DIVISIBILITY OF THE MIDDLE LUCASNOMIAL COEFFICIENT

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Abstract. Pomerance established several theorems about the number of integers n for which $n+k$ divides the binomial coefficient $\binom{2n}{n}$, k a given integer. We conduct a similar inquiry about the number of integers n for which U_{n+k} divides $\binom{2n}{n}_U$, where U is a fundamental Lucas sequence and $\binom{2n}{n}_U$ the corresponding middle Lucasnomial coefficient. In a final digression, we argue that central Fibonomials prime to 105 should be about as rare as middle binomial coefficients prime to 105, and we compute the first few examples.

1. Introduction

In a beautifully lucid arithmetical paper [13], Pomerance studied how often the middle binomial coefficient $\binom{2n}{n}$ is divisible by $n+k$, when k is a fixed arbitrary integer. Three main theorems were proved which we recall here. The first asserts the singularity of the Catalan numbers $C_n = \frac{1}{n+1}\binom{2n}{n}$ since for $k=1$, we have that $n+1$ divides $\binom{2n}{n}$ for all $n \geq 0$, whereas for $k \neq 1$, we have

Theorem 1.1. *Let k be an integer not equal to 1. There are infinitely many integers $n \geq 1$ such that $n+k$ does not divide $\binom{2n}{n}$.*

The second and third theorems indicate a drastic difference in behavior depending on whether $k \geq 1$, or $k \leq 0$.

Theorem 1.2. *Suppose $k \geq 1$. Then, the set of integers $n \geq 1$ such that $n+k$ divides $\binom{2n}{n}$ has asymptotic density one.*

Theorem 1.3. *Suppose $k \leq 0$. Then, the set of integers $n \geq 1$ such that $n+k$ divides $\binom{2n}{n}$ is infinite, but has upper asymptotic density at most $1-\log 2$.*

Pomerance [13, bottom of page 7] conjectured the existence of a positive lower asymptotic density for the set of positive integers n with $n+k \mid \binom{2n}{n}$, when $k \leq 0$, although this remains an open question. As a consequence of [13, Theorem 4], a positive lower density exists for all $k \leq 0$ iff it exists for $k=0$.

A fundamental Lucas sequence $U = U(P,Q)$ is a binary linear recurrent sequence defined by the initial values $U_0 = 0$, $U_1 = 1$ and the recurrence

$$U_{n+2} = PU_{n+1} - QU_n, \tag{1}$$

for all integers $n \geq 0$, where P and Q are nonzero integers. When $(P,Q) = (2,1)$, we find that $U_n = n$. We restrict ourselves to *nondegenerate* fundamental Lucas sequences, i.e., to those sequences with $U_n \neq 0$, for all $n \geq 1$. The condition $U_4U_6 \neq 0$ is a necessary and sufficient condition for the non-degeneracy of $U = (U_n)$ – see for instance [3, Section 2]. For $m \geq n \geq 1$, the *Lucasnomial* coefficient $\binom{m}{n}_U$ is defined as

$$\binom{m}{n}_U := \frac{U_mU_{m-1}\cdots U_{m-n+1}}{U_nU_{n-1}\cdots U_1}, \tag{2}$$

and as 1, if $m \geq 0$ and $n = 0$, and, otherwise, as 0. From the identity $U_m = U_{n+1}U_{m-n} - QU_nU_{m-n-1}$, one deduces [3, Section 5] that

$$\binom{m}{n}_U = U_{n+1}\binom{m-1}{n}_U - QU_{m-n-1}\binom{m-1}{n-1}_U, \text{ for } m \geq n \geq 1. \tag{3}$$

Using (3) and an induction, one sees that all Lucasnomials are integers. Indeed, identity (3) enables one to prove the integrality of Lucasnomials on the mth row of the Lucasnomial Pascal triangle from their integrality at row $m-1$. Note in passing that one may view (3) as a generalization of the binomial identity $\binom{m}{n} = \binom{m-1}{n} + \binom{m-1}{n-1}$. For $(P, Q) = (2, 1)$, $U_n = n$ and the RHS of (3) becomes

$$(n+1)\binom{m-1}{n} - (m-n-1)\binom{m-1}{n-1} = \binom{m-1}{n} + \binom{m-1}{n-1} + E_{m,n},$$

where

$$E_{m,n} = n\binom{m-1}{n} - (m-n)\binom{m-1}{n-1} = 0.$$

We were curious to know whether theorems similar to Theorems 1.1, 1.2, and 1.3 held with respect to middle Lucasnomial coefficients. The paper answers those questions. Besides this introduction, it contains five sections. In Section 2, Theorem 2.1 establishes the integrality of the *Lucasnomial Catalan* numbers $\frac{1}{U_{n+1}}\binom{2n}{n}_U$ for all nondegenerate fundamental Lucas sequence U and all $n \geq 0$. Section 3 is concerned with Theorem 1.1. Theorem 3.1 generalizes Theorem 1.1 to all nondegenerate Lucas sequences when $k \leq 0$. However, for $k \geq 2$, we find two exceptional sequences, namely $U(\pm 1, 2)$ for which, when $k = 2$ and 2 only, U_{n+k} divides $\binom{2n}{n}_U$ for all $n \geq 0$. Otherwise, for all other *regular* Lucas sequences, Theorem 1.1 also generalizes to the case $k \geq 2$ as we establish in Theorem 3.5.

A Lucas sequence $U(P, Q)$ is *regular* whenever it is nondegenerate and $\gcd(P, Q) = 1$. This latter condition is well-known to be logically equivalent to the property:

$$\gcd(U_m, U_n) = |U_{\gcd(m,n)}|, \quad \text{for all nonnegative } m \text{ and } n.$$

A *primitive* prime divisor of the nth term U_n of a Lucas sequence U is a prime factor of U_n, which does not divide $U_2U_3 \ldots U_{n-1}$.

