POLYNOMIAL EXTENSIONS OF THE LUCAS AND GINSBURG IDENTITIES REVISITED: ADDITIONAL DIVIDENDS I

THOMAS KOSHY

Abstract. In [11], we extended the fascinating identity

$$g_{n+k}^3 - (-1)^k l_k g_n^3 + (-1)^k g_{n-k}^3 = \begin{cases} f_k f_{2k} g_{3n} & \text{if } g_n = f_n \\ (x^2+4) f_k f_{2k} g_{3n} & \text{if } g_n = l_n, \end{cases}$$

to Jacobsthal, Vieta, and Chebyshev polynomial families [10]. We now extract from this identity additional Fibonacci, Lucas, Jacobsthal, Vieta, and Chebyshev dividends.

1. INTRODUCTION

In [11], we introduced the *extended gibonacci polynomials* $g_n(x)$ using the recurrence $g_{n+2}(x) = a(x)g_{n+1}(x) + b(x)g_n(x)$, where x is an arbitrary complex variable; $a(x), b(x), g_0(x)$, and $g_1(x)$ are arbitrary complex polynomials; and $n \geq 0$. We then presented Fibonacci, Lucas, Pell, Pell-Lucas, Jacobsthal, Jacobsthal-Lucas, Vieta, Vieta-Lucas, and Chebyshev polynomials of both types as subfamilies of the extended gibonacci family; they are denoted by $f_n(x)$, $l_n(x)$, $p_n(x)$, $q_n(x)$, $J_n(x)$, $j_n(x)$, $V_n(x)$, $v_n(x)$, $T_n(x)$, and $U_n(x)$, respectively [1, 4, 5, 6, 7, 8, 9, 12, 11, 15].

These subfamilies are closely linked by the relationships in Table 1, where $i = \sqrt{-1}$ [6, 12, 15, 16].

$J_n(x) = x^{(n-1)/2} f_n(1/\sqrt{x})$	$j_n(x) = x^{n/2} l_n(1/\sqrt{x})$
$V_n(x) = i^{n-1} f_n(-ix)$	$v_n(x) = i^n l_n(-ix)$
$V_n(x) = U_{n-1}(x/2)$	$v_n(x) = 2T_n(x/2)$
$xV_n(x^2+2) = f_{2n}(x)$	$xv_n(x^2+2) = l_{2n}(x)$
$J_{2n}(x) = x^{n-1} V_n\left(\frac{2x+1}{x}\right)$	$j_{2n}(x) = x^n v_n\left(\frac{2x+1}{x}\right)$

Table 1: Relationships Among the Gibonacci Subfamilies

The nth Fibonacci, Lucas, Pell, Pell-Lucas, Jacobsthal, and Jacobsthal-Lucas numbers are given by $F_n = f_n(1)$, $L_n = l_n(1)$, $P_n = p_n(1)$, $2Q_n = q_n(1)$, $J_n = J_n(2)$, and $j_n = j_n(2)$, respectively.

In the interest of brevity and convenience, we omit the argument in the functional notation, when there is *no* ambiguity; so g_n will mean $g_n(x)$. Again, for brevity, we let $g_n = f_n$ or l_n; $b_n = p_n$ or q_n; $c_n = J_n(x)$ or $j_n(x)$; $d_n = V_n(x)$ or $v_n(x)$; and $e_n = T_n(x)$ or $U_n(x)$; and correspondingly, we let $G_n = F_n$ or L_n; $B_n = P_n$ or Q_n; $C_n = J_n$ or j_n; and $d_n = V_n$ or v_n. We also omit a lot of basic algebra.

2. ADDITIONAL DIVIDENDS

In [10], we established the identity

$$g_{n+k}^3 - (-1)^k l_k g_n^3 + (-1)^k g_{n-k}^3 = \begin{cases} f_k f_{2k} g_{3n} & \text{if } g_n = f_n \\ \Delta^2 f_k f_{2k} g_{3n} & \text{if } g_n = l_n, \end{cases} \tag{2.1}$$

where $\Delta^2 = x^2 + 4$.

The next two theorems are direct consequences of this identity, and form the cornerstone of the discourse.

