AREAS OF TRIANGLES AND OTHER POLYGONS WITH VERTICES FROM VARIOUS SEQUENCES

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Abstract. Motivated by Elementary Problems B-1167 [13] and B-1172 [7], formulas for the areas of triangles and other polygons having vertices with coordinates taken from various sequences of integers are obtained.

1. Introduction

Finding the area of triangles with coordinates (F_n, F_{n+k}), (F_{n+2k}, F_{n+3k}) and (F_{n+4k}, F_{n+5k}) for both even and odd values of k was offered as a problem in *The Fibonacci Quarterly* [7]. In addition, solvers were asked to find similar formulas for triangles with coordinates of vertices as Lucas numbers. An earlier related problem in *The Fibonacci Quarterly* [13] invited solvers to find the area of a polygon with consecutive Fibonacci numbers as vertex coordinates beginning with (F_1, F_2) and running through (F_{2n-1}, F_{2n}). In this paper, formulas will be obtained for triangles and m-sided polygons with various sequential elements as vertices with the form $(p_n, p_{n+k}),\ (p_{n+2k}, p_{n+3k}),\ \ldots, (p_{n+(2m-2)k}, p_{n+(2m-1)k})$ where the p_i satisfy a recursion similar to that of the Fibonacci numbers. Similar, but different approaches to the topics presented here have been considered in [4] and [5].

2. Triangles with Fibonacci Type Coordinates

The area formulas for a triangle with vertices (F_n, F_{n+k}), (F_{n+2k}, F_{n+3k}) and (F_{n+4k}, F_{n+5k}) for both even and odd values of k were indicated in the statement of the problem given in [7]. Readers were invited to show that the area is

$$\frac{5F_k^4L_k}{2} \text{ if } k \text{ is even and } \frac{F_k^2L_k^3}{2} \text{ if } k \text{ is odd.}$$

The proof appears in a subsequent issue of *The Fibonacci Quarterly* [8]. We recall some facts from [8] which will be needed here. We work with a general recursion. Let $f(n)$ be defined on the integers by

$$f(n) = ar^n + b\frac{(-1)^{n+1}}{r^n} \tag{2.1}$$

where a, b and r are real numbers with $r \neq 0$. This is the general solution to the recursion

$$f(n) = pf(n-1) + f(n-2)$$

where p and r are related by $p = r - \frac{1}{r}$. Using the determinant formula from vector calculus for finding the area of a triangle, the area A of triangles with vertices $(f(n), f(n+k))$, $(f(n+2k), f(n+3k))$ and $(f(n+4k),\ f(n+5k))$ is (cf. [8])

$$A = \frac{ab(-1)^n}{2}\left(r^k - \frac{1}{r^k}\right)^3\left(r^k + \frac{1}{r^k}\right)\left(r^k + \frac{(-1)^{k+1}}{r^k}\right). \tag{2.2}$$

Using appropriate values for a, b and r, the general solution can be used to prove area formulas for triangles using Fibonacci type coordinates. The following are from [8]. If $a = b = \frac{1}{\sqrt{5}}$ and $r = \frac{1+\sqrt{5}}{2}$, then $f(n) = F_n$, and if $\overline{a} = 1$, $\overline{b} = -1$ and $r = \frac{1+\sqrt{5}}{2}$ then $f(n) = L_n$.

$$\text{If } k \text{ is even, then } \begin{cases} a\left(r^k - \dfrac{1}{r^k}\right) = a\left(r^k + \dfrac{(-1)^{k+1}}{r^k}\right) = F_k \\[2ex] \left(r^k + \dfrac{1}{r^k}\right) = \left(r^k + \dfrac{(-1)(-1)^{k+1}}{r^k}\right) = L_k \end{cases} \tag{2.3}$$

and

$$\text{if } k \text{ is odd, then } \begin{cases} a\left(r^k + \dfrac{1}{r^k}\right) = a\left(r^k + \dfrac{(-1)(-1)^{k+1}}{r^k}\right) = F_k \\[2ex] \left(r^k - \dfrac{1}{r^k}\right) = \left(r^k + \dfrac{(-1)(-1)^{k+1}}{r^k}\right) = L_k. \end{cases} \tag{2.4}$$

