

MARKOV EQUATION WITH FIBONACCI COMPONENTS

FLORIAN LUCA AND ANITHA SRINIVASAN

ABSTRACT. We find all triples of Fibonacci numbers $(x, y, z) = (F_i, F_j, F_n)$ satisfying the Markov equation $x^2 + y^2 + z^2 = 3xyz$.

1. INTRODUCTION

The Markov equation is

$$x^2 + y^2 + z^2 = 3xyz \tag{1.1}$$

in positive integers $x \leq y \leq z$. A Markov number is any positive integer that is a component of some solution to the Markov equation. Here is the sequence of Markov numbers

1, 2, 5, 13, 29, 34, 89, 169, 194, 233, 610, 985, 1325, ...

(sequence A002559 in [2]) appearing as coordinates of the Markov triples

$(1, 1, 1), (1, 1, 2), (1, 2, 5), (1, 5, 13), (2, 5, 29), (1, 13, 34), (1, 34, 89), (2, 29, 169),$
 $(5, 13, 194), (1, 89, 233), (5, 29, 433), (1, 233, 610), (2, 169, 985), (13, 34, 1325), \dots$

The Fibonacci sequence $\{F_m\}_{m \geq 0}$ starts as $F_0 = 0, F_1 = 1$ and satisfies the recurrence $F_{m+2} = F_{m+1} + F_m$ for all $m \geq 0$. Its first few terms are

0, 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, 233, 377, 610, 987, ...

(sequence A000045 in [2]). One notices that the Markov numbers seem to contain the odd indexed Fibonacci numbers. That this is so is a consequence of the formula

$$1 + F_{2k-1}^2 + F_{2k+1}^2 = 3F_{2k-1}F_{2k+1} \tag{1.2}$$

valid for all positive integers k . We ask whether there are other solutions $(x, y, z) = (F_i, F_j, F_n)$ to the Markov equation other than the ones arising from (1.2). Here is our main result.

Theorem 1.1. *If $(x, y, z) = (F_i, F_j, F_n)$ is a solution in positive integers to the Markov equation, then it is of the form shown in (1.2).*

Similar problems have been investigated before. For example, it is known that the set of three integers $\{F_{2n}, F_{2n+2}, F_{2n+4}\}$ has the property that the product of any two of them plus one is a square since

$$F_{2n}F_{2n+2} + 1 = F_{2n+1}^2, \quad F_{2n+2}F_{2n+4} + 1 = F_{2n+3}^2, \quad F_{2n}F_{2n+4} + 1 = F_{2n+2}^2.$$

In [1], it is shown that if $\{F_{2n}, F_{2n+2}, F_k\}$ has the property that the product of any two plus one is a square, then $k \in \{2n - 2, 2n + 4\}$ except for $n = 2$ when also $k = 1$ is possible.

F. L. was supported in part by grant CPRR160325161141 and an A-rated scientist award from the NRF of South Africa and by grant no. 17-02804S of the Czech Granting Agency.

2. PRELIMINARY RESULTS

Lemma 2.1. *If $(a, b, c) \neq (1, 1, 1)$ satisfies the Markov equation and $a \leq b \leq c$, then $3ab < b + c$.*

Proof. If $b = c$, then

$$a^2 = 3ab^2 - 2b^2 = b^2(3a - 2)$$

and $b^2(3a - 2) > b^2 \geq a^2$ if $a > 1$, which is a contradiction. Thus, $a = 1$ leading to $a = b$, showing that $a = b = c = 1$, which is excluded. Thus, $c > b$, therefore $3abc = a^2 + b^2 + c^2 < 3c^2$, which gives $ab < c$ and hence $a^2 < c$ (as $a \leq b$). Next, from $c(b - 1) \geq (b + 1)(b - 1) = b^2 - 1$, we have $bc - c \geq b^2$. Therefore, $a^2 + b^2 < c + b^2 \leq bc$. It follows that $\frac{a^2 + b^2}{c} < b$, so that $3ab = \frac{a^2 + b^2}{c} + c < b + c$. \square