The discriminant Δ of a Lucas sequence $U(P, Q)$ is $P^2 - 4Q$. Theorem 3.5 utilizes the primitive prime divisor theorems of [1, 6]. That is, for regular nonzero-discriminant Lucas sequences $U = (U_n)$, U_n has a primitive prime divisor, not dividing Δ, for all $n > 30$. Moreover, these theorems tell us exactly what Lucas sequences U and what values of $n \leq 30$ are such that U_n does not have a primitive divisor. One easily checks that the only regular Lucas sequence with $\Delta = 0$ is $U_n = n$. But, in this paper, we call a regular Lucas sequence, Δ-*regular*, if it is not $U_n = n$. The distinction is important, as depending on whether our theorems address regular, or Δ-regular sequences, we may (at best) get a generalization, or an analog, of the theorems for ordinary binomial coefficients.

Theorem 4.3 of Section 4 generalizes Theorem 1.2 to all regular Lucas sequences when $k \geq 1$. However, Theorem 5.1 in Section 5 shows that division of $\binom{2n}{n}_U$ by U_{n+k}, if $k \leq 0$, occurs for only finitely many integers n when U is Δ-regular. Thus, we only have an analog of Theorem 1.3. The contrast between the cases $k \geq 1$ and $k \leq 0$ is even sharper than for ordinary binomial coefficients. The sixth and last section lists a few open questions and extends a heuristic of Pomerance about the number of integers n for which $\binom{2n}{n}$ is prime to 105 to middle Fibonomials. The Fibonacci sequence F is the Lucas sequence $U(1, -1)$. Lucasnomials with respect to F are called *Fibonomials* and have been the object of more studies than general

Lucasnomials. The paper is sprinkled with special Fibonacci examples of the general theorems (see Propositions 3.2 and 4.2, Example 5.2, and Problem 4 of Section 6).

Notation. Extending the notation used in [13, Section 7], given a Lucas sequence U and an integer k, we write $D_{U,k}$ for the set of integers $n \geq 1$ such that U_{n+k} divides $\binom{2n}{n}_U$. We let $\bar{D}_{U,k}$ denote the complementary set of $D_{U,k}$ in the positive integers.

The proofs of the theorems in Pomerance's article rely much on Kummer's rule [12], which gives the p-adic valuation of the binomial $\binom{m+n}{n}$ as the number of carries in the addition of m and n performed in base p, p a prime. This rule was extended to Lucasnomials; see [11] and [3, Section 4]. The crucial point is that $n \in D_{U,k}$ iff for each prime p, the p-adic valuation of $\binom{2n}{n}_U$ is at least that of U_{n+k}. The rank $\rho_U(p)$, or ρ, of a prime p in a Lucas sequence U is the smallest positive index t such that p divides U_t. It is guaranteed to exist when $p \nmid Q$. The next two propositions are basic working tools of this paper.

Putting together statement (4.4) and Theorem 4.1 of [2, Section 4], we obtain the p-adic valuation of all terms of a fundamental Lucas sequence $U = U(P, Q)$ for all primes $p \nmid Q$.

Proposition 1.4. *Let $U = U(P, Q)$ be a nondegenerate fundamental Lucas sequence and $p \nmid Q$ be a prime of rank ρ in U. Then, for all nonnegative integers m and n, we have*

$$\rho \mid p - \left(\frac{\Delta}{p}\right), \qquad \text{if } p \text{ is odd},$$

$$p \mid U_n \iff \rho \mid n,$$

$$\nu_p(U_{\rho m}) = \nu_p(U_\rho) + \nu_p(m) + \delta = a + x + \delta,$$

where $\Delta = P^2 - 4Q$, $\left(\frac{*}{*}\right)$ is the Legendre symbol, $a = \nu_p(U_\rho)$, $x = \nu_p(m)$ and

$$\delta = \delta_{p,P,Q,m} = \begin{cases} \nu_2\big((P^2 - 3Q)/2\big), & \text{if } p = 2,\ PQ \text{ is odd, and } m \text{ is even}, \\ 0, & \text{otherwise.} \end{cases}$$

The notation x, a, and δ is used consistently with the utilization of Proposition 1.4 throughout the paper.

The most general Kummer rule for Lucasnomials can be stated as follows [2, Section 4].

Proposition 1.5. [1] *Let $U = U(P, Q)$ be a nondegenerate Lucas sequence and $p \nmid Q$ be a prime of rank ρ in U. Let m and n be two positive integers. Then the p-adic valuation of the Lucasnomial $\binom{m+n}{n}_U$ is equal to the number of carries that occur to the left of the radix point when m/ρ and n/ρ are added in base-p notation, plus $\nu_p(U_\rho)$ if a carry occurs across the radix point. However, if p is 2, P is odd and $Q \equiv -1 \pmod 4$, one must add $\delta = \nu_2\big((P^2 - 3Q)/2\big)$ to the previous counts if a carry occurs from the first to the second digit to the left of the radix point.*

This theorem suggests we distinguish carries across or to the left of the radix point from the other carries. Thus, as in [2], when adding $n/\rho + n/\rho$ in base p, we will call a carry *relevant* when it occurs across or to the left of the radix point.

Although we usually write the p-adic valuation of an integer m, i.e., the highest exponent e of p such that p^e divides m as $\nu_p(m)$, we omit the parentheses when m is a Lucasnomial coefficient $\binom{\ell}{k}_U$ and write instead $\nu_p\binom{\ell}{k}_U$. The paper assumes familiarity with Lucas sequences.

[1] See Proposition 6.1 at the end of the paper for an alternative wording of Proposition 1.5

2. Lucasnomial Catalan Numbers

Perhaps Gould [9], [17, A003150] was the first to introduce Fibonomial Catalan numbers and to prove their integrality. The integrality proof found in [10, Section 7] would readily generalize to all regular Lucas sequences. The Lucasnomial Catalan numbers were considered in the paper [15, Section 6.4] as special values of Lucas-Catalan polynomials, but an earlier talk [16, Last remark] mentions them, as did the subsequent note [7]. We are not aware of other appearances. The term 'generalized Catalan number' usually refers to the numbers $\frac{1}{an+1}\binom{an+n}{n}$, $a \geq 1$.

