ITERATIONS OF A MODIFIED SISYPHUS FUNCTION

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ABSTRACT. The *Sisyphus function* is defined and, after a slight modification, we determine the smallest nonnegative integer n requiring a specified number of iterations of the function that must be applied to n until the sequence generated by the iterations of this modified function becomes stable or cycles.

1. INTRODUCTION

The Sisyphus function originated in [7], an article by John Schram in 1987. The function very likely reappeared later that year in Michael Ecker's REC newsletter[1] communicated to him by Martin Gardner (listed as a contributing editor of the REC). The function appeared again in 1993 [8] where a slightly different but equivalent definition was used. Ecker later discussed the function in the first tribute book to Martin Gardner in 1999 [3]. In all references it is treated as a recreational novelty. We add a little more mathematical substance with this analysis. The function, despite its seemingly simple definition, provides a surprisingly rich problem provided a slight modification of the definition is made.

The function is, of course, named after Sisyphus, the deceitful king from Greek mythology who frequently killed travelers and guests [6]. After tricking a number of the Greek gods and cheating death, Sisyphus was punished by Zeus to carry or push a large boulder to the top of a large hill, but the boulder was cursed to always be dropped or released before reaching the top and subsequently roll back down the hill. It is common to describe a task as *Sisyphean* if it is pointless or entails performing a procedure over and over. Schram named this function with the latter interpretation in mind.

Schram's initial presentation of the function in [7] was descriptive. We provide a more formal representation of his definition so it will be clear as to the modifications that must be made.

Let $\Sigma = \{0, 1, \ldots, 9\}$ be the alphabet consisting of the decimal digits and let Σ^* be the set of all finite strings over Σ. The *Sisyphus function on digit strings*, denoted $\mathcal{S}_{str}$, is the function $\mathcal{S}_{str} : \Sigma^* \to \Sigma^*$ that counts the number of even digits, the number of odd digits, the total number of digits, and concatenates these numbers (from left to right) as one single string of digits. For example, $\mathcal{S}_{str}(123468) = 426$ because the argument consists of four even digits, two odd digits, and a total of six digits.

Schram's interest was in the observation that any digit string iterated through the function would always converge to the same digit string. For example, iterating 1335557777 through the function gives

$$1335557777 \to 01010 \to 325 \to 123 \to 123 \to \cdots , \tag{1.1}$$

indicating that 123 is a stable point of the function.

[1] Copies of the issues of the newsletter are difficult to find. The author has not been able to secure editions of the newsletter before 1989. Zerger in [8] mentions the Sisyphus function and cites an REC newsletter from 1987: [1].

Schram's article listed some examples and challenged the readers to devise a proof that all digit strings iterate to 123. An outline is provided in [8]. After a slight redefinition of the function, we provide a formal proof that all nonnegative integers converge to the stable point.

2. The Modified Sisyphus Function

One template for analyzing the iteration dynamics of discrete functions is given in Guy's *Unsolved Problems in Number Theory* [5, problem E34]. The function used by Guy takes any positive integer and maps it to the sum of the squares of its decimal digits. Guy considers only the subset of the domain whose elements iterate to the stable point (that is, the *happy numbers*). Among the deluge of questions proposed, he asks, for a fixed nonnegative integer k, to determine the least nonnegative integer that converges to the stable point in exactly k iterations.[2] The purpose of this paper is to do the same for the Sisyphus function.

To that end, we introduce a modified function so that the domain and range are in terms of integers rather than digit strings. This modified definition will allow us to make a technical adjustment to avoid the use of a leading zero appearing in (1.1). Although we can certainly use the convention to drop leading zeros, a canonical definition can be adopted by writing the total number of digits (which must be nonzero) of the integer on the left rather than on the right, guaranteeing no leading zeros.

