CIRCULAR BALANCING NUMBERS

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Abstract. Circular balancing numbers are introduced and several special cases are explored.

1. Introduction

A balancing number is a natural number n such that if it is removed from first m ($m > n$ and m depends on n) natural numbers arranged in a line, then the sum of numbers to the left of n is equal to the sum to its right [1, 6]. Several generalizations of balancing numbers have been studied by many authors [2, 3, 4, 5, 6, 7, 8]. In this paper, our focus is on another exciting generalization of balancing numbers, which we call circular balancing numbers.

Instead of arranging numbers in a line as in the case of balancing and cobalancing numbers [1, 3], consider an arrangement of m natural numbers equally spaced on a circle. Fix a number k on this circle. By deleting two numbers corresponding to a chord whose one end is k and other end is $x(> k)$, the circular arrangement of numbers will be divided into two arcs. If the sums of numbers on those two arcs are the same, then we call x a k-circular balancing number. More precisely, we can define circular balancing numbers as follows.

Definition 1.1. *Let k be a fixed positive integer. We call a positive integer n, a k-circular balancing number if the Diophantine equation*

$$(k+1)+(k+2)+\cdots+(n-1)=(n+1)+(n+2)+\cdots+m+(1+2+\cdots+k-1) \quad (1.1)$$

holds for some natural number m.

It is possible to simplify equation (1.1) as

$$T_m + k^2 = n^2, \quad k+2 < n < m$$

where T_m is the mth triangular number. The Diophantine equation $T_m + k^2 = n^2$ is a variant of the Pythagorean equation $x^2 + y^2 = z^2$ with one square replaced by a triangular number. However, unlike the Pythagorean equation, it is difficult to find a compact form of solutions for the equation $T_m + k^2 = n^2$.

Observe that if $k = 0$, then the circular balancing numbers coincide with the balancing numbers [1, 6]. If $k = 1$, then the circular balancing numbers are almost balancing numbers [5].

Example 1.2. *Since $2+3=5$, 4 is a 1-circular balancing number. Similarly, since $11+\cdots+19=21+\cdots+24+(1+\cdots+9)$, 20 is a 10-circular balancing number.*

2. 2-Circular Balancing Numbers

By definition, a natural number x is a 2-circular balancing number if

$$3+4+\cdots+(x-1)=(x+1)+\cdots+m+1$$

holds for some natural number m. Equivalently, a natural number $x > 2$ is a 2-circular balancing number if and only if $8x^2 - 31$ is a perfect square. Setting $8x^2 - 31 = y^2$, the calculation of 2-circular balancing numbers reduces to solving the generalized Pell equation

$$y^2 - 8x^2 = -31. \tag{2.1}$$

It is easy to see that the fundamental solution of the Pell equation $y^2 - 8x^2 = 1$ is $3 + \sqrt{8}$ and $1 + 2\sqrt{8}$ is a fundamental solution of (2.1). Using the theory of generalized Pell equations, one class of 2-circular balancing numbers can be obtained from

$$y_n + \sqrt{8}x_n = (1 + 2\sqrt{8})(3 + \sqrt{8})^{n-1}, \quad n = 1, 2, \ldots.$$

Thus, the nth member of this class of 2-circular balancing numbers is given by

$$x_n = \frac{(1 + 4\sqrt{2})(3 + 2\sqrt{2})^{n-1} - (1 - 4\sqrt{2})(3 - 2\sqrt{2})^{n-1}}{4\sqrt{2}}, \quad n = 1, 2, \ldots.$$