We provide two proofs, one algebraic, the other arithmetic, of the integrality of Lucasnomial Catalan numbers. Finding a combinatorial proof would require a combinatorial interpretation of these numbers. Several papers [4, 14, 5] have found, or explained, combinatorial interpretations for Lucasnomials. However, the search of a combinatorial interpretation for Lucasnomial Catalan numbers does not seem to have succeeded yet [15, 16] in spite of the many interpretations ordinary Catalan numbers possess. Proving their integrality came immediately out of the classical formula (3), but, as it turned out and not surprisingly, we had been anticipated [10, 16, 7].

Theorem 2.1. *Let $U = (U_n)_{n\geq 0}$ be a fundamental Lucas sequence with $U_4U_6 \neq 0$. The Lucasnomial Catalan rational number,*

$$\frac{1}{U_{n+1}}\binom{2n}{n}_U,$$

is integral for all $n \geq 0$.

Proof 1. Putting $m = 2n$ in (3) and dividing through by U_{n+1}, we obtain

$$\frac{1}{U_{n+1}}\binom{2n}{n}_U = \binom{2n-1}{n}_U - Q\,\frac{U_{n-1}}{U_{n+1}}\binom{2n-1}{n-1}_U = \binom{2n-1}{n}_U - Q\binom{2n-1}{n-2}_U, \tag{4}$$

which is an integer. □

The second proof is less general as we assume, for simplicity, the regularity of U.

Proof 2. A prime p that divides Q does not divide any term of U because, by the recurrence (1), $U_n \equiv P^{n-1} \pmod{p}$. However, $p \nmid P$ by the regularity assumption. If a prime $p \nmid Q$ of rank ρ divides U_{n+1}, then, there is an integer $\lambda > 0$ prime to p such that $n+1 = \lambda p^x \rho$. Then, with the notation of Proposition 1.4, $\nu_p(U_{n+1}) = a + x + \delta$. Now,

$$\frac{n}{\rho} = \lambda p^x - \frac{1}{\rho} = (\lambda p^x - 1) + \frac{\rho - 1}{\rho}.$$

In the base-p addition of n/ρ to itself, we see there is a carry across the radix point since the fractional part of n/ρ, $(\rho-1)/\rho$, is at least $1/2$. The first x digits of n/ρ to the left of the radix point are all $p-1$. Thus, there are at least x carries in the addition of $\lambda p^x - 1$ to itself. By Proposition 1.5, we see that the p-adic valuation of $\binom{2n}{n}_U$ is at least that of $\nu_p(U_{n+1})$. This holds for all primes p that divide U_{n+1}. □

Remark. Proof 2 actually shows that $\nu_p\binom{2n}{n}_U \geq \nu_p(U_{n+1})$ for all $p \nmid \gcd(P,Q)$ and all nondegenerate $U(P,Q)$.

3. Theorem 1.1 and Lucasnomials

Theorem 3.1. *Let U be a nondegenerate fundamental Lucas sequence and $k \geq 0$ be an integer. Then, there are infinitely many integers $n \geq 1$ such that U_{n-k} does not divide $\binom{2n}{n}_U$.*

Proof. Because U is nondegenerate, there are only finitely many primes of rank $\le 2k$. Choose any odd prime p of rank $\rho > 2k$. Let $n = p^x\rho + k \ge 1$ for some $x \ge 0$ large enough. By Proposition 1.4, p^{a+x} divides U_{n-k} with $a = \nu_p(U_\rho)$. But $\lfloor n/\rho \rfloor = p^x$ and $\{n/\rho\} = k/\rho < 1/2$, so there are no relevant carries in the base-p addition of n/ρ to itself. By Proposition 1.5, $p \nmid \binom{2n}{n}_U$. Hence, $U_{n-k} \nmid \binom{2n}{n}_U$. □

We treat separately the Fibonacci case $F = U(1, -1)$ and central Fibonomial coefficients. This serves as an introduction to the general case. The next result shows that $k = 1$ enjoys the same status it has with respect to ordinary middle binomial coefficients.

Proposition 3.2. *Let k be an integer. For $k \ne 1$, there are infinitely many values of n such that F_{n+k} does not divide $\binom{2n}{n}_F$. Otherwise, F_{n+1} divides $\binom{2n}{n}_F$ for all $n \ge 1$.*

Proof. The cases $k \le 0$ and $k = 1$ are given respectively by Theorems 3.1 and 2.1.

If $k = 2$, then consider an n of the form $3 \cdot 2^x - 2$, $x \ge 1$. Then, by Proposition 1.4, $\nu_2(F_{n+2}) = x + 1 + \delta = x + 2$. However, the base-2 addition of $n/3$ to itself, i.e., of $2^x - 2/3 = 2^x - 1 + 1/3$ yields no carry across the radix point, since $2\{n/3\} = 2/3 < 1$, and exactly x carries to the left of that point. Thus, by Proposition 1.5, $\nu_2\binom{2n}{n}_F = x + \delta = x + 1$. Hence, $F_{n+2} \nmid \binom{2n}{n}_F$.

Suppose $k \ge 3$. Then, $F_k > 1$. Consider a prime divisor p of F_k of rank ρ. Thus $\rho \mid k$. Suppose $n \ge 1$ is an integer of the form $p^x\rho - k$ with $x \ge 1$ and large enough. Then, $\nu_p(F_{n+k}) = a + x$, or $x + 2$ in case $p = 2$, where again $a = \nu_p(F_\rho)$. But, $n/\rho = p^x - k/\rho$ is a positive integer. Thus, by Proposition 1.5, the p-adic valuation of $\binom{2n}{n}_F$ is at most x, unless $p = 2$ when it can potentially be $x + 1$. In all cases, $\nu_p\binom{2n}{n}_F < \nu_p(F_{n+k})$ so that $F_{n+k} \nmid \binom{2n}{n}_F$. □