Define the *modified Sisyphus function on nonnegative integers* $\mathcal{S}_{int} : \mathbb{N} \to \mathbb{Z}^+$ (using $\mathbb{N}$ to represent the nonnegative integers and $\mathbb{Z}^+$ to represent the positive integers) as taking any nonnegative integer n and writing an integer whose digits are constructed by concatenating from left to right the total number of digits of n, the total number of even digits of n, and the total number of odd digits of n. For example, $\mathcal{S}_{int}(123\,589) = 624$ (we use half-spaces rather than commas to group the digits by threes starting on the right). For brevity, we will refer to this function simply as the *Sisyphus function* and use $\mathcal{S}$ to represent it.

A few more definitions are presented before proceeding. For $n \in \mathbb{N}$, the *iteration sequence* of n (the *orbit*) is the sequence of the iterates of $\mathcal{S}$ generated by n (the *seed*). For example, the iteration sequence of $123\,589$ through $\mathcal{S}$ is

$$123\,589 \to 624 \to 330 \to 312 \to 312 \to \cdots, \tag{2.1}$$

indicating that 312 is a stable point for the new function (we will see later that 312 is the only fixed point and that there are no nontrivial cycles). Moreover, the *height* of the iteration sequence generated by n, denoted $\mathcal{S}^{\#}(n)$, is the number of iterations of $\mathcal{S}$ that must be applied to n until the iteration sequence reaches either a fixed point or a cycle. That is, $\mathcal{S}^{\#}(n) = \min\left\{j \in \mathbb{N} : \mathcal{S}^{(j)}(n) = 312\right\}$. For example, $\mathcal{S}^{\#}(123\,589) = 3$ since $123\,589$ requires three iterations of $\mathcal{S}$ until reaching the fixed point.

Using $\mathcal{S}$, we can naturally partition $\mathbb{N}$ by defining, for each $k \in \mathbb{N}$, the kth *iterated set* of $\mathcal{S}$, denoted I_k, as

$$I_k := \{n \in \mathbb{N} : \mathcal{S}^{\#}(n) = k\}$$

to be the set of nonnegative integers whose elements reach the fixed point in exactly k iterations. For example, $123\,589 \in I_3$. We let

$$\tau_k := \min(I_k) \tag{2.2}$$

denote the minimum value that maps to the fixed point in exactly k iterations. From our previous example, we see that an upper bound for τ_3 is $123\,589$ (the actual number is given in Proposition 3.1). We call $(\tau_0, \tau_1, \tau_2, \ldots)$ the *minimal height sequence associated to* $\mathcal{S}$.

[2]Answers to Guy's questions for problem E34 are supplied in [4].

Taking inspiration from Guy, we determine the values of τ_k through $\mathcal{S}$ for as many values of k that are possible, but a few more preliminary definitions are needed first.

For $n \in \mathbb{N}$, we denote by δ_n, the number of decimal digits in n. More explicitly, $\delta_0 = 1$ and, for $n > 0$, $\delta_n = 1 + \lfloor \log_{10} n \rfloor$, where $\lfloor \cdot \rfloor$ is the floor function. Given two integers $m, n \in \mathbb{N}$, the *integer concatenation* of m and n, denoted using the bracketed vector $\langle m, n \rangle$, is $\langle m, n \rangle := m10^{\delta_n} + n$. For the concatenation of many integers $n_1, n_2, \ldots, n_j \in \mathbb{N}$, the *bracketed vector notation* for the (left-to-right) integer concatenation is defined recursively as $\langle n_1 \rangle = n_1$ and $\langle n_1, \ldots, n_i \rangle = \langle \langle n_1, \ldots n_{i-1} \rangle, n_i \rangle$ for $1 < i \leq j$. In many cases, the bracket vector notation is not necessary, but we use it for convenience to help distinguish the digital components of the image of $\mathcal{S}$.

For $n \in \mathbb{N}$, define ε_n and ω_n as the number of, respectively, even and odd decimal digits of n (so $\delta_n = \varepsilon_n + \omega_n$). Thus the Sisyphus function can be written as $\mathcal{S}(n) = \langle \delta_n, \varepsilon_n, \omega_n \rangle$. For example, $\mathcal{S}(123\,589) = \langle 6, 2, 4 \rangle = 624$.