Using the Binet form for balancing numbers [see [1],[6]], one can have

$$x_n = 2B_n - 5B_{n-1}, \quad n = 1, 2, \ldots.$$

It is well-known that $y_{-n} + \sqrt{8}x_{-n}$ is also a solution of the generalized Pell equation (2.1). Since $x_n = 2B_n - 5B_{n-1}$ and $B_{-n} = -B_n$, it follows that

$$x'_n = x_{-n} = 2B_{-n} - 5B_{-n-1} = 5B_{n+1} - 2B_n$$

which is positive and greater than 2 for $n = 0, 1, 2, \ldots$ and therefore, represents another class of 2-circular balancing numbers. Using the theory of generalized Pell's equation, one can easily verify that there is no other class of 2-circular balancing numbers. Hence, the set

$$\{2B_n - 5B_{n-1},\ 5B_n - 2B_{n-1} \ : \ n = 1, 2, \ldots\}$$

is an exhaustive list of 2-circular balancing numbers. Each of the two classes of 2-circular balancing numbers can be recursively calculated by a binary recurrence identical to that for balancing numbers. In particular, these recurrences are

$$x_{n+1} = 6x_n - x_{n-1}$$

and

$$x'_{n+1} = 6x'_n - x'_{n-1}$$

with initial values $x_0 = 5$, $x_1 = 2$, $x'_0 = 2$, $x'_0 = 5$. We can summarize the above discussion in the following theorem.

Theorem 2.1. *The 2-circular balancing numbers are solutions in x of the generalized Pell equation $y^2 - 8x^2 = -31$. These solutions partition into two classes given by $x_n = 2B_n - 5B_{n-1}$, $x'_n = 5B_n - 2B_{n-1} : n = 1, 2, \ldots$ and satisfy the binary recurrences $x_{n+1} = 6x_n - x_{n-1}$ and $x'_{n+1} = 6x'_n - x'_{n-1}$ with initial values $x_0 = 5$, $x_1 = 2$, $x'_0 = 2$, and $x'_1 = 5$.*

3. 3-Circular Balancing Numbers

In view of the Definition 1.1, a natural number x is a 3-circular balancing number if

$$4 + 5 + (x - 1) = (x + 1) + \cdots + m + 1 + 2$$

holds for some natural number m. After simplification, we can conclude that a natural number $x > 3$ is a 3-circular balancing number if and only if $8x^2 - 71$ is a perfect square. Writing $8x^2 - 71 = y^2$, the calculation of 3-circular balancing numbers requires solving of the generalized Pell equation

$$y^2 - 8x^2 = -71. \tag{3.1}$$

In the last section, we have already noticed that the fundamental solution of the Pell equation $y^2 - 8x^2 = 1$ is $3 + \sqrt{8}$ and a fundamental solution of (3.1) is $1 + 3\sqrt{8}$. Using the theory of generalized Pell equations, one class of 3-circular balancing numbers is contained in

$$y_n + \sqrt{8}x_n = (1 + 3\sqrt{8})(3 + \sqrt{8})^{n-1}, \quad n = 1, 2, \ldots.$$

Thus, the nth member of this class is given by

$$x_n = \frac{(1 + 6\sqrt{2})(3 + 2\sqrt{2})^{n-1} - (1 - 6\sqrt{2})(3 - 2\sqrt{2})^{n-1}}{4\sqrt{2}}, \quad n = 1, 2, \ldots.$$

Using the Binet form for balancing numbers [6], it is easy to see that

$$x_n = 3B_n - 8B_{n-1}, \quad n = 1, 2, \ldots.$$

It is well-known that $y_{-n} + \sqrt{8}x_{-n}$ is also a solution of the generalized Pell equation (3.1). Since $x_n = 3B_n - 8B_{n-1}$ and $B_{-n} = -B_n$, it follows that

$$x'_n = x_{-n} = 3B_{-n} - 8B_{-n-1} = 8B_{n+1} - 3B_n$$

is positive and greater than 3 for $n = 0, 1, 2, \ldots$ and hence, represents another class of 3-circular balancing numbers. One can verify that there are just two fundamental solutions of (3.1). Hence, the set

$$\{3B_n - 8B_{n-1}, \quad 8B_n - 3B_{n-1} : n = 1, 2, \ldots\}$$

contains all the 3-circular balancing numbers. The two classes of 3-circular balancing numbers can also be expressed by means of the binary recurrences

$$x_{n+1} = 6x_n - x_{n-1}$$

and

$$x'_{n+1} = 6x'_n - x'_{n-1}$$

with initial values $x_0 = 8$, $x_1 = 3$, $x'_0 = 3$, $x'_1 = 8$. The above discussion proves the following theorem.