The area of the triangle with coordinates of vertices as Fibonacci numbers given above follows directly from 2.2, 2.3 and 2.4. Similarly, the area of the triangle with Lucas vertices (L_n, L_{n+k}), (L_{n+2k}, L_{n+3k}) and (L_{n+4k}, L_{n+5k}) is

$$\frac{25}{2}F_k^4 L_k, \quad \text{if } k \text{ is even and } \quad \frac{5F_k^2 L_k^3}{2}, \text{ if } k \text{ is odd.}$$

We turn our attention to generalized Fibonacci numbers. Consider the sequence $[G_n]$ where $G_0 = t - s$, $G_1 = s$, $G_2 = t$ and $G_{n+1} = G_n + G_{n-1}$. Using (2.1) and solving the linear system, $f(0) = (t - s)$ and $f(1) = s$, we find that if $a = \dfrac{s + \frac{(t-s)}{r}}{\sqrt{5}}$, $b = \dfrac{s + (s-t)r}{\sqrt{5}}$ and $r = \frac{1+\sqrt{5}}{2}$, then $f(n) = G_n$. Note that $ab = \dfrac{s^2 + st - t^2}{5}$.

Theorem 2.1. *The area of the triangle with generalized Fibonacci vertices (G_n, G_{n+k}), (G_{n+2k}, G_{n+3k}) and (G_{n+4k}, G_{n+5k}) is*

$$\frac{5(s^2 + st - t^2)}{2}F_k^4 L_k \text{ if } k \text{ is even and } \left(\frac{s^2 + st - t^2}{2}\right)F_k^2 L_k^3 \text{ if } k \text{ is odd.}$$

Proof. If k is even, using (2.2) and (2.3),

$$A = \frac{ab}{2}\left(r^k - \frac{1}{r^k}\right)^4\left(r^k + \frac{1}{r^k}\right) = \frac{5(s^2 + st - t^2)}{2}F_k^4 L_k.$$

If k is odd, using (2.2) and (2.4)

$$\begin{aligned} A &= \frac{ab}{2}\left(r^k - \frac{1}{r^k}\right)^3\left(r^k + \frac{1}{r^k}\right)^2 \\ &= \frac{(s^2 + s - t^2)}{2(\sqrt{5})^2}\left(r^k + \frac{(-1)^{k+1}}{r^k}\right)^2\left(r^k + \frac{-1(-1)^{k+1}}{r^k}\right)^3. \end{aligned}$$

Hence $A = \left(\dfrac{s^2 + st - t^2}{2}\right)F_k^2 L_k^3$. $\square$

It is also possible to find the formula for the area of a triangle if the coordinates of the vertices are Pell numbers or Pell-Lucas numbers. The Pell and Pell-Lucas numbers have Binet forms: [2, 10, 12]

$$P_n = \frac{(1+\sqrt{2})^n - (1-\sqrt{2})^n}{2\sqrt{2}} = \frac{\alpha^n - \beta^n}{\alpha - \beta}$$

and

$$Q_n = (1+\sqrt{2})^n + (1-\sqrt{2})^n = \alpha^n + \beta^n.$$

Returning to the general function (2.1), $f(n) = ar^n + b\frac{(-1)^{n+1}}{r^n}$, if $r = 1+\sqrt{2}$ and $a = b = \frac{1}{2\sqrt{2}}$, then $f(n) = P_n$. If $r = 1+\sqrt{2}$, $\bar{a} = 1$ and $\bar{b} = -1$, then $f(n) = Q_n$.

$$\text{If } k \text{ is even, then } \begin{cases} a\left(r^k - \dfrac{1}{r^k}\right) = a\left(r^k + \dfrac{(-1)^{k+1}}{r^k}\right) = P_k \\[2ex] \left(r^k + \dfrac{1}{r^k}\right) = \left(r^k + \dfrac{(-1)(-1)^{k+1}}{r^k}\right) = Q_k \end{cases} \tag{2.5}$$

and

$$\text{if } k \text{ is odd, then } \begin{cases} a\left(r^k + \dfrac{1}{r^k}\right) = a\left(r^k + \dfrac{(-1)(-1)^{k+1}}{r^k}\right) = P_k \\[2ex] \left(r^k - \dfrac{1}{r^k}\right) = \left(r^k + \dfrac{(-1)(-1)^{k+1}}{r^k}\right) = Q_k. \end{cases} \tag{2.6}$$

