A-CASSINI SEQUENCES AND THEIR SPECTRUM

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ABSTRACT. We solve a generalization of the familiar non-linear Cassini relation for its linear sequences.

1. A-Cassini Relation and Sequences

We consider the non-linear recurrence

$$g_{n+1}g_{n-1} = g_n^2 + (-1)^n A$$

with non-zero initial values $g_1 = a$, $g_2 = b$, and given A. This is a generalization of the familiar Cassini relation for the standard Fibonacci sequence when $A = 1$. We are interested in A, a, and b so that the sequences (called A-Cassini sequences) satisfying this non-linear relation are integer valued. The methods are similar to those in [1] but are more closely aligned to Fibonacci sequences and polynomials.

Theorem 1.1. *Let* $\mu = \frac{b^2-a^2+A}{ab}$. *Then for* $n \geq 3$, $\frac{g_n - g_{n-2}}{g_{n-1}} = \mu$.

Proof. The proof is by induction on n. For $n = 3$, $g_3 = \frac{b^2+A}{a}$ so $\frac{g_3-g_1}{g_2} = \frac{b^2-a^2+A}{ab}$. Now, assume the result is true for $n \geq 3$. We will show that $\frac{g_{n+1}-g_{n-1}}{g_n} = \frac{b^2-a^2+A}{ab}$. Using the induction hypothesis, let $\delta = \frac{g_{n+1}-g_{n-1}}{g_n} - \mu = \frac{g_{n+1}-g_{n-1}}{g_n} - \frac{g_n-g_{n-2}}{g_{n-1}}$. Finding a common denominator and using the non-linear recurrence twice, we get

$$\begin{aligned}\delta &= \frac{g_{n+1}g_{n-1} - g_{n-1}^2 - g_n^2 + g_n g_{n-2}}{g_n g_{n-1}} \\ &= \frac{g_n g_{n-2} - g_{n-1}^2 + g_{n+1}g_{n-1} - g_n^2}{g_n g_{n-1}} \\ &= \frac{(-1)^{n-1}A + (-1)^n A}{g_n g_{n-1}} = 0.\end{aligned}$$

□

The next result follows immediately from Theorem 1.1 and well-known results on linear recurrences.

Corollary 1.2. *For initial values* $g_1 = a$ *and* $g_2 = b$, $g_n = \mu g_{n-1} + g_{n-2}$ *is a second order linear sequence solution to the non-linear A-Cassini relation* $g_{n+1}g_{n-1} = g_n^2 + (-1)^n A$. *The growth of this sequence is* $\lim_{n\to\infty} \frac{g_n}{g_{n-1}} = \sigma$ *where* σ *is a root of* $x^2 - \mu x - 1$

A polynomial in x, y, ... and their inverses $\frac{1}{x}$, $\frac{1}{y}$, ... is called a Laurent polynomial.

Corollary 1.3. *For indeterminates* a *and* b *the sequence* $g_n = \mu g_{n-1} + g_{n-2}$ *is a Laurent polynomial in* a *and* b.

Proof. As shown above, g_3 has denominator a. Since μ has denominator ab, the result follows by an easy induction that g_{n+1} has denominator $a^{n-2}b^{n-3}$ for $n \geq 3$. □

If $a = b = 1$, then $\mu = A$; this yields an integer sequence called the standard A-Cassini sequence. If $A = 1$ then this is the Fibonacci sequence 1, 1, 2, 3, ….

We now prove the converse to our Corollary 1.2.

Proposition 1.4. *Suppose that* $g_{n+1} = Mg_n + g_{n-1}$ *with non-zero initial values* $g_1 = a$, $g_2 = b$, *and given* M. *Let* $A = a^2 + Mab - b^2$. *Then, this sequence satisfies the A-Cassini relation* $g_{n+1}g_{n-1} = g_n^2 + (-1)^n A$.

Proof. By the Corollary above, the solution to the A-Cassini relation with $A = a^2 + Mab - b^2$ with initial values $h_1 = a$ and $h_2 = b$ is $h_{n+1} = Mh_n + h_{n-1}$ since $\mu = M$. Thus, $h_n = g_n$ for all n. □

Recall that the Fibonacci polynomials are given by $f_1 = 1$, $f_2 = x$, and $f_n = xf_{n-1} + f_{n-2}$ when $n \geq 3$; they are of degree $n - 1$, see [2]. The first few Fibonacci polynomials are 1, x, $x^2 + 1$, $x(x^2 + 2)$, $x^4 + 3x^2 + 1$.

