{r-1})}\left(\frac{(r-1)x\ln c+\ln a}{\ln a+\ln b}\right)\frac{t^n}{n!}\\
&\qquad\times\frac{u^{k_1+k_2+\ldots+k_{r-1}}}{k_1!k_2!\ldots k_{r-1}!}\\
&\quad= e^{\left(\frac{(r-1)y\ln c+\ln a}{\ln a+\ln b}\right)u}e^{(r-1)\left(\frac{(r-1)x\ln c+\ln a}{\ln a+\ln b}\right)t}\sum_{k_1+k_2+\ldots+k_{r-1}\geq 0}\frac{2\mathrm{Li}_{(-k_1,-k_2,\ldots,-k_{r-1})}(1-e^{-t})}{(1+e^t)^{r-1}}\\
&\qquad\times\frac{u^{k_1+k_2+\ldots+k_{r-1}}}{k_1!k_2!\ldots k_{r-1}!}\\
&\quad= \frac{2e^{\left(\frac{(r-1)y\ln c+\ln a}{\ln a+\ln b}\right)u}e^{(r-1)\left(\frac{(r-1)x\ln c+\ln a}{\ln a+\ln b}\right)t}}{(1+e^t)^{r-1}}\sum_{0<m_1<m_2<\ldots<m_{r-1}}(1-e^{-t})^{m_{r-1}}\mathcal{S}(u,m_1,\ldots,m_{r-1})
\end{aligned}$$

where

$$\begin{aligned}
\mathcal{S}(u,m_1,\ldots,m_{r-1}) &= \sum_{k_1+\ldots+k_{r-1}\geq 0}\frac{(um_1)^{k_1}\ldots(um_{r-1})^{k_{r-1}}}{k_1!\ldots k_{r-1}!}\\
&= \sum_{\widehat{m}\geq 0}\frac{1}{\widehat{m}!}\sum_{k_1+k_2+\ldots+k_{r-1}=\widehat{m}}\binom{\widehat{m}}{k_1,\ldots k_{r-1}}(um_1)^{k_1}\ldots(um_{r-1})^{k_{r-1}}\\
&= \sum_{\widehat{m}\geq 0}\frac{(um_1+\ldots+um_{r-1})^{\widehat{m}}}{\widehat{m}!}\\
&= e^{u(m_1+\ldots+m_{r-1})}.
\end{aligned}$$

Thus,

$$\begin{aligned}
&\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\mathcal{D}_n^{(m)}(x,y;a,b,c)\frac{t^n}{n!}\frac{u^m}{m!}\\
&\quad= \frac{2e^{\left(\frac{(r-1)y\ln c+\ln a}{\ln a+\ln b}\right)u}e^{(r-1)\left(\frac{(r-1)x\ln c+\ln a}{\ln a+\ln b}\right)t}}{(1+e^t)^{r-1}}\sum_{0<m_1<m_2<\ldots<m_{r-1}}(1-e^{-t})^{m_{r-1}}e^{u(m_1+\ldots+m_{r-1})}\\
&\quad= \frac{2e^{\left(\frac{(r-1)y\ln c+\ln a}{\ln a+\ln b}\right)u}e^{(r-1)\left(\frac{(r-1)x\ln c+\ln a}{\ln a+\ln b}\right)t}}{(1+e^t)^{r-1}}\\
&\qquad\times\frac{e^u(1-e^{-t})}{1-e^u(1-e^{-t})}\frac{e^{2u}(1-e^{-t})}{1-e^{2u}(1-e^{-t})}\cdots\frac{e^{(r-1)u}(1-e^{-t})}{1-e^{(r-1)u}(1-e^{-t})}\\
&\quad= \frac{2e^{\left(\frac{(r-1)y\ln c+\ln a}{\ln a+\ln b}\right)u}e^{(r-1)\left(\frac{(r-1)x\ln c+\ln a}{\ln a+\ln b}\right)t}e^{\binom{r}{2}u}(1-e^{-t})^{r-1}}{(1+e^t)^{r-1}\prod_{i=1}^{r-1}(1-e^{iu}(1-e^{-t}))}\\
&\quad= \frac{2e^{\left(\frac{(r-1)y\ln c+\ln a}{\ln a+\ln b}\right)u}e^{(r-1)\left(\frac{(r-1)x\ln c+\ln a}{\ln a+\ln b}\right)t}e^{\binom{r}{2}u+(r-1)t}(1-e^{-t})^{r-1}}{(1+e^t)^{r-1}\prod_{i=1}^{r-1}(e^t+e^{iu}-e^{t+iu})}.
\end{aligned}$$

□

Note that equation (3.3) can easily be deduced from equation (3.6) by taking $r = 1$. One may also try to obtain the explicit formula for $\mathcal{D}_n^{(m)}(x, y; a, b, c)$ using the method in obtaining (3.4).

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