Theorem 2.1.

$$\begin{aligned} f_k f_{2k} g_{n+k+1}^3 &= f_{k+1} f_{2k+2} g_{n+k}^3 - (-1)^k (f_k f_{2k} l_{k+1} + f_{k+1} f_{2k+2} l_k) g_n^3 \\ &\quad + \ (-1)^k f_{k+1} f_{2k+2} g_{n-k}^3 + (-1)^k f_k f_{2k} g_{n-k-1}^3. \end{aligned} \tag{2.2}$$

Proof. Suppose $g_n = l_n$. It then follows from identity (2.1) that

$$l_{n+k}^3 - (-1)^k l_k l_n^3 + (-1)^k l_{n-k}^3 = \Delta^2 f_k f_{2k} l_{3n} \tag{2.3}$$

$$l_{n+k+1}^3 + (-1)^k l_{k+1} l_n^3 - (-1)^k l_{n-k-1}^3 = \Delta^2 f_{k+1} f_{2k+2} l_{3n}. \tag{2.4}$$

Multiplying equation (2.3) with $f_{k+1} f_{2k+2}$ and equation (2.4) with $f_k f_{2k}$, we get

$$f_{k+1} f_{2k+2} \left[l_{n+k}^3 - (-1)^k l_k l_n^3 + (-1)^k l_{n-k}^3 \right] = \Delta^2 f_k f_{k+1} f_{2k} f_{2k+2} l_{3n}; \tag{2.5}$$

$$f_k f_{2k} \left[l_{n+k+1}^3 + (-1)^k l_{k+1} l_n^3 - (-1)^k l_{n-k-1}^3 \right] = \Delta^2 f_k f_{k+1} f_{2k} f_{2k+2} l_{3n}, \tag{2.6}$$

respectively.

Equating the two left sides yields

$$f_{k+1} f_{2k+2} \left[l_{n+k}^3 - (-1)^k l_k l_n^3 + (-1)^k l_{n-k}^3 \right] = f_k f_{2k} \left[l_{n+k+1}^3 + (-1)^k l_{k+1} l_n^3 - (-1)^k l_{n-k-1}^3 \right].$$

Regrouping the terms, we get the desired identity when $g_n = l_n$.

Identity (2.2) with $g_n = f_n$ follows by a similar derivation. □

Since $f_1 = 1$, $f_2 = x = l_1$, and $f_4 = x^3 + 2x = x l_2$, the next result follows from equation (2.2) by letting $k = 1$.

Corollary 2.2.

$$g_{n+2}^3 = x(x^2+2) g_{n+1}^3 + (x^2+1)(x^2+2) g_n^3 - x(x^2+2) g_{n-1}^3 - g_{n-2}^3. \tag{2.7}$$

□

Identity (2.7) implies that the cubes of Fibonacci and Lucas polynomials satisfy the fourth-order recurrence

$$a_{n+2} = x(x^2+2) a_{n+1} + (x^2+1)(x^2+2) a_n - x(x^2+2) a_{n-1} - a_{n-2},$$

where $a_n = a_n(x)$ and $n \geq 2$. When $a_n = f_n^3, a_0 = 0, a_1 = 1, a_2 = x^3$, and $a_3 = (x^2+1)^3$; and when $a_n = l_n^3, a_0 = 8, a_1 = x^3, a_2 = (x^2+2)^3$, and $a_3 = (x^3+3x)^3$.

Consequently, the cubes of Pell and Pell-Lucas polynomials satisfy the recurrence

$$b_{n+2} = 4x(2x^2+1) b_{n+1} + 2(2x^2+1)(4x^2+1) b_n - 4x(2x^2+1) b_{n-1} - b_{n-2},$$

where $b_n = b_n(x)$ and $n \geq 2$. When $b_n = p_n^3$, $b_0 = 0$, $b_1 = 1$, $b_2 = 8x^3$, and $b_3 = (4x^2+1)^3$; and when $b_n = q_n^3$, $b_0 = 8$, $b_1 = 8x^3$, $b_2 = (4x^2+2)^3$, and $b_3 = (8x^3+6x)^3$.