Theorem 3.1. *The values of x satisfying the generalized Pell equation $y^2 - 8x^2 = -71$ partition into two classes given by $x_n = 3B_n - 8B_{n-1}$ and $x'_n = 8B_n - 3B_{n-1} : n = 1, 2, \ldots$ that represent all the 3-circular balancing numbers. These two classes of solutions satisfy the binary recurrences $x_{n+1} = 6x_n - x_{n-1}$ and $x'_{n+1} = 6x'_n - x'_{n-1}$ with initial values $x_0 = 8$, $x_1 = 3$, $x'_0 = 3$, and $x'_1 = 8$.*

4. 4-Circular Balancing Numbers

By virtue of Definition 1.1, a natural number x is a 4-circular balancing number if

$$5 + 6 + (x - 1) = (x + 1) + \cdots + m + 1 + 2 + 3$$

holds for some natural number m. After simplification, it follows that a natural number $x > 4$ is a 4-circular balancing number if and only if $8x^2 - 127$ is a perfect square. Setting $8x^2 - 127 = y^2$, the calculation of 4-circular balancing numbers requires solving the generalized Pell equation.

$$y^2 - 8x^2 = -127. \tag{4.1}$$

We already know that the fundamental solution of the Pell equation $y^2 - 8x^2 = 1$ is $3 + \sqrt{8}$ and a fundamental solution of (4.1) is $1 + 4\sqrt{8}$. Thus, one class of 4-circular balancing numbers is contained in

$$y_n + \sqrt{8}x_n = (1 + 4\sqrt{8})(3 + \sqrt{8})^{n-1}, \quad n = 1, 2, \ldots$$

Using this equation, the nth member of this class of 4-circular balancing numbers can be written as

$$x_n = \frac{((1+8\sqrt{2})(3+2\sqrt{2})^{n-1} - (1-8\sqrt{2})(3-2\sqrt{2})^{n-1})}{4\sqrt{2}}, \quad n = 1, 2, \ldots$$

and referring to the Binet form for balancing numbers, one can get

$$x_n = 4B_n - 11B_{n-1}, \quad n = 1, 2, \ldots.$$

As usual, $y_{-n} + \sqrt{8}x_{-n}$ is also a solution of the generalized Pell equation (4.1). Since $x_n = 4B_n - 11B_{n-1}$ and $B_{-n} = -B_n$, it follows that

$$x'_n = x_{-n} = 4B_{-n} - 11B_{-n-1} = 11B_{n+1} - 4B_n$$

is positive and greater than 4 for $n = 0, 1, 2, \ldots$ and hence, represents another class of 4-circular balancing numbers. One can verify that (4.1) has just two fundamental solutions. Therefore, the set

$$\{4B_n - 11B_{n-1}, \quad 11B_n - 4B_{n-1} : n = 1, 2, \ldots\}$$

gives the complete list of 4-circular balancing numbers. The two classes of 4-circular balancing numbers can also be recursively expressed as

$$x_{n+1} = 6x_n - x_{n-1}$$

and

$$x'_{n+1} = 6x'_n - x'_{n-1}, \quad n = 1, 2, \ldots$$

with initial values $x_0 = 11$, $x_1 = 4$, $x'_0 = 4$, and $x'_1 = 11$. In view of the above discussion, we have the following theorem

Theorem 4.1. *The* 4*-circular balancing numbers are solutions in* x *of the generalized Pell equation* $y^2 - 8x^2 = -127$ *and can be realized in two classes as* $x_n = 4B_n - 11B_{n-1}$ *and* $x'_n = 11B_n - 4B_{n-1}$*;* $n = 1, 2, \ldots$*. Further, the two classes of* 4*-circular balancing numbers obey the recurrence relations* $x_{n+1} = 6x_n - x_{n-1}$ *and* $x'_{n+1} = 6x'_n - x'_{n-1}$ *with initial values* $x_0 = 11$*,* $x_1 = 4$*,* $x'_0 = 4$*, and* $x'_1 = 11$*.*