Theorem 2.2. *The area of the triangle with Pell vertices (P_n, P_{n+k}), (P_{n+2k}, P_{n+3k}) and (P_{n+4k}, P_{n+5k}) is*

$$4P_k^4 Q_k, \ \textit{if } k \textit{ is even and } \frac{P_k^2 Q_k^3}{2}, \ \textit{if } k \textit{ is odd.}$$

Proof. If k is even, using (2.2) and (2.5), we get

$$A = \frac{ab}{2}\left(r^k - \frac{1}{r^k}\right)^4\left(r^k + \frac{1}{r^k}\right) = \frac{(2\sqrt{2})^2}{2(2\sqrt{2})^4}\left(r^k - \frac{1}{r^k}\right)^4\left(r^k + \frac{1}{r^k}\right) = 4P_k^4 Q_k.$$

If k is odd, using (2.2) and (2.6), we get

$$A = \frac{ab}{2}\left(r^k - \frac{1}{r^k}\right)^3\left(r^k + \frac{1}{r^k}\right)^2 = \frac{1}{2(2\sqrt{2})^2}\left(r^k - \frac{1}{r^k}\right)^3\left(r^k + \frac{1}{r^k}\right)^2 = \frac{P_k^2 Q_k^3}{2}.$$

□

Theorem 2.3. *The area of the triangle with Lucas-Pell vertices (Q_n, Q_{n+k}), (Q_{n+2k}, Q_{n+3k}) and (Q_{n+4k}, Q_{n+5k}) is*

$$32P_k^4 Q_k, \ \textit{if } k \textit{ is even and } 8P_k^2 Q_k^3, \ \textit{if } k \textit{ is odd.}$$

Proof. If k is even then

$$A = \frac{\bar{a}\bar{b}}{2}\left(r^k - \frac{1}{r^k}\right)^4\left(r^k + \frac{1}{r^k}\right) = \frac{-(2\sqrt{2})^4}{2(2\sqrt{2})^4}\left(r^k - \frac{1}{r^k}\right)^4\left(r^k + \frac{1}{r^k}\right).$$

The absolute value gives the area as $32P_k^4 Q_k$.

If k is odd then

$$A = \frac{\overline{a}\overline{b}}{2}\left(r^k - \frac{1}{r^k}\right)^3\left(r^k + \frac{1}{r^k}\right)^2 = \frac{-(2\sqrt{2})^2}{2(2\sqrt{2})^2}\left(r^k - \frac{1}{r^k}\right)^3\left(r^k + \frac{1}{r^k}\right)^2.$$

The absolute value gives the area as $8P_k^2Q_k^3$. □

For triangles with Jacobsthal and Jacobsthal-Lucas [11] sequences as vertices, the eigenvalues do not lend themselves to the structure using (2.1). Using the Binet forms for the Jacobsthal and Jacobsthal-Lucas numbers [9], we get

$$J_n = \frac{1}{3}(2^n - (-1)^n)$$

and

$$j_n = 2^n + (-1)^n.$$

Theorem 2.4. *The points with Jacobsthal coordinates*

$$(J_n, J_{n+k}), (J_{n+2k}, J_{n+3k}), (J_{n+4k}, J_{n+5k}), \dots, (J_{n+2mk}, J_{n+(2m+1)k}),$$

are colinear as are the points with Jacobsthal-Lucas coordinates

$$(j_n, j_{n+k}), (j_{n+2k}, j_{n+3k}), (j_{n+4k}, j_{n+5k}), \dots, (j_{n+2mk}, j_{n+(2m+1)k}).$$

Proof. The equation of the line through the first two points with Jacobsthal coordinates is

$$y = \frac{J_{n+3k} - J_{n+k}}{J_{n+2k} - J_n}(x - J_n) + J_{n+k}.$$

For $x = J_{n+2mk}$ with $m \geq 4$ we have $y = \frac{J_{n+3k} - J_{n+k}}{J_{n+2k} - J_n}(J_{n+2mk} - J_n) + J_{n+k}$. Using the Binet form, this can be written as

$$y = \left(\frac{2^{n+3k} - 2^{n+k}}{2^{n+2k} - 2^n}\right)\left(\frac{2^{n+2mk} - 2^n}{3}\right) + \left(\frac{2^{n+k} - (-1)^{n+k}}{3}\right)$$

which simplifies to $y = \frac{2^{n+(2m+1)k} - (-1)^{n+k}}{3}$. Since $n+k$ and $n+(2m+1)k$ are congruent mod 2, $y = J_{n+(2m+1)k}$. Thus all the points in the set are colinear. The proof for the set of points with Jacobsthal-Lucas coordinates is similar. □

As these points are colinear they form a degenerate polygon which will have area $A = 0$.