Lemma 3.3. *Let $U(P, Q)$ be a regular Lucas sequence and $k \ge 2$ be an integer. Suppose some prime p of rank ρ divides U_k. Then,*

$$U_{n+k} \nmid \binom{2n}{n}_U, \text{ for all } n = \rho p^x - k, \text{ with } x \ge 1 \text{ large enough so } n > 0.$$

Proof. Assume $n = \rho p^x - k \ge 1$ for some $x \ge 1$. Suppose $p \mid U_k$. As seen at the beginning of the second proof of Theorem 2.1, $p \nmid Q$. Hence, by Proposition 1.4, $\rho \mid k$. Thus, $n/\rho = p^x - k/\rho$ is an integer $\le p^x - 1$. Therefore, there are at most x relevant carries when adding n/ρ to n/ρ in base p. We conclude that $\nu_p\binom{2n}{n}_U \le x + \delta$, whereas, with the notation of Proposition 1.4, $\nu_p(U_{n+k}) = a + x + \delta > x + \delta$. Therefore, $U_{n+k} \nmid \binom{2n}{n}_U$. □

Remark. Theorem 1.1 is a corollary of Theorems 3.1 and Lemma 3.3. Indeed, $U_n = n$ is the regular sequence $U(2, 1)$ and U_k has a prime divisor for all $k \ge 2$.

Lemma 3.4. *Let $U(P, Q)$ be a regular Lucas sequence and $k \ge 2$ be an integer. Suppose one of the three conditions i) $|U_k| \ge 2$, or ii) U_{k+1} has a primitive prime divisor, or iii) U_{k+2} has a primitive prime divisor, holds. Then, unless $k = 2$ and $U = U(\pm 1, 2)$, there are infinitely many integers $n \ge 1$ such that*

$$U_{n+k} \nmid \binom{2n}{n}_U.$$

For $(P, Q) = (\pm 1, 2)$, we find that $\frac{1}{U_{n+2}}\binom{2n}{n}_U$ is integral for all $n \ge 0$.

Proof. Let p, a prime of rank ρ, designate either a factor of U_k, or a primitive divisor of U_{k+1}, or of U_{k+2}. Suppose $m \ge 1$ is of the form $\rho p^x - k$, $x \ge 1$. Note that, in all cases, $p \nmid Q$. By

Lemma 3.3, if $p \mid U_k$, then $m \in \bar{D}_{U,k}$ for all $x \geq 1$. So we assume $U_k = \pm 1$. If p has rank $\rho = k + \ell$, $\ell = 1$ or 2, then $m/\rho = (p^x - 1) + \ell/\rho$. The fractional part of m/ρ is

$$\frac{\ell}{\rho} = \begin{cases} \frac{1}{k+1} \leq \frac{1}{3}, & \text{if } \ell = 1, \\ \frac{2}{k+2}, & \text{if } \ell = 2, \end{cases}$$

which is $< 1/2$, unless $k = 2$ and $\ell = 2$. So unless $k = 2$ and $\ell = 2$, the base-p addition $m/\rho + m/\rho$ produces exactly x carries to the left of the radix point and none across that point. Hence, we find that $\nu_p(U_{m+k}) > \nu_p\binom{2m}{m}_U$. Thus, by the foregoing argument, for $\bar{D}_{U,k}$ to be finite, we need to have $k = 2$, $\rho = 4$ and both U_2 and U_3 equal to ± 1. Indeed, a prime dividing U_3 would have to be of rank 3. But $U_2 = P$ and $U_3 = P^2 - Q$. Since $P^2 - Q = 1$ would imply $Q = 0$, we see that $P^2 - Q = -1$ and $Q = 2$. Now the nth terms of $U(1, 2)$ and $U(-1, 2)$ have the same absolute values for all $n \geq 0$. If $P = 1$, then the first few terms of U are $0, 1, 1, -1, -3, -1, 5, 7, -3, -17, -11$.

It remains to show that U_{n+2} divides $\binom{2n}{n}_U$ for all $n \geq 0$, when $U = U(1, 2)$. We will do so by proving that for all primes p, $\nu_p(U_{n+2}) \leq \nu_p\binom{2n}{n}_U$. There is nothing to prove if the prime p does not divide U_{n+2}. Suppose p divides U_{n+2} for some $n \geq 0$. Then, as all terms of U are odd, p is odd. By Proposition 1.4, as $p \nmid Q$, $n + 2$ is of the form $\lambda p^x \rho$, ($p \nmid \lambda$, $x \geq 0$) and

$$\nu_p(U_{n+2}) = a + x, \quad (\text{with } a = \nu_p(U_\rho)).$$

Now, $n/\rho = \lambda p^x - 1 + (\rho - 2)/\rho$ and $\{n/\rho\} = (\rho - 2)/\rho$. In the addition of n/ρ to itself in base p, there is a carry across the radix point because, as $\rho \geq 4$, $2\{n/\rho\} \geq 1$. Moreover, as the least x significant digits of $\lambda p^x - 1$ are all $p - 1$, there are at least x additional carries to the left of the radix point. By Proposition 1.5,

$$\nu_p\binom{2n}{n}_U \geq a + x.$$

Hence, U_{n+2} divides $\binom{2n}{n}_U$. □

In [6], a Δ-regular Lucas sequence U was defined to be *k-defective* iff for all primes $p \nmid Q\Delta$, the rank of p in U is not k, that is, if U_k has no primitive prime divisor, where primes dividing Δ were not considered as primitive divisors. Their main result is that for all $k > 30$ no Δ-regular Lucas sequence is k-defective. In addition they listed, up to equivalence, all Δ-regular Lucas sequences that are k-defective in the range $2 \leq k \leq 30$. Two sequences $U = U(P, Q)$ and $U' = U(P', Q')$ are equivalent iff $P = \pm P'$ and $Q = Q'$. In case of equivalence, we have $|U_n| = |U'_n|$ for all $n \geq 0$. To be more precise, Table 1 of [6] lists, up to equivalence, all k-defective Lucas sequences with $k = 5$ and $7 \leq k \leq 30$. Table 3 of [6] lists the remaining values $2 \leq k \leq 4$ and $k = 6$, but contains a few errors and misses some sequences. One can find a corrected table, Table 3 in [1, Théorème 4.1]. The sequences in those lists are written in the form (P, Δ), where $\Delta = P^2 - 4Q$.