It also follows from Corollary 2.2 that

$$\begin{aligned} G_{n+2}^3 &= 3G_{n+1}^3 + 6G_n^3 - 3G_{n-1}^3 - G_{n-2}^3 \\ b_{n+2}^3 &= 4x(2x^2+1) b_{n+1}^3 + 2(2x^2+1)(4x^2+1) b_n^3 - 4x(2x^2+1) b_{n-1}^3 - b_{n-2}^3 \\ B_{n+2}^3 &= 12B_{n+1}^3 + 30B_n^3 - 12B_{n-1}^3 - B_{n-2}^3 \end{aligned} \tag{2.8}$$

$$g_{n+2}^3 + g_{n-2}^3 \equiv 0 \pmod{x^2+2}$$

$$B_{n+2}^3 + B_{n-2}^3 \equiv 6B_n^2 \pmod{12}.$$

Identity (2.8) with $G_n = F_n$ appears in [17].

The next theorem paves the way for extracting additional dividends from (2.1).

Theorem 2.3.

$$f_k f_{2k} g^3_{n+k+1} + f_{k+1} f_{2k+2} g^3_{n+k} + (-1)^k (f_k f_{2k} l_{k+1} - f_{k+1} f_{2k+2} l_k) g^3_n \tag{2.9}$$
$$+ (-1)^k f_{k+1} f_{2k+2} g^3_{n-k} - (-1)^k f_k f_{2k} g^3_{n-k-1} = \begin{cases} 2 f_k f_{k+1} f_{2k} f_{2k+2} g_{3n} & \text{if } g_n = f_n \\ 2\Delta^2 f_k f_{k+1} f_{2k} f_{2k+2} g_{3n} & \text{if } g_n = l_n. \end{cases}$$

Proof. Adding equations (2.5) and (2.6), we get identity (2.9) when $g_n = l_n$. A similar technique yields the identity with $g_n = f_n$. □

The following result is a direct consequence of this theorem.

Corollary 2.4.

$$g^3_{n+2} + x(x^2+2) g^3_{n+1} + (x^2-1)(x^2+2) g^3_n$$
$$- x(x^2+2) g^3_{n-1} + g^3_{n-2} = \begin{cases} 2x^2(x^2+2) g_{3n} & \text{if } g_n = f_n \\ 2x^2\Delta^2(x^2+2) g_{3n} & \text{if } g_n = l_n. \end{cases} \tag{2.10}$$

It follows from equation (2.10) that

$$G^3_{n+2} + 3G^3_{n+1} - 3G^3_{n-1} + G^3_{n-2} = \begin{cases} 6G_{3n} & \text{if } G_n = F_n \\ 30G_{3n} & \text{if } G_n = L_n; \end{cases} \tag{2.11}$$

$$b^3_{n+2} + 4x(2x^2+1) b^3_{n+1} + 2(2x^2+1)(4x^2-1) b^3_n$$
$$- 4x(2x^2+1) b^3_{n-1} + b^3_{n-2} = \begin{cases} 16x^2(2x^2+1) b_{3n} & \text{if } b_n = p_n \\ 64x^2(x^2+1)(2x^2+1) b_{3n} & \text{if } b_n = q_n; \end{cases}$$

$$B^3_{n+2} + 12B^3_{n+1} + 18B^3_n - 12B^3_{n-1} + B^3_{n-2} = \begin{cases} 48B_{3n} & \text{if } B_n = P_n \\ 96B_{3n} & \text{if } B_n = Q_n. \end{cases}$$

Identities (2.8) and (2.11) together imply that

$$G^3_{n+1} + G^3_n - G^3_{n-1} = \begin{cases} G_{3n} & \text{if } G_n = F_n \\ 5G_{3n} & \text{if } G_n = L_n. \end{cases} \tag{2.12}$$

The Fibonacci version of this identity appears in Dickson's classic work, *History of the Theory of Numbers*, Vol. 1 [2, 8, 10, 11, 13, 14]; and Long discovered its Lucas counterpart [10, 13].

Identity (2.11), coupled with (2.12), gives yet another interesting cubic identity:

$$G^3_{n+2} + 2G^3_{n+1} - G^3_n - 2G^3_{n-1} + G^3_{n-2} = \begin{cases} 5G_{3n} & \text{if } G_n = F_n \\ 25G_{3n} & \text{if } G_n = L_n. \end{cases} \tag{2.13}$$

It follows by identities (2.11), (2.12), and (2.13) that

$$G^3_{n+2} - 3G^3_n + G^3_{n-2} = \begin{cases} 3G_{3n} & \text{if } G_n = F_n \\ 15G_{3n} & \text{if } G_n = L_n. \end{cases} \tag{2.14}$$

This result, with $G_n = F_n$, is Ginsburg's identity [3, 10, 11].

