RELATIONSHIPS BETWEEN k-GONAL NUMBERS THAT ARE CENTERED k-GONAL, AND LUCAS AND RELATED NUMBERS

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Abstract. Relationships between k-gonal numbers, centered k-gonal numbers and Lucas numbers, and between k-gonal, centered k-gonal numbers and Pell-Lucas numbers are explored in this paper.

1. Introduction

For a positive integer $k \geq 3$, the nth term of the kth polygonal number, $G(n;k)$, and the nth term of the kth centered polygonal number, $C(n;k)$, are given by

$$G(n;k) = \frac{(k-2)n^2 + (4-k)n}{2}$$

and

$$C(n;k) = \frac{kn^2 - kn + 2}{2},$$

respectively (see [5]). It was shown in [1] that the numbers that are both k-gonal and centered k-gonal are given by

$$u(n;k) = \frac{k}{16(k-2)} \left\{ \frac{-2k^2 + 18k - 32}{k} + \left[k - 1 + \sqrt{k(k-2)}\right]^{2n+1} + \left[k - 1 - \sqrt{k(k-2)}\right]^{2n+1} \right\} \tag{1.1}$$

for $n \geq 0$ and $k \geq 3$. The Pell numbers, P_n, are defined recursively by $P_0 = 0$, $P_1 = 1$, and $P_{n+2} = 2P_{n+1} + P_n$ for all $n \geq 0$. The Pell-Lucas numbers, Q_n, are defined recursively by $Q_0 = 2$, $Q_1 = 2$, and $Q_{n+2} = 2Q_{n+1} + Q_n$ for all $n \geq 0$. The Binet formulas for P_n and Q_n are

$$P_n = \frac{1}{\sqrt{8}} (r^n - s^n)$$

and

$$Q_n = r^n + s^n,$$

respectively, for $n \geq 0$ where $r = 1 + \sqrt{2}$ and $s = 1 - \sqrt{2}$.

Relationships between k-gonal numbers, Fibonacci, Lucas, and related numbers have not been widely investigated. We highlight two sources. In [4], it was shown that the only Fibonacci numbers that are also triangular numbers are 0, 1, 3, 21, and 55. Also, in [2], Chapter 7, some results about pentagonal Pell-Lucas numbers, and heptagonal Pell numbers are given. In this paper, we prove some relationships between k-gonal, centered k-gonal numbers and Lucas numbers, and between k-gonal, centered k-gonal numbers and Pell-Lucas numbers. Other relationships to related numbers are also indicated.

2. k-gonal, Centered k-gonal Numbers and Lucas Numbers

The idea is based on the observation that, in (1.1), $\sqrt{k(k-2)}$ can be an integral multiple of $\sqrt{5}$. In fact, we consider equation

$$k(k-2) = 5j^2 \tag{2.1}$$

for some integer j. Equation (2.1) can be written as

$$(k-1)^2 - 5j^2 = 1. \tag{2.2}$$

Equation (2.2) is a Pell equation of the form $x^2 - 5y^2 = 1$ with $x = k-1$ and $y = j$. To solve (2.2) we find that $(k-1, j) = (9, 4)$ is the smallest solution with $k - 1 > 1$. The generator is $\theta = 9 + 4\sqrt{5} = \alpha^6$ with $\alpha = \frac{1+\sqrt{5}}{2}$. Thus all solutions are given by

$$\begin{aligned}\theta^l &= (9+4\sqrt{5})^l \\ &= \alpha^{6l} \\ &= \frac{L_{6l}}{2} + \frac{F_{6l}}{2}\sqrt{5}.\end{aligned}$$

It follows that $k = x + 1 = \frac{1}{2}L_{6l} + 1$ for $l \geq 1$. And so, after simplification, we have the following relationship.