Theorem 3.5. *Suppose $U(P, Q)$ is a regular Lucas sequence and k is a fixed integer not 1. Then there exist infinitely many integers n such that*

$$U_{n+k} \nmid \binom{2n}{n}_U,$$

unless $(P, Q) = (\pm 1, 2)$ and $k = 2$, in which case

$$U_{n+2} \mid \binom{2n}{n}_U \qquad \textit{for all } n \geq 0.$$

Proof. By Theorem 3.1, we need only consider the cases $k \geq 2$. By Theorem 1.1, we assume U is Δ-regular. By Lemma 3.4, if U_{n+k} divides $\binom{2n}{n}_U$ for all but finitely many n, then, unless $k = 2$ and $(P, Q) = (\pm 1, 2)$, the following conditions must hold

$$U_k = \pm 1 \text{ and } U \text{ is simultaneously } k,\ (k+1) \text{ and } (k+2) - \text{defective.} \tag{5}$$

Therefore, we consult the above-mentioned tables in [1] and [6] and search for Lucas sequences U that satisfy the conditions (5) with $2 \leq k \leq 28$. However, the only values of k that contain defective sequences for k, $k+1$ and $k+2$ are $k = 2$, $k = 3$, $k = 4$, $k = 5$, and $k = 6$. Inspecting those tables further, we see that there are no 4-defective Lucas sequences with $P = \pm 1$. Since $U_2 = P$, the conditions (5) cannot be met when $k = 2$. Conditions (5) are not met either when $k = 3$ or $k = 4$. Indeed, there is only a finite list of seven sequences with $P > 0$ that are 5-defective. Four of them, namely $(P, \Delta) = (1, 5)$, $(1, -7)$, $(1, -11)$, and $(1, -15)$, have $P = 1$, but again there are no 4-defective sequences with $P = 1$. The three remaining ones are $(2, -40)$, $(12, -76)$, and $(12, -1364)$. However, 4-defective sequences with P even are all of the form $(P, \pm 4 - P^2)$ and our three 5-defective sequences with P even do not match this form. Now, the only sequence with $P > 0$ that is both 5 and 7-defective is $(P, \Delta) = (1, -7)$. It is also the only sequence simultaneously 7 and 8-defective. But 6-defective sequences with $P = 1$ have discriminant $\Delta = (4(-2)^\nu - 1)/3$, $(\nu \geq 1)$, and -7 is not of this form. Hence, (5) cannot hold for either $k = 5$ or $k = 6$. ☐

Corollary 3.6. *For the two Lucas sequences $U(P, Q) = U(\pm 1, 2) = (U_n)_{n \geq 0}$, the rational numbers*

$$\frac{1}{U_{n+1} U_{n+2}} \binom{2n}{n}_U \qquad \text{are integers for all } n \geq 0. \tag{6}$$

For no other regular Lucas sequences do we have that the numbers in (6) are integers for all n sufficiently large.

Proof. Combine Theorems 2.1 and 3.5 with the fact that if U is regular, then $\gcd(U_{n+1}, U_{n+2}) = 1$ for all $n \geq 0$. ☐

Remark. The sequence $U(1, 2)$ is somewhat of an anomaly. Remarkably, if it were not for the two exceptional sequences $U(\pm 1, 2)$, all Δ-regular Lucas sequences (U_n) would have a primitive prime divisor for all $n > 12$, rather than 30; see [6, Table 1].

4. Theorem 1.2 and Lucasnomials, (i.e., The Case $k \geq 1$)

A subset S of the positive integers has asymptotic density d iff $\lim_{z \to +\infty} \#S(z)/z = d$, where $S(z) = S \cap [1, z]$. Lower and upper asymptotic densities are obtained by replacing the limit by (resp.) the liminf and the limsup.

We proceed in several steps before proving a generalization of Theorem 1.2 to all regular Lucas sequences.

Proposition 4.1. *Suppose $U(P, Q)$ is a regular Lucas sequence and $k \geq 1$ is a fixed integer. Then the set of positive integers n such that U_{n+k} divides $\binom{2n}{n}_U$ has a positive lower asymptotic density.*

Proof. Suppose p, a prime, divides U_{n+k}. As seen in the second proof of Theorem 2.1, we may assume $p \nmid Q$. We know $n + k$ is of the form $\lambda p^x \rho$, where λ is prime to p, $x \geq 0$ and ρ is the rank of p. By Proposition 1.4, $\nu_p(U_{n+k}) = a + x + \delta$, where $a = \nu_p(U_\rho)$. Now,

$$\frac{n}{\rho} = \frac{\lambda p^x \rho - k}{\rho} = (\lambda p^x - 1) + \frac{\rho - k}{\rho}. \tag{7}$$

If $\frac{\rho-k}{\rho} \geq \frac{1}{2}$, i.e., if $\rho \geq 2k$, then, by Proposition 1.5, the p-adic valuation of the Lucasnomial $\binom{2n}{n}_U$ is at least $a+x+\delta$. We conclude that $\{n;\ p \mid U_{n+k} \implies \rho(p) \geq 2k\}$ is a subset of $D_{U,k}$. That is, $D_{U,k}$ contains the set $\{n;\ \rho(p) \nmid n+k$ for all p's such that $\rho(p) < 2k\}$. In particular, $D_{U,k}$ contains the set $\{n;\ q \nmid n+k$ for all primes q with $q < 2k\}$. This latter set is a union of $\prod_{q<2k}(q-1)$ distinct arithmetic progressions with common difference $\prod_{q<2k} q$, (q prime). This set has asymptotic density equal to $\prod_{q<2k}(q-1) \times (\prod_{q<2k} q)^{-1} = \prod_{q<2k}(1-1/q) > 0$, ($q$ prime). Thus, the lower asymptotic density of $D_{U,k}$ is positive. □

We look at the particular instance of the Fibonacci sequence F and $k=2$ and prove that $D_{F,2}$ has asymptotic density 1. The exceptional integers not in $D_{F,2}$ are those outlined while proving the case $k=2$ of Proposition 3.2.