3. ADDITIONAL IMPLICATIONS

Next, we investigate the implications of identities (2.7) and (2.10) to the Jacobsthal, Vieta, and Chebyshev subfamilies. To this end, the relationships in Table 1 will come in handy.

3.1. **Jacobsthal Byproducts.** Since $J_n(x) = x^{(n-1)/2} f_n(u)$, it follows from identity (2.7) that

$$f_{n+2}^3 = \frac{2x+1}{x\sqrt{x}} f_{n+1}^3 + \frac{(x+1)(2x+1)}{x^2} f_n^3 - \frac{2x+1}{x\sqrt{x}} f_{n-1}^3 - f_{n-2}^3,$$

where $u = 1/\sqrt{x}$ and $f_n = f_n(u)$. Multiplying this equation with $x^{3(n+1)/2}$ yields

$$J_{n+2}^3(x) = (2x+1)J_{n+1}^3(x) + x(x+1)(2x+1)J_n^3(x) - x^3(2x+1)J_{n-1}^3(x) - x^6 J_{n-2}^3(x).$$

Likewise, since $j_n(x) = x^{n/2} l_n(u)$, it follows from identity (2.7) that

$$j_{n+2}^3(x) = (2x+1)j_{n+1}^3(x) + x(x+1)(2x+1)j_n^3(x) - x^3(2x+1)j_{n-1}^3(x) - x^6 j_{n-2}^3(x).$$

Combining these two equations, we get the cubic identity

$$c_{n+2}^3 = (2x+1)c_{n+1}^3 + x(x+1)(2x+1)c_n^3 - x^3(2x+1)c_{n-1}^3 - x^6 c_{n-2}^3. \tag{3.1}$$

Consequently, the cubes of Jacobsthal and Jacobsthal-Lucas polynomials satisfy the recurrence

$$z_{n+2} = (2x+1)z_{n+1} + x(x+1)(2x+1)z_n - x^3(2x+1)z_{n-1} - x^6 z_{n-2},$$

where $z_n = z_n(x) = c_n^3$; when $z_n = J_n^3(x)$, $z_0 = 0$, $z_1 = 1 = z_2$, and $z_3 = (x+1)^3$; and when $z_n = j_n^3(x)$, $z_0 = 8$, $z_1 = 1$, $z_2 = (2x+1)^3$, and $z_3 = (3x+1)^3$.

Identity (3.1) implies that

$$C_{n+2}^3 = 5C_{n+1}^3 + 30C_n^3 - 40C_{n-1}^3 - 64C_{n-2}^3 \tag{3.2}$$

$$c_{n+2}^3 + x^6 c_{n-2}^3 \equiv 0 \pmod{2x+1}$$

$$C_{n+2}^3 \equiv C_{n-2}^3 \pmod{5}.$$

Identity (2.10) also has Jacobsthal consequences. Replacing x with $1/\sqrt{x}$ and multiplying both sides of the resulting equation with $x^{3(n+1)/2}$ yields

$$\begin{aligned} &c_{n+2}^3 + (2x+1)c_{n+1}^3 - x(x-1)(2x+1)c_n^3 \\ &\quad - x^3(2x+1)c_{n-1}^3 + x^6 c_{n-2}^3 = \begin{cases} 2(2x+1)c_{3n} & \text{if } c_n = J_n(x) \\ 2(2x+1)(4x+1)c_{3n} & \text{if } c_n = j_n(x). \end{cases} \end{aligned} \tag{3.3}$$

Consequently,

$$C_{n+2}^3 + 5C_{n+1}^3 - 10C_n^3 - 40C_{n-1}^3 + 64C_{n-2}^3 = \begin{cases} 10C_{3n} & \text{if } C_n = J_n \\ 90C_{3n} & \text{if } C_n = j_n. \end{cases} \tag{3.4}$$

Identity (3.4), coupled with (3.2), implies

$$C_{n+1}^3 + 2C_n^3 - 8C_{n-1}^3 = \begin{cases} C_{3n} & \text{if } C_n = J_n \\ 9C_{3n} & \text{if } C_n = j_n, \end{cases} \tag{3.5}$$

as in [11].