3. Area of m-sided polygons with Fibonacci type vertices

Problem B-1167 in the Fibonacci Quarterly [13] asked to find the area of a polygon with vertices $(F_1, F_2), (F_3, F_4), \dots, (F_{n-1}, F_n)$. In this section, we explore a more general approach to find the area of an m-sided polygons whose vertices have Fibonacci type coordinates

$$(f_n, f_{n+k}),\ (f_{n+2k}, f_{n+3k}),\ (f_{n+4k}, f_{n+5k}), \dots, (f_{n+(m-2)k}, f_{n+(m-1)k}).$$

As before we start with a general case and use the result to find the area of m-sided polygons with other Fibonacci type sequences as coordinates of the vertices. Using the surveyor's formula [3], the area of polygons with vertex coordinates $P_1(x_1, y_1)$, $P_2(x_2, y_2), \dots, P_n(x_n, y_n)$ is

$$A = \frac{1}{2}\left|\sum_{i=1}^{n-1} x_i y_{i+1} + x_n y_1 - \sum_{i=1}^{n-1} x_{i+1} y_i - x_1 y_n\right|. \tag{3.1}$$

Let $f(n)$ be given by (2.1). Using equation (3.1), the area of an m-sided polygon can be expressed as

$$\begin{aligned} A =& \frac{1}{2}\Big(f(n)f(n+3k) + f(n+2k)f(n+5k) + \dots \\ &+ f(n+(2m-4)k)f(n+(2m-1)k) \\ &+ f(n+(2m-2)k)f(n+k) \\ &- f(n+2k)f(n+k) - f(n+4k)f(n+3k) - \dots \\ &- f(n+(2m-2)k)f(n+(2m-3)k) - f(n)f(n+(2m-1)k)\Big). \end{aligned}$$

Reordering the terms, we have

$$\begin{aligned} A =& \frac{1}{2}\Big([f(n)f(n+3k) - f(n+2k)f(n+k)] \\ &+ [f(n+2k)f(n+5k) - f(n+4k)f(n+3k)] + \dots \\ &+ [f(n+(2m-4)k)f(n+(2m-1)k) - f(n+(2m-2)k)f(n+(2m-3)k)] \\ &+ [f(n+(2m-2)k)f(n+k) - f(n)f(n+(2m-1)k)]\Big). \end{aligned} \tag{3.2}$$

Note that the first $(m-1)$ pairs have the form $f(t)f(t+3k) - f(t+k)f(t+2k)$. We evaluate these terms using equation (2.1),

$$\begin{aligned} f(t)f(t+3k) - f(t+k)f(t+2k) &= \left(ar^t + \frac{b(-1)^{t+1}}{r^t}\right)\left(ar^{t+3k} + \frac{b(-1)^{t+k+1}}{r^{t+3k}}\right) \\ &\quad - \left(ar^{t+k} + \frac{b(-1)^{t+k+1}}{r^{t+k}}\right)\left(ar^{t+2k} + \frac{b(-1)^{t+1}}{r^{t+2k}}\right) \\ &= ab(-1)^{t+1}\left[\frac{(-1)^k}{r^{3k}} + r^{3k} - \frac{1}{r^k} - \frac{(-1)^k}{r^k}\right]. \end{aligned}$$

If k is odd, we have

$$ab(-1)^{t+1}\left[\frac{-1}{r^{3k}} + r^{3k} - \frac{1}{r^k} + \frac{1}{r^k}\right] = ab(-1)^{t+1}\left(r^k + \frac{1}{r^k}\right)\left(r^{2k} - \frac{1}{r^{2k}}\right).$$

For even k, we have

$$ab(-1)^{t+1}\left[\frac{1}{r^{3k}} + r^{3k} - \frac{1}{r^k} - \frac{1}{r^k}\right] = ab(-1)^{t+1}\left(r^k - \frac{1}{r^k}\right)\left(r^{2k} - \frac{1}{r^{2k}}\right).$$

There will be $m-1$ of these terms. Note that t has the same parity as n.