Theorem 2.1. *Let $u(n,k)$ be as in (1.1) and L_i be the ith Lucas number. Then*

$$u(n; 1 + \frac{1}{2}L_{6l}) = \frac{1}{-16 + 8L_{6l}} \left\{ -17 + 7L_{6l} - \frac{1}{2}L_{12l} + (1 + \frac{1}{2}L_{6l})L_{6l(2n+1)} \right\}. \tag{2.3}$$

Remark 2.2. *Using the identities $L_{2m} = L_m^2 - 2(-1)^m$ and*

$$L_m^2 = 5F_m^2 + 4(-1)^m, \tag{2.4}$$

a relationship between $u(n; 1 + \frac{1}{2}L_{6l})$ and Fibonacci numbers follows from (2.3).

The following remarks illustrate a partial converse to the observation at the beginning of this section that lead to theorem (2.1).

Remark 2.3. *Assume m is even and (L_m, y_m) is a solution to the Pell equation*

$$x^2 - k(k-2)y^2 = 4. \tag{2.5}$$

We show that this implies $k(k-2) = 5j^2$ for some integer j (compare with the fact that the Pell equation $x^2 - 5y^2 = 4$ is solvable in positive integers if and only if $x = L_{2n}$ and $y = F_{2n}$, for $n \geq 1$ [3]).

Since m is even, (2.4) and (2.5) imply $5F_m^2 = k(k-2)y_m^2$. If $k(k-2)$ is not a multiple of 5, then y_m^2 must be a multiple of 5, and so y_m is a multiple of 5. Let $y_m = 5^l j$, where j is not a multiple of 5. Then

$$5F_m^2 = k(k-2)5^{2l}j^2,$$

and so

$$F_m^2 = 5^{2l-1}k(k-2)j^2.$$

Thus, $k(k-2)5^{2l-1}$ is a perfect square. Since $k(k-2)$ is not a multiple of 5, $k(k-2)5^{2l-1}$ cannot be a perfect square. This is a contradiction. Thus, $k(k-2)$ is a multiple of 5. Let $k(k-2) = 5a$ for some integer a. It follows $5F_m^2 = k(k-2)y_m^2 = 5ay_m^2$. Thus, $F_m^2 = ay_m^2$, and so $a = j^2$ for some integer j. Note that if, in addition, L_m is even as is the case in (2.3), then (2.5) can be written in the form (2.2).

A similar argument holds if m is odd and (L_m, y_m) is a solution to $x^2 - k(k-2)y^2 = -4$.

Remark 2.4. *Assume m is odd and (L_m, y_m) is a solution to (2.5). We show that $k = 2b+1$, where b is an odd integer.*

From (2.4) and (2.5), we obtain

$$2L_m^2 = 5F_m^2 + k(k-2)y_m^2, \tag{2.6}$$

and so $5F_m^2$ and $k(k-2)y_m^2$ have the same parity. This implies that k is odd. For if it were even, then $5F_m^2$ and $k(k-2)y_m^2$ would become both even. Since m is odd, $F_m = 2f_m$, where f_m is odd, and $L_m = 2^2 l_m$, where l_m is odd. Now (2.6) implies

$$2 \times 2^2 \times l_m^2 = 5f_m^2 + k'(k'-1)y_m^2, \tag{2.7}$$

where $k = 2k'$. Since $k'(k'-1)$ is even and f_m is odd, (2.7) yields a contradiction, and so k must be odd.

If $5F_m^2$ and $k(k-2)y_m^2$ are both odd, then, from (2.4) and (2.5), we obtain

$$5F_m^2 = k(k-2)y_m^2 + 8,$$

and equivalently, $5(F_m^2 - 1) = k(k-2)y_m^2 + 3$. Let $F_m = 2i+1$ and $y_m = 2j+1$ for some integers i and j, respectively. It follows that

$$\begin{aligned} 5(4i^2 + 4i) &= k(k-2)(4j^2 + 4j + 1) + 3 \\ &= k(k-2)(4j^2 + 4j) + k(k-2) + 3 \\ &= k(k-2)(4j^2 + 4j) + (k-1)^2 + 2. \end{aligned}$$