It follows by identities (3.4) and (3.5) that

$$C_{n+2}^3 + 4C_{n+1}^3 - 12C_n^3 - 32C_{n-1}^3 + 64C_{n-2}^3 = \begin{cases} 9C_{3n} & \text{if } C_n = J_n \\ 81C_{3n} & \text{if } C_n = j_n. \end{cases}$$

Consequently, $C_{n+2}^3 \equiv C_{3n} \pmod 4$.

Next, we pursue the implications of Corollaries 2.2 and 2.4 to the Vieta family.

3.2. **Vieta Byproducts.** Since $V_n(x) = i^{n-1} f_n(-ix)$, replace x with $-ix$ in identity (2.7) and then multiply the resulting equation with $i^{3(n+1)}$. This yields

$$V_{n+2}^3 = x(x^2-2)V_{n+1}^3 - (x^2-1)(x^2-2)V_n^3 + x(x^2-2)V_{n-1}^3 - V_{n-2}^3.$$

Using the link $v_n(x) = i^n l_n(-ix)$, it follows likewise from (2.7) that

$$v_{n+2}^3 = x(x^2-2)v_{n+1}^3 - (x^2-1)(x^2-2)v_n^3 + x(x^2-2)v_{n-1}^3 - v_{n-2}^3.$$

Thus,

$$d_{n+2}^3 = x(x^2-2)d_{n+1}^3 - (x^2-1)(x^2-2)d_n^3 + x(x^2-2)d_{n-1}^3 - d_{n-2}^3. \tag{3.6}$$

Identity (2.10) similarly yields

$$d_{n+2}^3 + Axd_{n+1}^3 - A(x^2+1)d_n^3 + Axd_{n-1}^3 + d_{n-2}^3 = \begin{cases} 2Ax^2 d_{3n} & \text{if } d_n = V_n(x) \\ 2Ax^2(x^2-4)d_{3n} & \text{if } d_n = v_n(x), \end{cases} \tag{3.7}$$

where $A = x^2 - 2$.

3.2.1. *Fibonacci and Lucas Implications.* Identities (3.6) and (3.7) have Fibonacci and Lucas implications. Using the relationships $xV_n(x^2+2) = f_{2n}$ and $xv_n(x^2+2) = l_{2n}$, we have $xd_n(x^2+2) = g_{2n}$. It then follows from identity (3.6) that

$$\begin{aligned} g_{2n+4}^3 &= (x^2+2)(x^4+4x^2+2)g_{2n+2}^3 - (x^2+1)(x^2+3)(x^4+4x^2+2)g_{2n}^3 \\ &\quad + (x^2+2)(x^4+4x^2+2)g_{2n-2}^3 - g_{2n-4}^3. \end{aligned} \tag{3.8}$$

This implies

$$\begin{aligned} G_{2n+4}^3 &= 21G_{2n+2}^3 - 56G_{2n}^3 + 21G_{2n-2}^3 - G_{2n-4}^3 \\ b_{2n+4}^3 &= 4(2x^2+1)(8x^4+8x^2+1)b_{2n+2}^3 - 2(4x^2+1)(4x^2+3)(8x^4+8x^2+1)b_{2n}^3 \\ &\quad + 4(2x^2+1)(8x^4+8x^2+1)b_{2n-2}^3 - b_{2n-4}^3 \\ B_{2n+4}^3 &= 204B_{2n+2}^3 - 1190B_{2n}^3 + 204B_{2n-2}^3 - B_{2n-4}^3 \\ g_{2n+4}^3 + g_{2n-4}^3 &\equiv 0 \pmod{x^4+4x^2+2}. \end{aligned}$$

With $E = x^4 + 4x^2 + 2$, identity (3.7) yields

$$\begin{aligned} g_{2n+4}^3 &+ E(x^2+2)g_{2n+2}^3 - E(E+3)g_{2n}^3 \\ &+ E(x^2+2)g_{2n-2}^3 + g_{2n-4}^3 = \begin{cases} 2Ex^2(x^2+2)^2 g_{6n} & \text{if } g_n = f_n \\ 2Ex^4(x^2+2)^2(x^2+4)g_{6n} & \text{if } g_n = l_n. \end{cases} \end{aligned} \tag{3.9}$$