Proposition 4.2. *The set $D_{F,2}$ of integers n such that F_{n+2} divides $\binom{2n}{n}_F$ contains all natural numbers but those of the form $3 \cdot 2^x - 2$, $x \geq 1$. Hence, $D_{F,2}$ has asymptotic density 1.*

Proof. By the proof of Proposition 4.1, for F_{n+2} to divide $\binom{2n}{n}_F$ it suffices that all the ranks of the prime factors of F_{n+2} be ≥ 4. But, only the prime 2 has Fibonacci rank less than 4. So, the only potential exceptions are n's satisfying $n+2 = \lambda 2^x \cdot 3$ for some $x \geq 0$, (λ odd). Say $\lambda = 2\mu+1$, and assume $\mu \geq 1$. Then, using (7), we see that

$$\lfloor \frac{n}{\rho} \rfloor = \lambda 2^x - 1 = \mu 2^{x+1} + 2^x - 1 \ \text{ and } \ \{\frac{n}{\rho}\} = \frac{\rho-k}{\rho} = \frac{3-2}{3} = \frac{1}{3}.$$

So, although there is no carry across the radix point when adding $n/3$ to itself in base 2, there are at least $x+1$ carries in the base-2 addition of the integral part of $n/3$ to itself and, in particular, there is one from the first to the second digit to the right of the radix point if $x \geq 1$. Thus, by Proposition 1.5, the 2-adic valuation of $\binom{2n}{n}_F$ is at least $x+1+\delta$, which equals $x + \nu_2(F_3) + \delta = \nu_2(F_{n+2})$. But, when $\lambda = 1$ and $x \geq 1$, according to the proof of Proposition 3.2 for $k=2$, F_{n+2} does not divide $\binom{2n}{n}_F$. Finally, one checks the case $x=0$, i.e., that F_3 divides $\binom{6}{3}_F$. □

It turns out that $D_{U,k}$, for $k \geq 1$ and U regular, always has asymptotic density one as we prove next. The proof resembles those of Lemmas 1 and 2 in [13]. As in [13, Lemma 1], given a prime p, we write $\theta_p = \log(\frac{p+1}{2})/\log p$.

Theorem 4.3. *Suppose $U(P,Q)$ is a regular Lucas sequence and $k \geq 1$ is a fixed integer. Then the set of positive integers n such that U_{n+k} divides $\binom{2n}{n}_U$ has asymptotic density 1.*

Proof. We will show that $\bar{D}_{U,k}$ has asymptotic density 0. If $n \notin D_{U,k}$, then there must exist a prime p with $\nu_p(U_{n+k}) > \nu_p\binom{2n}{n}_U$. However, by the proof of Proposition 4.1, if p has rank $\rho \geq 2k$, then $\nu_p(U_{n+k}) \leq \nu_p\binom{2n}{n}_U$. Therefore, $\bar{D}_{U,k}$ is equal to the finite union $\bigcup_{p\in P} A_p$, where P is the set of primes of rank $< 2k$, and

$$A_p := \left\{n;\ \nu_p(U_{n+k}) > \nu_p\binom{2n}{n}_U\right\}.$$

Thus, it suffices to prove that each A_p, $p \in P$, has asymptotic density 0. So we fix some $p \in P$. Now if $p \mid U_{n+k}$, as is the case of integers n in A_p, then $n+k = m\rho = \lambda p^x \rho$, for some positive integer λ prime to p and $x \geq 0$. Hence, $n/\rho = m - k/\rho$. Using euclidean division, one may write $k = \ell\rho + r$, $0 \leq r < \rho$. Thus,

$$\frac{n}{\rho} = (m-1) - \ell + \frac{\rho - r}{\rho} = (\lambda-1)p^x + (p^x - 1) - \ell + \frac{\rho - r}{\rho}. \tag{8}$$

Let $z > 0$ be a large real number. Suppose $n \le z$. Define $y := z/\rho$ and $D := \lfloor 1 + \log y/\log p \rfloor$, the maximal number of p-ary digits $\lfloor n/\rho \rfloor$ may have. Write

$$(\lambda - 1)p^x = \lambda_D p^D + \cdots + \lambda_x p^x, \quad 0 \le \lambda_i < p, \text{ for all } i,\ x \le i \le D. \tag{9}$$

Let u be the least integer t with $p^t > \ell$. For $x \ge 0$, define A_p^x as the set of integers n in A_p of the form $n + k = \lambda p^x \rho$, $p \nmid \lambda$.

Suppose first $x \ge u$. Subtracting ℓ from $p^x - 1$, whose p-ary digits are all $p - 1$, may alter at most the u least significant digits of $p^x - 1$. If the fractional part of n/ρ, i.e., $(\rho - r)/\rho$, is less than $1/2$, then the base-p addition of n/ρ to itself, may produce as few as $x - u$ relevant carries. Thus, a deficit of $a_p := u + \delta + a$ carries for the p-adic valuation of $\binom{2n}{n}_U$ to at least match that of U_{n+k}. Define B_p^x as the set of integers n such that

$$n + k = \lambda p^x \rho,\ p \nmid \lambda, \text{ and less than } a_p \text{ carries occur in adding } (\lambda - 1)p^x \text{ to itself.}$$

Since $A_p^x \subset B_p^x$, it suffices to show that the union $\bigcup_{u \le x \le D} B_p^x$ has asymptotic density 0 to see that $\bigcup_{u \le x \le D} A_p^x$ also has 0-density. By Kummer's rule for Lucasnomials, for n to be in $B_p^x(z)$, all digits λ_i, up to at most $a_p - 1$ of them, have to lie in the interval $[0, p/2)$. Thus, adopting an overabundant way of counting, the cardinality of $B_p^x(z)$ is bounded above by

$$\binom{D - x + 1}{D - x + 1 - (a_p - 1)} \left\lfloor \frac{p+1}{2} \right\rfloor^{D+2-x-a_p} p^{a_p - 1} \ll_p D^{a_p - 1} \cdot \frac{\left(\frac{p+1}{2}\right)^D}{\left\lfloor \frac{p+1}{2} \right\rfloor^x}.$$