5. k-Circular Balancing Numbers

By virtue of Definition 1.1, a natural number $x > k$ is a k-circular balancing number if and only if $8x^2 - 8k^2 + 1$ is a perfect square. Writing $8x^2 - 8k^2 + 1 = y^2$, the k-circular balancing numbers are values of x satisfying the generalized Pell equation

$$y^2 - 8x^2 = -8k^2 + 1. \tag{5.1}$$

A fundamental solution of the above equation is $1+k\sqrt{8}$. Thus, one class of k-circular balancing numbers can obtained from

$$y_n + \sqrt{8}x_n = (1 + k\sqrt{8})(3 + \sqrt{8})^{n-1}, \quad n = 1, 2, \ldots.$$

The nth member of this class is given by

$$x_n = \frac{(1+2k\sqrt{2})(3+2\sqrt{2})^{n-1} - (1-2k\sqrt{2})(3-2\sqrt{2})^{n-1}}{4\sqrt{2}}.$$

Using the Binet form for balancing numbers, it is easy to see that

$$x_n = kB_n - (3k-1)B_{n-1}, \quad n = 1, 2, \ldots.$$

It is well-known that $y_{-n} + \sqrt{8}x_{-n}$ is also a solution of (5.1). Since $x_n = kB_n - (3k-1)B_{n-1}$ and $B_{-n} = -B_n$, it follows that $x'_n = x_{-n} = kB_{-n} - (3k-1)B_{-n-1} = (3k-1)B_{n+1} - kB_n$

is positive and greater than k for $n = 0, 1, 2, \ldots$ and hence, gives another class of k-circular balancing numbers. Thus, the set

$$\{kB_n - (3k-1)B_{n-1}, \quad (3k-1)B_n - kB_{n-1} : n = 1, 2, \ldots\}$$

represents two classes of k-circular balancing numbers. These two classes can be recursively defined as

$$x_{n+1} = 6x_n - x_{n-1}$$

and

$$x'_{n+1} = 6x'_n - x'_{n-1}$$

with initial values $x_0 = 3k-1$, $x_1 = k$, $x'_0 = k$, $x'_1 = 3k-1$. The above discussion can be summarized as follows.

Theorem 5.1. *For any arbitrary positive integer k, the k-circular balancing numbers are solutions in x of the generalized Pell equation $y^2 - 8x^2 = -8k^2 + 1$. It is always possible to extract two classes of k-circular balancing numbers given by $x_n = kB_n - (3k-1)B_{n-1}$, $x'_n = (3k-1)B_n - kB_{n-1} : n = 1, 2, \ldots$. These two classes can be described in terms of binary recurrences as $x_{n+1} = 6x_n - x_{n-1}$ and $x'_{n+1} = 6x'_n - x'_{n-1}$ with initial terms $x_0 = 3k-1$, $x_1 = k$, $x'_0 = k$, and $x'_1 = 3k-1$.*

6. Scope for Future Work

It is important to note that two classes of k-circular balancing numbers appearing in Theorem 5.1 may not provide an exhaustive list for some values of k. In particular, the 6-circular balancing numbers are solutions of $y^2 - 8x^2 = -287$ and these solutions partition into four classes and hence there are four classes of 6-circular balancing numbers. One can verify that these four classes constitute the set

$$\{6B_n - 17B_{n-1}, \quad 17B_n - 6B_{n-1}, \quad 8B_n - 9B_{n-1}, \quad 9B_n - 8B_{n-1} : n = 1, 2, \ldots\}.$$

It is not possible to explore all classes of circular balancing numbers for an arbitrary positive integer k as it requires solving the generalized parametrized Pell's equation (5.1). However, there is ample scope for exploring all k-circular balancing numbers at least for certain subclasses of natural numbers. We leave this as an open problem for future researchers.

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