Consequently,

$$G_{2n+4}^3 + 21G_{2n+2}^3 - 70G_{2n}^3 + 21G_{2n-2}^3 + G_{2n-4}^3 = \begin{cases} 126G_{6n} & \text{if } G_n = F_n \\ 630G_{6n} & \text{if } G_n = L_n; \end{cases}$$

$$\begin{aligned} &b_{2n+4}^3 + 4F(2x^2+1)b_{2n+2}^3 - 2F(16x^4+16x^2+5)b_{2n}^3 \\ &\quad + 4F(2x^2+1)b_{2n-2}^3 + b_{2n-4}^3 = \begin{cases} 64Fx^2(2x^2+1)^2 b_{6n} & \text{if } b_n = p_n \\ 256Fx^2(x^2+1)(2x^2+1)^2 b_{6n} & \text{if } b_n = q_n; \end{cases} \end{aligned}$$

$$B_{2n+4}^3 + 204B_{2n+2}^3 - 1258B_{2n}^3 + 204B_{2n-2}^3 + B_{2n-4}^3 = \begin{cases} 9,792B_{6n} & \text{if } B_n = P_n \\ 19,584B_{6n} & \text{if } B_n = Q_n, \end{cases}$$

where $F = 8x^4 + 8x^2 + 1$.

3.2.2. *Jacobsthal Implications.* As can be predicted, identities (3.6) and (3.7), together with the relationships $J_{2n}(x) = x^{n-1}V_n(u)$ and $j_{2n(}x) = x^n v_n(u)$, have Jacobsthal consequences, where $u = \dfrac{2x+1}{x}$. To begin with, it follows from identity (3.6) that

$$\begin{aligned} x^4 d_{n+2}^3 &= x(2x+1)(2x^2+4x+1)d_{n+1}^3 - (2x^2+4x+1)(3x^2+4x+1)d_n^3 \\ &\quad + x(2x+1)(2x^2+4x+1)d_{n-1}^3 - x^4 d_{n-2}^3, \end{aligned} \tag{3.10}$$

where $d_n = d_n(u)$.

Using the above Vieta-Jacobsthal links, this yields the Jacobsthal identity

$$\begin{aligned} c_{2n+4}^3 &= (2x+1)(2x^2+4x+1)c_{2n+2}^3 - x^2(2x^2+4x+1)(3x^2+4x+1)c_{2n}^3 \\ &\quad + x^6(2x+1)(2x^2+4x+1)c_{2n-2}^3 - x^{12}c_{2n-4}^3. \end{aligned}$$

This implies

$$C_{2n+4}^3 = 85C_{2n+2}^3 - 1428C_{2n}^3 + 5440C_{2n-2}^3 - 4096C_{2n-4}^3. \tag{3.11}$$

A similar derivation from identity (3.7) yields

$$\begin{aligned} &x^4 d_{n+2}^3 + Ax(2x+1)d_{n+1}^3 \\ &- A(5x^2+4x+1)d_n^3 + Ax(2x+1)d_{n-1}^3 + x^4 d_{n-2}^3 = \begin{cases} 2A(2x+1)^2 d_{3n} & \text{if } d_n = V_n \\ 2A(4x+1)(2x+1)^2 d_{3n} & \text{if } d_n = v_n, \end{cases} \end{aligned}$$

where $d_n = d_n(u)$ and $A = 2x^2 + 4x + 1$.

Consequently,

$$\begin{aligned} &c_{2n+4}^3 + A(2x+1)c_{2n+2}^3 - Ax^2(5x^2+4x+1)c_{2n}^3 \\ &\quad + Ax^6(2x+1)c_{2n-2}^3 + x^{12}c_{2n-4}^3 = \begin{cases} 2A(2x+1)^2 c_{6n} & \text{if } c_n = J_n(x) \\ 2A(4x+1)(2x+1)^2 c_{6n} & \text{if } c_n = j_n(x). \end{cases} \end{aligned}$$

In particular, we have

$$C_{2n+4}^3 + 85C_{2n+2}^3 - 1972C_{2n}^3 + 5440C_{2n-2}^3 - 4096C_{2n-4}^3 = \begin{cases} 850C_{6n} & \text{if } C_n = J_n \\ 7,650C_{6n} & \text{if } C_n = j_n. \end{cases} \tag{3.12}$$

Finally, we present the consequences of Corollaries 2.2 and 2.4 to the Chebyshhev family.