Thus, summing over all x, $u \le x \le D$, we obtain

$$\# \bigcup_{u \le x \le D} B_p^x(z) \ll_p D^{a_p} \cdot \left(\frac{p+1}{2}\right)^D \ll_p (\log y)^{a_p} \left(\frac{p+1}{2}\right)^{\frac{\log y}{\log p}} = (\log y)^{a_p} y^{\theta_p} = o(z), \tag{10}$$

where $\theta_p = \log \frac{p+1}{2} / \log p < 1$.

Suppose now $x < u$. Then $\nu_p(U_{n+k}) \le x + \delta + a \le a_p := u + \delta + a$. Hence, by the Kummer rule for Lucasnomials, for $n \in A_p^x$, the number of carries in the base-p addition of n/ρ to itself must be less than a_p. Let v be the largest index $i > x$ such that λ_i is affected by the subtraction of $\ell + 1 - p^x$ from $(\lambda - 1)p^x$ (see (8) and (9)). We claim that v is at most equal to $u + a_p$. Indeed, if $\lambda_u \ge 1$, then $v \le u$. If $\lambda_u = 0$ and $(\ell + 1) - p^x > \lambda_{u-1}p^{u-1} + \cdots + \lambda_x p^x$, then v is the least index $j > u$ with $\lambda_j \ge 1$. (It must exist if $n \ge \rho$ by (8).) The subtraction of $(\ell + 1) - p^x$ then decrements λ_v by 1, puts all digits λ_i, $u < i < v$, to $p - 1$ and again possibly alters the remaining digits λ_i, $x \le i \le u$. Thus, the base-p addition of n/ρ to itself produces a minimum of $v - u - 1$ relevant carries. Hence, if $v - u > a_p$, then $n \notin A_p$. Thus, $v \le u + a_p$. Define C_p as the set of integers n such that at most $a_p - 1$ carries occur in the base-p addition of n/ρ to n/ρ between the places $a_p + u + 1$ and D left of the radix point. Then, choosing at most $a_p - 1$ p-ary digits λ_i, $a_p + u + 1 \le i \le D$, outside the interval $[0, p/2)$, we see, overcounting again some elements of $C_p(z)$, that

$$\#C_p(z) \le \binom{D - a_p - u}{D - 2a_p - u + 1}_U \left\lfloor \frac{p+1}{2} \right\rfloor^{D - 2a_p - u + 1} p^{a_p - 1} \ll_p D^{a_p} \left(\frac{p+1}{2}\right)^D,$$

which, as seen in (10), is $o(z)$. Since $A_p^x \subset C_p$ for all x, $0 \le x < u$, we find that $\bigcup_{0 \le x < u} A_p^x$ has 0 asymptotic density. Combining this case with the case $x \ge u$, we obtain that A_p has asymptotic density 0. $\square$

5. Theorem 1.3 and Lucasnomials, (i.e., The Case $k \le 0$)

The divisibility of middle Lucasnomial coefficients when U is Δ-regular and $k \le 0$ is different from the behavior of ordinary middle binomial coefficients as described in Theorem 1.3.

Theorem 5.1. *Suppose $U(P,Q)$ is a Δ-regular Lucas sequence and $k \ge 0$ is a fixed integer. Then, there are at most finitely many n such that U_{n-k} divides $\binom{2n}{n}_U$.*

Proof. For n larger than M_k :=max$\{3k, 30+k\}$, U_{n-k} has a primitive prime divisor, say p, by the primitive prime divisor theorem [6]. Since its rank ρ satisfies $30 < n-k = \rho \le p+1$, we find that $p \ge 31$. Now,

$$\frac{n}{\rho} = \frac{n}{n-k} = 1 + \frac{k}{n-k}.$$

But, $n > 3k$ implies that $k/(n-k) < 1/2$. Thus, in the base-p addition of n/ρ to itself, there is no relevant carry as $1+1 < 31 \le p$. By Proposition 1.5, p does not divide $\binom{2n}{n}_U$. Therefore, U_{n-k} does not divide $\binom{2n}{n}_U$ for all $n > M_k$. □

Example 5.2. *The set $D_{F,0} = \{1,2,3\}$, where F is the Fibonacci sequence.*

In this example, we may replace M_k by 12 as for the Fibonacci sequence every term larger than 12 has a primitive prime divisor. However, for $n = 4,5,7,8,9,10$, and 11, F_n has an odd primitive prime divisor, namely and respectively $3,5,13,7,17,11$, and 89. It remains to check the cases $n = 1,2,3,6$, and 12.

6. The Middle Fibonomial Coefficient and 105, Plus Some Open Questions

We propose a few open problems.

Problem 1. Find a combinatorial interpretation of the Lucasnomial Catalan numbers, a quest Sagan [16] seems to have worked at. But, would there also be an interpretation for the numbers $\frac{1}{U_{n+2}}\binom{2n}{n}_U$ and $\frac{1}{U_{n+1}U_{n+2}}\binom{2n}{n}_U$, when $U = U(\pm 1, 2)$?

Problem 2. Do Theorems 3.5, 4.3, and 5.1 extend to all nondegenerate Lucas sequences?

Define $\Omega_{U,m}$ as the set of integers n such that $\binom{2n}{n}_U$ is prime to m, where U is a Lucas sequence and $m \ge 1$ an integer. Let I_n denote the Lucas sequence $I_n = n$, i.e., $I = U(2,1)$.