Recall that

$$F_k = \frac{\alpha^k - \beta^k}{\alpha - \beta} \quad \text{where} \quad (\alpha, \beta) = \left(\frac{1 + \sqrt{5}}{2}, \frac{1 - \sqrt{5}}{2} \right) \quad (2.1)$$

for all $k \geq 0$. In particular,

$$\alpha^{k-2} \leq F_k \leq \alpha^{k-1} \quad \text{holds for all} \quad k \geq 1. \quad (2.2)$$

3. THE PROOF OF THEOREM 1.1

Assume $x = F_i$, $y = F_j$, $z = F_n$ with $x \leq y \leq z$. Since $F_1 = F_2 = 1$, we assume that $2 \leq i \leq j \leq n$. Then,

$$z - 3xy = -\frac{x^2 + y^2}{z}. \quad (3.1)$$

Inserting the values of x , y , z in the left side of (3.1), we get

$$\frac{\alpha^n}{\sqrt{5}} - \frac{3}{5}\alpha^{i+j} = -\frac{F_i^2 + F_j^2}{F_n} + \frac{\beta^n}{\sqrt{5}} - \frac{3}{5}(\alpha^i\beta^j + \alpha^j\beta^i - \beta^{i+j}).$$

Taking absolute values and using

$$\begin{aligned} \frac{F_i^2 + F_j^2}{F_n} &\leq \frac{2F_j^2}{F_n} \leq 2\alpha^{2j-n} \leq 2\alpha^j, \\ \left| \frac{\beta^n}{\sqrt{5}} \right| &\leq \frac{\alpha^{-j}}{\sqrt{5}} < \frac{\alpha^j}{5}, \\ \left| \frac{3}{5}(\alpha^i\beta^j + \alpha^j\beta^i - \beta^{i+j}) \right| &\leq \frac{3}{5}(2\alpha^j + 1) \leq \frac{9\alpha^j}{5}, \end{aligned}$$

we get that

$$\left| \frac{\alpha^n}{\sqrt{5}} - \frac{3}{5}\alpha^{i+j} \right| \leq \alpha^j \left(2 + \frac{1}{5} + \frac{9}{5} \right) = 4\alpha^j.$$

Dividing across by $\alpha^{i+j}/\sqrt{5}$ we get

$$\left| \alpha^{n-i-j} - \frac{3}{\sqrt{5}} \right| < \frac{4\sqrt{5}}{\alpha^i}. \quad (3.2)$$

Certainly,

$$1 < \frac{3}{\sqrt{5}} < \alpha$$

and

$$\min_{k \in \mathbb{Z}} \left| \alpha^k - \frac{3}{\sqrt{5}} \right| = \left| \alpha - \frac{3}{\sqrt{5}} \right| > 0.2763,$$

so (3.2) shows that

$$0.2763 < \frac{4\sqrt{5}}{\alpha^i},$$

which gives $\alpha^i < 4\sqrt{5}/0.2763$, or $i \leq 7$. We record what we have proved.

Lemma 3.1. *If $(x, y, z) = (F_i, F_j, F_n)$ satisfies (1.1) with $i \leq j \leq n$, then $i \in \{2, 3, 4, 5, 6, 7\}$. Of these, only $i = 2, 3, 5, 7$ lead to $F_i = 1, 2, 5, 13$, which are Markov numbers.*

Lemma 3.2. *If $(x, y, z) = (F_i, F_j, F_n)$ satisfies (1.1) with $i \leq j \leq n$, then n is odd, $j = n - 2$ and $F_i = 1$.*

Proof. We shall treat the case $F_i = 1$ at the end.