The m^{th} term of the area equation (3.2) is

$$\begin{aligned} &f(n+(2m-2)k)f(n+k) - f(n+(2m-1)k)f(n) \\ &\quad = \left(ar^{n+(2m-2)k} + \frac{b(-1)^{n+1}}{r^{n+(2m-2)k}}\right)\left(ar^{n+k} + \frac{b(-1)^{n+k+1}}{r^{n+k}}\right) \\ &\quad - \left(ar^{n+(2m-1)k} + \frac{b(-1)^{n+k+1}}{r^{n+(2m-1)k}}\right)\left(ar^n \frac{b(-1)^{n+1}}{r^n}\right) \\ &\quad = ab(-1)^n\left[(-1)^{k+1}r^{(2m-3)k} - \frac{1}{r^{(2m-3)k}} + r^{(2m-1)k} - \frac{(-1)^{k+1}}{r^{(2m-1)k}}\right]. \end{aligned}$$

If k is odd, we have

$$ab(-1)^n\left[r^{(2m-3)k}-\frac{1}{r^{(2m-3)k}}+r^{(2m-1)k}-\frac{1}{r^{(2m-1)k}}\right]$$
$$=ab(-1)^n\left(r^k+\frac{1}{r^k}\right)\left(r^{(2m-2)k}-\frac{1}{r^{(2m-2)k}}\right).$$

If k is even, we have

$$ab(-1)^n\left[-r^{(2m-3)k}-\frac{1}{r^{(2m-3)k}}+r^{(2m-1)k}+\frac{1}{r^{(2m-1)k}}\right]$$
$$=ab(-1)^n\left(r^k-\frac{1}{r^k}\right)\left(r^{(2m-2)k}-\frac{1}{r^{(2m-2)k}}\right).$$

Therefore, using $(-1)^{t+1}=(-1)^{n+1}$, the formula for the area of a polygon with m sides and vertices $(f(n),f(n+k)),(f(n+2k),f(n+3k)),\cdots,(f(n+(2m-2)k),f(n+(2m-1)k))$ will be as follows. If k is even, we have

$$\frac{1}{2}\left|(-1)^{n+1}\right|\left|ab\left[(m-1)\left(r^k-\frac{1}{r^k}\right)\left(r^{2k}-\frac{1}{r^{2k}}\right)-\left(r^k-\frac{1}{r^k}\right)\left(r^{(2m-2)k}-\frac{1}{r^{(2m-2)k}}\right)\right]\right|. \tag{3.3}$$

If k is odd, the result is

$$\frac{1}{2}\left|(-1)^{n+1}\right|\left|ab\left[(m-1)\left(r^k+\frac{1}{r^k}\right)\left(r^{2k}-\frac{1}{r^{2k}}\right)-\left(r^k+\frac{1}{r^k}\right)\left(r^{(2m-2)k}-\frac{1}{r^{(2m-2)k}}\right)\right]\right|. \tag{3.4}$$

Theorem 3.1. *The area for any m-sided polygon with Fibonacci coordinates*

$$(F_n,F_{n+k}),\ (F_{n+2k},F_{n+3k}),\ (F_{n+4k},F_{n+5k}),\ldots,(F_{n+(m-2)k},F_{n+(m-1)k})$$

is given by

$$\frac{1}{2}\left|(m-1)F_kF_{2k}-F_kF_{(2m-2)k}\right|.$$

The area for any m-sided polygon with Lucas coordinates

$$(L_n,L_{n+k}),\ (L_{n+2k},L_{n+3k}),\ (L_{n+4k},L_{n+5k}),\ldots,(L_{n+(m-2)k},L_{n+(m-1)k})$$

is given by

$$\frac{5}{2}\left|(m-1)F_kF_{2k}-F_kF_{(2m-2)k}\right|.$$

The proof follows directly from equations (2.3), (2.4), (3.3) and (3.4).