3.3. **Chebyshev Byproducts.** Since $U_{n-1}(x) = V_n(2x)$ and $2T_n(x) = v_n(2x)$, it follows from identities (3.6) and (3.7) that

$$e_{n+2}^3 = 4x(2x^2-1)e_{n+1}^3 - 2(2x^2-1)(4x^2-1)e_n^3 + 4x(2x^2-1)e_{n-1}^3 - e_{n-2}^3.$$

Likewise,

$$e_{n+2}^3 + 4Bxe_{n+1}^3 - 2B(4x^2+1)e_n^3 + 4Bxe_{n-1}^3 + e_{n-2}^3 = \begin{cases} 16Bx^2 e_{3n+2} & \text{if } e_n = U_n(x) \\ 64Bx^2(x^2-1)e_{3n} & \text{if } e_n = T_n(x), \end{cases}$$

where $B = 2x^2 - 1$.

4. Acknowledgment

The author thanks Z. Gao for his computational help, and to the reviewer for helpful suggestions and comments that enhanced the exposition of the original version.

References

[1] M. Bicknell, *A primer for the Fibonacci numbers: Part VII*, The Fibonacci Quarterly, **8.5** (1970), 407–420.
[2] L. E. Dickson, *History of the Theory of Numbers*, Vol I, Chelsea, New York, 1966.
[3] J. Ginsburg, *A relationship between cubes of Fibonacci numbers*, Scripts Mathematica, 1953, 242.
[4] A. F. Horadam, *Jacobsthal representation numbers*, The Fibonacci Quarterly, **34.1** (1996), 40–54.
[5] A. F. Horadam, *Jacobsthal representation polynomials*, The Fibonacci Quarterly, **35.2** (1997), 137–148.
[6] A. F. Horadam, *Vieta polynomials*, The Fibonacci Quarterly, **40.3** (2002), 223–232.
[7] A. F. Horadam and Bro. J. M. Mahon, *Pell and Pell-Lucas polynomials*, The Fibonacci Quarterly, **23.1** (1985), 7–20.
[8] T. Koshy, *Fibonacci and Lucas Numbers with Applications*, Vol. I, Wiley, New York, 2018.
[9] T. Koshy, *Pell and Pell-Lucas Numbers with Applications*, Springer, New York, 2014.
[10] T. Koshy, *Polynomial extensions of the Lucas and Ginsburg identities*, The Fibonacci Quarterly, **52.2** (2014), 141–147.
[11] T. Koshy, *Polynomial extensions of the Lucas and Ginsburg identities: Revisited*, The Fibonacci Quarterly, **55.2** (2017), 147–151.
[12] T. Koshy, *Vieta polynomials and their close relatives*, The Fibonacci Quarterly, **54.2** (2016), 141–148.
[13] C. T. Long, *Discovering Fibonacci identities*, The Fibonacci Quarterly, **24.2** (1986), 160–166.
[14] R. S. Melham, *Families of identities involving sums of powers of the Fibonacci and Lucas numbers*, The Fibonacci Quarterly, **37.4** (1999), 315–319.
[15] N. Robbins, *Vieta's triangular array and a related family of polynomials*, International Journal of Mathematics and Mathematical Sciences, **14** (1991), 239–244.
[16] A. G. Shannon and A. F. Horadam, *Some relationships among Vieta, Morgan-Voyce and Jacobsthal polynomials*, Applications of Fibonacci Numbers (ed. F. T. Howard), Vol. 8, Kluwer, Dordrecht, 1999, 307–323.
[17] D. Zeitlin and F. D. Parker, *Problem H-14*, The Fibonacci Quarterly, **1.2** (1963), 54.

MSC2010: 11B37, 11B39, 11B50

Department of Mathematics, Framingham State University, Framingham, Massachusetts 01701, USA
E-mail address: tkoshy@emeriti.framingham.edu