Theorem 1.5. *Let* $A = a^2 + Mab - b^2$ *for given* a, b, *and* M. *Then, the sequence satisfying the A-Cassini relation* $g_{n+1}g_{n-1} = g_n^2 + (-1)^n A$ *is determined from the Fibonacci polynomial sequence* $\{f_n | \ n \geq 1\}$ *as*

$$g_{n+2} = bf_{n+1}(M) + af_n(M), \ n \geq 1.$$

Proof. The base cases are:

$$\begin{aligned} & bf_2(M) + af_1 = bM + a = g_3 \\ \text{and} \quad & bf_3(M) + af_2(M) = b(M^2 + 1) + aM = bM^2 + aM + b \\ & \quad = M(bM + a) + b = Mg_3 + g_2 = g_4. \end{aligned}$$

By the induction hypothesis and $g_{n+1} = Mg_n + g_{n-1}$, we obtain

$$\begin{aligned} g_{n+2} &= M(bf_n(M) + af_{n-1}(M)) + bf_{n-1}(M) + af_{n-2}(M) \\ &= b(Mf_n(M) + f_{n-1}(M)) + a(Mf_{n-1}(M) + f_{n-2}(M)) \\ &= bf_{n+1}(M) + af_n(M). \end{aligned}$$

Therefore, by induction, the result is true for all $n \geq 1$. □

2. Spectrum

We want to determine for a given integer M the possible integer values of A so that $M = \mu$ for some integers a and b. We call this the A-Cassini spectrum of M denoted ASpec(M).

Let M be an integer and $A = 1$. Then for $a = 1$ and $b = M$ we have $\mu = M$. Therefore, $1 \in$ ASpec(M) for any integer M.

Let $M = 1$. In this case, Cassini sequences are generalized Fibonacci sequences. If $A = 1$, the equation $\mu = M$ becomes $a^2 + ab - b^2 = 1$. The values $a = b = 1$ satisfy this equation. This gives the Fibonacci sequence. So, $1 \in$ ASpec(1). If $A = 5$, we obtain the Lucas sequence as an example. Thus, $5 \in$ ASpec(1). However, $A = 2$ is not possible. The solution of the equation $b^2 - ab - a^2 + 2 = 0$ is $b = \frac{1}{2}(a \pm \sqrt{5a^2 - 8})$; $5a^2 - 8 = c^2$ for some integer c is impossible, considering the equation mod 5. Hence, $2 \notin$ ASpec(1). Similarly, $A = 3$ is impossible. However, for $A = 4$, we get the doubled Fibonacci sequence. By similar arguments, any $A = \pm 2 \mod 5$ is impossible. When $A = 11$, we obtain $a = 1$ and $b = 4$ which is the Cassini-Fibonacci sequence 1, 4, 5, 9, 14, 23, ... So, $11 \in \text{ASpec}(1)$.

In general, the equation $5a^2 - 4A = c^2$ or $\frac{c+\sqrt{5}a}{2}\frac{c-\sqrt{5}a}{2} = -A$ (that is $-A$ is a norm from the ring $\mathbf{Z}[\frac{1+\sqrt{5}}{2}]$).

For a given integer A, the possible integers M for which $b^2 - a^2 + A = Mab$ has integer solutions a and b is denoted MSpec(A).

Let $A = 1$ and M be any integer. The equation $\mu = M$ becomes $b^2 - a^2 + 1 = Mab$. An integer solution to this equation is $a = 1$ and $b = M$. Therefore, MSpec(1) contains all integers.

For $A = 5$, consider $b^2 - a^2 + 5 = Mab$. It is easy to see that -5, -4, -1, 0, 1, 4, $5 \in \text{MSpec}(5)$.

For a given (M, A) with $A \in \text{ASpec}(M)$ the set of integers (a, b) which yield integer Cassini sequences is denoted PSpec(M, A), called the parameter spectrum.

If $M = 1$ and $A = 1$, what are all integer solutions to $b^2 - ab - a^2 = -1$. This is a familiar Pell's equation. Complete the square to get $(b - a/2)^2 - 5a^2/4 = -1$; so all solutions can be determined using the odd powers k of the fundamental unit $u = \frac{1+\sqrt{5}}{2}$ via $\frac{(2b-a)+\sqrt{5}a}{2} = u^k$.

References

[1] R. C. Alperin, *Integer sequences generated by* $x_{n+1} = (x_n^2 + A)/x_{n-1}$, The Fibonacci Quarterly, **49.4** (2011), 362–365

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