Problem 3. If $U(P,Q)$ is nondegenerate with Q odd, then, by [2, Theorem 5.2], $\Omega_{U,2}$ is either empty or a singleton. If $p \nmid Q$ is an odd prime, then, by Kummer's rule for Lucasnomials, it is easy to see $\Omega_{U,p}$ is infinite. As mentioned in [13], the set $\Omega_{I,pq}$ is known [8] to be infinite, whenever p and q are two odd primes. Is the set $\Omega_{F,pq}$ also infinite, if F is the Fibonacci sequence? Would it be true that $\Omega_{U,pq}$ is infinite, if $U(P,Q)$ is an arbitrary nondegenerate Lucas sequence, as long as p and q do not divide $\gcd(P,Q)$ and are odd primes?

Problem 4. Are there infinitely many integers n with $\binom{2n}{n}_F$ coprime to 105?

Ron Graham offers a reward of \$1000 for settling the question of whether the set of integers n such that $\binom{2n}{n}$ is prime to 105 is finite or infinite, observing on heuristic grounds that this set should be infinite. Hence, a fifth problem: Would the Fibonacci Association offer \$987 for settling Problem 4?

Pomerance [13] proposed a heuristic argument in favor of the infinitude of Graham's 105-set. As we understood it, the argument went more or less as follows. If p is an odd prime, then $\#\Omega_{I,p}(z)$ is of the order of z^{θ_p}, as z tends to ∞, with $\theta_p = \log(\frac{p+1}{2})/\log p$. Indeed, imagine for the sake of simplicity that $z = p^t$, t large. Suppose $n = \sum_{i=0}^{t-1} n_i p^i$, where the n_i's are the base-p digits of n and $0 \le n < z$. By Kummer's rule, n belongs to $\Omega_{I,p}$ iff $n_i \in [0,(p-1)/2]$, for all i, $0 \le i < t$. Thus, we obtain exactly

$$\#\Omega_{I,p}(z) = \#\{p^t\} + \left(\frac{p+1}{2}\right)^t - \#\{0\} = z^{\theta_p}. \tag{11}$$

Thus, the probability that a random integer in $[1, z]$ be in $\Omega_{I,p}$ is $1/z^{1-\theta_p}$. Assuming probabilistic independence of the base-p and the base-q representations of a general integer, p and q being distinct odd primes, we obtain that the probability for an integer in $[1, z]$ to lie in $\Omega_{I,105}$ is

$$\prod_{p\in\{3,5,7\}} \frac{1}{z^{1-\theta_p}} = \frac{1}{z^{\theta}},$$

where θ is about 0.974. Hence, we expect at least $z^{0.025}$ integers in $\Omega_{I,105}(z)$, as $z \to \infty$.

A similar heuristic applies to the Fibonomial case. Suppose p is an odd prime of rank ρ. Say $z = \rho p^t$ to simplify matters. Then, any integer n, $0 \le n < z$, has a unique representation in the mixed-base $(\rho p^{t-1}, \rho p^{t-2}, \ldots, \rho p, \rho, 1)$ of the form

$$n = n_{t-1}p^{t-1} + n_{t-2}p^{t-2} + \cdots + n_0\rho + d, \tag{12}$$

where $0 \le n_i < p$, $(0 \le i < t)$, and $0 \le d < \rho$.

Using Proposition 1.5, which we can be rephrased as in Proposition 6.1 below, we find that n belongs to $\Omega_{F,p} \cap [0, z)$ iff each $n_i \in [0, p/2)$ and $d \in [0, \rho/2)$. Since $\rho_F(p) = p + 1$ for $p = 3$ and $p = 7$, we see that

$$\#\Omega_{F,p}(z) = \frac{p+1}{2} \cdot \left(\frac{p+1}{2}\right)^t = \frac{p+1}{2} \cdot \left(\frac{z}{\rho}\right)^{\theta_p} = \frac{1}{2}(p+1)^{1-\theta_p} \cdot z^{\theta_p}. \tag{13}$$

For $p = 5$, since $\rho_F(5) = \rho_I(5) = 5$, we still expect about z^{θ_p} integers in $\Omega_{F,p}(z)$.

Conclusion. Although $\frac{1}{2}(3+1)^{1-\theta_3} \cdot \frac{1}{2}(7+1)^{1-\theta_7}$ is about 0.758, we can hardly assert that the proportion of middle Fibonomials prime to 105 is about 3/4 that of middle binomial coefficients. Indeed, other choices of z than p^t in (11) and ρp^t in (13) would yield cardinalities for both $\Omega_{I,p}(z)$ and $\Omega_{F,p}(z)$ of the type $c_z z^{\theta_p}$ with $c_z = O_p(1)$, but $c_z \neq 1$ in general [13, Lemma 1]. However, these heuristics suggest that $\#\Omega_{I,105}(z)$ and $\#\Omega_{F,105}(z)$ are both of the order of $z^{1-\theta}$, as z tends to infinity.

The first few middle Fibonomials prime to 105. The only values of $n \le 20{,}000$ with $\binom{2n}{n}$ prime to 105 are well known to be 1, 10, 756, 757, 3160, 3186, 3187, 3250, 7560, 7561, and 7651 [17, A030979]. Tinkering with PARI/GP, we believe we found all $n \le 4050$ with $\binom{2n}{n}_F$ prime to 105. They are

$$1, 1312, 3256, 3257, 3936, 3937, 4000, 4001, \text{ and } 4032.$$

Unsurprisingly, we find similar clustering phenomena in both sequences.

Proposition 6.1. *Let $U(P,Q)$ be a nondegenerate Lucas sequence and $p \nmid Q$ a prime of rank ρ in U. Let m and n be two positive integers written in the mixed-base $\{\rho p^i\}$, $i \ge 0$, as in (12). Then the p-adic valuation of the Lucasnomial $\binom{m+n}{n}_U$ is obtained by counting carries in this mixed-base addition of m and n, where a carry at the first place has a weight of $\nu_p(U_\rho)$, a carry at the second place a weight of $1+\delta$, with δ equal to $\nu_2\big((P^2-3Q)/2\big)$, if p is 2 and PQ is odd, zero, otherwise, and other carries count for* 1.

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