Theorem 3.2. *The area for any m-sided polygon with Pell coordinates*

$$(P_n,P_{n+k}),\ (P_{n+2k},P_{n+3k}),\ (P_{n+4k},P_{n+5k}),\ldots,(P_{n+(m-2)k},P_{n+(m-1)k})$$

is given by

$$\frac{1}{2}\left|(m-1)P_kP_{2k}-P_kP_{(2m-2)k}\right|.$$

The area for any m-sided polygon with Pell-Lucas coordinates

$$(Q_n,Q_{n+k}),\ (Q_{n+2k},Q_{n+3k}),\ (Q_{n+4k},Q_{n+5k}),\ldots,(Q_{n+(m-2)k},Q_{n+(m-1)k})$$

is given by

$$4\left|(m-1)P_kP_{2k}-P_kP_{(2m-2)k}\right|.$$

The proof follows directly from equations (2.5), (2.6), (3.3) and (3.4).

Theorem 3.3. *The area for any m-sided polygon with generalized Fibonacci coordinates*

$$(G_n, G_{n+k}),\ (G_{n+2k}, G_{n+3k}),\ (G_{n+4k}, G_{n+5k}), \dots, (G_{n+(m-2)k}, G_{n+(m-1)k})$$

is given by

$$\frac{s^2+st-t^2}{2}\left|(m-1)F_kF_{2k} - F_kF_{(2m-2)k}\right|.$$

The proof follows directly from equations (2.3), (2.4), (3.3) and (3.4).

4. Triangles with Polygonal Number Co-ordinates

The general polygonal numbers are defined by the formula [6]

$$P_n^{(r)} = \frac{n(n(r-2)-(r-4))}{2}, \text{ for } r \geq 3.$$

Theorem 4.1. *The area of any triangle with polygonal number vertices (P_n, P_{n+k}), (P_{n+2k}, P_{n+3k}) and (P_{n+4k}, P_{n+5k}) is*

$$A = 4(r-2)^2k^4. \tag{4.1}$$

Proof. Analogous to the proof of the Fibonacci case, the area is the absolute value of the determinant

$$A = \frac{1}{2}\begin{vmatrix} P_{n+2k}-P_n & P_{n+3k}-P_{n+k} \\ P_{n+4k}-P_n & P_{n+5k}-P_{n+k} \end{vmatrix}$$

which upon expansion yields the following.

$$\begin{aligned} A &= \frac{1}{2}\left((P_{n+2k}-P_n)(P_{n+5k}-P_{n+k}) - (P_{n+4k}-P_n)(P_{n+3k}-P_{n+k})\right) \\ &= \frac{1}{2}[(k[(r-2)(2n+2k)-(r-4)])(2k[(r-2)(2n+6k)-(r-4)]) \\ &\quad - (k[(r-2)(2n+4k)-(r-4)])(2k[(r-2)(2n+4k)-(r-4)])] \\ &= \frac{1}{2}[(2k^2[(r-2)^2(4n^2+16nk+12k^2)+(r-4)^2-(r-4)(r-2)(4n+8k)]) \\ &\quad - (2k^2[(r-2)^2(4n^2+16nk+16k^2)+(r-4)^2-(r-4)(r-2)(4n+8k)])] \\ &= -\frac{8k^4(r-2)^2}{2}. \end{aligned} \tag{4.2}$$

The area of the triangle is therefore $A = 4(r-2)^2k^4$. □

5. Area of m-sided polygons with Polygonal Number Co-ordinates

The following table summarizes results for the area of triangles, quadrilaterals, pentagons, hexagons and heptagons using polygonal vertices in the given pattern.

m	Triangular	Square	Pentagonal	Hexagonal	Heptagonal
3	$4k^4$	$16k^4$	$36k^4$	$64k^4$	$100k^4$
4	$16k^4$	$64k^4$	$144k^4$	$256k^4$	$400k^4$
5	$40k^4$	$160k^4$	$360k^4$	$640k^4$	$1000k^4$
6	$80k^4$	$320k^4$	$720k^4$	$1280k^4$	$2000k^4$
7	$140k^4$	$560k^4$	$1260k^4$	$2240k^4$	$3500k^4$

Note that the coefficients of the m-sided polygon is 4 times the sequence $\{1, 4, 10, 20, 35, ...\}$ which can be written as $\dfrac{m(m-1)(m-2)}{6}$ and is the general term for the tetrahedral (or triangle pyramid) number sequence having OEIS number A000292 [14]. This leads to a conjecture which is answered by

Theorem 5.1. *The area of any m-sided polygon ($m \geq 3$)with polygonal number vertices*

$$(P_n, P_{n+k}), (P_{n+2k}, P_{n+3k}), \ldots, (P_{n+(2m-2)k}, P_{n+(2m-1)k})$$

is $\dfrac{4m(m-1)(m-2)(r-2)^2k^4}{6}$.