Assume that $F_i = 2$. Then, we have

$$4 + F_j^2 + F_n^2 = 6F_jF_n$$

or

$$4 + F_j^2 = F_n(6F_j - F_n),$$

which gives $F_n < 6F_j$. From Lemma 2.1, we have $6F_j < F_j + F_n$, which gives $5F_j < F_n$. Hence, we have

$$5F_j < F_n < 6F_j.$$

This is false because

$$F_{j+3} = F_{j+2} + F_{j+1} = 2F_{j+1} + F_j = 3F_j + 2F_{j-1} < 5F_j$$

(since $j \geq i \geq 3$), while

$$F_{j+4} = F_{j+3} + F_{j+2} = 2F_{j+2} + F_{j+1} = 3F_{j+1} + 2F_j = 5F_j + 3F_{j-1} > 6F_j,$$

since the last inequality is equivalent to $3F_{j-1} > F_j$, which holds because $F_j = F_{j-1} + F_{j-2}$ and $2F_{j-1} > F_{j-2}$. Thus, for $j \geq 3$, the interval $(5F_j, 6F_j)$ does not contain any Fibonacci number. Similarly, in the cases $F_i = 5$ and $F_i = 13$, we get

$$14F_j < F_n < 15F_j,$$

and

$$38F_j < F_n < 39F_j,$$

respectively, which are false as

$$F_{j+5} = 8F_j + 5F_{j-1} < 14F_j \quad \text{and} \quad F_{j+6} = 13F_j + 8F_{j-1} > 15F_j \quad (j \geq 5)$$

and

$$F_{j+8} = 21F_j + 13F_{j-1} < 38F_j \quad \text{and} \quad F_{j+9} = 34F_j + 21F_{j-1} > 39F_j \quad (j \geq 7),$$

so that the intervals $(14F_j, 15F_j)$ and $(38F_j, 39F_j)$ do not contain a Fibonacci number for $j \geq 5$ and $j \geq 7$, respectively.

So we have $F_i = 1$. In this case, we get

$$F_j < F_n < 3F_j,$$

which gives $n = j + 1$ or $n = j + 2$. In the first case, $n \geq 3$ and we have

$$1 + F_{n-1}^2 + F_n^2 = 3F_{n-1}F_n$$

or

$$1 + (F_n - F_{n-1})^2 = F_{n-1}F_n,$$

or

$$1 + F_{n-2}^2 = F_{n-1}F_n,$$

which is possible only in the case $n = 3$. But, this gives the solution $(1, 1, 2) = (F_1, F_1, F_3)$, so it satisfies the conclusion of the theorem. Finally, when $j = n - 2$, we get

$$1 + F_{n-2}^2 + F_n^2 = 3F_{n-2}F_n,$$

which implies that n is odd (otherwise one of $n - 2$ or n is a multiple of 4, so one of F_{n-2} or F_n is divisible by 3, in which case the above relation is impossible modulo 3). \square

ACKNOWLEDGEMENTS

We thank the referee for a careful reading of the manuscript.

REFERENCES

- [1] B. He, F. Luca, and A. Togbé, *Diophantine triples of Fibonacci numbers*, Acta Arithmetica, **175** (2016), 57–70.
- [2] N. J. A. Sloane, *On-Line Encyclopedia of Integer Sequences*, <https://oeis.org/>

MSC2010: 11D99, 11B39

SCHOOL OF MATHEMATICS, UNIVERSITY OF THE WITWATERSRAND, PRIVATE BAG X3, WITS 2050, SOUTH AFRICA;

CENTRO DE CIENCIAS MATEMÁTICAS, UNAM, MORELIA, MEXICO

DEPARTMENT OF MATHEMATICS, FACULTY OF SCIENCES, UNIVERSITY OF OSTRAVA, 30 DUBNA 22, 701 03 OSTRAVA 1, CZECH REPUBLIC

E-mail address: florian.luca@wits.ac.za

SAINT LOUIS UNIVERSITY, MADRID CAMPUS, AVENIDA DEL VALLE 34, 28003 MADRID, SPAIN

E-mail address: rsrinivasan.anitha@gmail.com