Since k is odd, $(k-1)^2$ is a multiple of 4. Thus, $5(4i^2+4i) = k(k-2)(4j^2+4j) + (k-1)^2 + 2$ implies that 2 is a multiple of 4; a contradiction. Thus, $5F_m^2$ and $k(k-2)y_m^2$ are both even. Hence, F_m^2 and y_m^2 are multiples of 4, and so (2.6) implies that L_m is even. It is known that if m is odd and F_m is even, then $F_m = 2f_m$, where f_m is odd, and if m is odd and L_m is even, then $L_m = 2^2 l_m$, where l_m is odd. It follows from (2.6) that $2^3 l_m^2 = 5f_m^2 + k(k-2)z_m^2$, where $y_m = 2z_m$ and z_m is odd. Writing $f_m = 2r_m + 1$ and $z_m = 2s_m + 1$, we obtain

$$2^3 l_m^2 = 5(r_m^2 + r_m) + k(k-2)(s_m^2 + s_m) + \frac{(k-1)^2}{4} + 1.$$

This implies that $\frac{(k-1)^2}{4}$ must be odd. It follows that $k = 2b+1$, where b is odd.

3. k-gonal, Centered k-gonal Numbers and Pell-Lucas Numbers

An argument similar to the one in Section 2 yields a relationship between k-gonal, centered k-gonal numbers and Pell-Lucas numbers. We need only consider when $\sqrt{k(k-2)}$ can be an integral multiple of $\sqrt{2}$. This leads to the Pell equation

$$x^2 - 2y^2 = 1$$

with $x = k-1$ and $y = m$. Since the generator in this case is $\phi = 3+2\sqrt{2}$, the general solution is given by

$$\begin{aligned}\phi^l &= (3+2\sqrt{2})^l \\ &= \left[(1+\sqrt{2})^2\right]^l \\ &= (1+\sqrt{2})^{2l} \\ &= r^{2l} \\ &= \frac{1}{2}Q_{2l} + \frac{1}{2}P_{2l}\sqrt{2}.\end{aligned}$$

Now $k = x+1 = 1+\frac{1}{2}Q_{2l}$ and so, after simplification, we have the following relationship.

Theorem 3.1. *Let $u(n,k)$ be as in (1.1) and Q_i be the ith Pell-Lucas number. Then*

$$u(n;1+\frac{1}{2}Q_{2l}) = \frac{1}{-16+8Q_{2l}}\left\{-2(1+\frac{1}{2}Q_{2l})^2 + 18(1+\frac{1}{2}Q_{2l}) - 32 + (1+\frac{1}{2}Q_{2l})Q_{2l(2n+1)}\right\}. \tag{3.1}$$

Remark 3.2. *Using the identities $Q_{2m}^2 = Q_m^2 - 2(-1)^m$ and $Q_n^2 = 8P_m^2 + 4(-1)^m$, a relationship between $u(n;1+\frac{1}{2}Q_{2l})$ and Pell numbers follows from (3.1).*

The following remarks illustrate a partial converse to the observation at the beginning of this section that lead to Theorem 3.1.

Remark 3.3. *An argument identical to the one used in remark (2.3) yields the following. Let (Q_m, y_m) be a solution to $x^2 - k(k-2)y^2 = 4(-1)^n$. Then $k(k-2) = 2j^2$ for some integer j.*

Remark 3.4. *If m is odd, then (2.5) cannot have a solution of the form (Q_m, y_m).*

In fact, if it did, then (2.5) and the identity $Q_m^2 = 2P_m^2 - 4$ imply

$$2Q_m^2 = 2P_m^2 + k(k-2)y_m^2. \tag{3.2}$$

Thus, $k(k-2)y_m^2$ must be even. In fact, whether k is even or odd, $k(k-2)y_m^2$ is even implies $k(k-2)y_m^2 = 4c$ for some integer c. It follows from (3.2) that

$$Q_m^2 = P_m^2 + 2c. \tag{3.3}$$

Since Q_m is even for all values of m and P_m is odd when m is odd, (3.3) yields a contradiction.

References

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