Proof. For the general case, it is convenient to re-write (3.1) as

$$\frac{1}{2}\left|\sum_{i=1}^{m-1} x_i y_{i+1} + x_m y_1 - \sum_{i=1}^{m-1} x_{i+1} y_i - x_1 y_m\right|. \tag{5.1}$$

The proof is based on induction on m. The statement has been shown to be true for $m = 3$. Assume that the area of a polygon with $m = t$ is $\dfrac{4t(t-1)(t-2)(r-2)^2k^4}{6}$. Note that the area of the $t+1$ sided polygons will be the area of the t sided polygon plus the area of the triangle with coordinates $(P_n,\ P_{n+k})$, $(P_{n+(2t-2)k},\ P_{n+(2t-1)k})$ and$(P_{2t},\ P_{n+(2t+1)k})$ which is

$$A = \frac{1}{2}\begin{vmatrix} P_{n+(2t-2)k} - P_n & P_{n+(2t-1)k} - P_{n+k} \\ P_{n+(2t)k} - P_n & P_{n+(2t+1)k} - P_{n+k} \end{vmatrix}.$$

On expansion, this yields

$$\begin{aligned}
A &= \frac{1}{2}\left((P_{n+(2t-2)k} - P_n)(P_{n+(2t+1)k} - P_{n+k}) - (P_{n+(2t-1)k} - P_{n+k})(P_{n+(2t-1)k} - P_{n+k})\right) \\
&= \frac{1}{2}[(t-1)k[2(r-2)((t-1)k+n) - (r-4)]kt[2(r-2)((t+1)k+n) - (r-4)] \\
&\qquad -k(t-1)[2(r-2)(tk+n) - (r-4)]kt[2(r-2)(tk+n) - (r-4)] \\
&= \frac{1}{2}k^2t(t-1)[[2(r-2)((t-1)k+n) - (r-4)][2(r-2)((t+1)k+n) - (r-4)] \\
&\qquad -[2(r-2)(tk+n) - (r-4)][2(r-2)(tk+n) - (r-4)] \\
&= \frac{1}{2}k^2t(t-1)(-4(r-2)^2k^2) = \frac{-4(r-2)^2k^4(t-1)(3t)}{6}.
\end{aligned}$$

The absolute value yields

$$\frac{4(r-2)^2k^4(t-1)(3t)}{6}.$$

Adding this area to the t-sided polygon gives the area of the $t+1$-sided polygon as

$$\frac{4(t+1)(t)(t-1)(r-2)^2k^4}{6}$$

as required. □

6. Concluding Comments

Whereas the second order sequences seem to lend themselves to obtainable formulas for polygonal areas, the third order sequences do not appear to be as obliging. For example, three popular third order sequences yield the following areas of triangle with the indicated sequential coordinates.

k	Tribonacci	Perrin	Padovan
1	3	$\frac{9}{2}$	0
2	64	$\frac{47}{2}$	1
3	$849 = 3(283)$	$\frac{31}{9}$	$15 = 3 \cdot 5$
4	$23360 = 5 \cdot 64 \cdot 73$	$\frac{298}{2} = 149$	$44 = 4 \cdot 11$
5	$509729 = 11 \cdot 149 \cdot 311$	$\frac{1629}{2} = \frac{9 \cdot 181}{2}$	$95 = 5 \cdot 19$
6	$10049160 = 3 \cdot 5 \cdot 8 \cdot 11 \cdot 23 \cdot 331$	$\frac{9640}{2} = \frac{5 \cdot 8 \cdot 241}{2} = 4820$	$810 = 2 \cdot 5 \cdot 81$

Area of Triangles with $n = 1$ and the indicated value for k.

Comparing the Perrin numbers with the Padovan numbers yields no patterns. Nor does a comparison of the Tribonacci numbers with themselves. We leave the investigation of the quadrilateral cases to the interested reader.

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