The following lemma describes the iteration structure of $\mathcal{S}$, if n is limited to five or fewer digits and is cited in the proof of all four theorems in this paper.

Lemma 2.1. *Let $n \in \mathbb{N}$ be a seed consisting of five or fewer digits. Then the iteration sequence of n converges to 312 in three or fewer iterations of $\mathcal{S}$. More specifically,*

(I) *if n consists of one or five digits, then $\mathcal{S}^{\#}(n) = 2$,*
(II) *if n consists of two or four digits, then $\mathcal{S}^{\#}(n)$ is 2 or 3, and*
(III) *if n consists of three digits, then $\mathcal{S}^{\#}(n) \leq 2$.*

Proof. If n consists of five digits, then $\mathcal{S}(n) = \langle 5, \varepsilon_n, \omega_n \rangle$ where $\varepsilon_n + \omega_n = 5$. Then the parity of ε_n and ω_n must be different. So $n \to \langle 5, \varepsilon_n, \omega_n \rangle \to 312$ implying $\mathcal{S}^{\#}(n) = 2$. Similar arguments for n consisting of a single digit or three digits will give the respective result that $\mathcal{S}^{\#}(n) = 2$ or $\mathcal{S}^{\#}(n) \leq 2$.

If n consists of four digits, then $\mathcal{S}(n) = \langle 4, \varepsilon_n, \omega_n \rangle$, where $\varepsilon_n + \omega_n = 4$. Then the parity of ε_n and ω_n are the same. If both are odd, then $n \to \langle 4, \varepsilon_n, \omega_n \rangle \to 312$. If both are even, then $n \to \langle 4, \varepsilon_n, \omega_n \rangle \to 330 \to 312$. Thus, $\mathcal{S}^{\#}(n)$ is 2 or 3. A similar argument for n consisting of two digits will give the identical result that $\mathcal{S}^{\#}(n)$ is 2 or 3. $\square$

To answer Schram's challenge for our function, we have the following.

Theorem 2.2. *All iteration sequences through $\mathcal{S}$ converge to 312.*

In the proof, we demonstrate a lower bound for which $\mathcal{S}$ is decreasing for all values larger than the lower bound.

Proof. Given n, let $\kappa \equiv \kappa(n) = \delta_n$ be the number of decimal digits in n. Since the largest that both ε_n and ω_n can be is δ_n, then

$$\begin{aligned}
\mathcal{S}(n) &= \langle \delta_n, \varepsilon_n, \omega_n \rangle < \langle \delta_n, \delta_n, \delta_n \rangle = \langle \kappa, \kappa, \kappa \rangle \\
&= \kappa \left(100^{\delta_\kappa} + 10^{\delta_\kappa} + 1 \right) = \kappa \left(100^{1 + \lfloor \log_{10} \kappa \rfloor} + 10^{1 + \lfloor \log_{10} \kappa \rfloor} + 1 \right) \\
&\leq \kappa \left[100 \cdot 100^{\log_{10} \kappa} + 10 \cdot 10^{\log_{10} \kappa} + 1 \right] = 100\kappa^3 + 10\kappa^2 + \kappa.
\end{aligned}$$

It is a straightforward induction to show that $100\kappa^3 + 10\kappa^2 + \kappa < 10^{\kappa - 1}$ for every integer $\kappa \geq 6$. Hence, we have

$$\mathcal{S}(n) < 10^{\kappa - 1} = 10^{\delta_n - 1} = 10^{\lfloor \log_{10} n \rfloor} \leq 10^{\log_{10} n} = n.$$

Thus, $\mathcal{S}$ is decreasing if n consists of 6 or more digits.

This, along with the results of Lemma 2.1, implies the conclusion. $\square$

3. Zero through Four Iterations

Proposition 3.1. *The smallest nonnegative integers of heights 0 to 3 through $\mathcal{S}$ are, respectively, $\tau_0 = 312$, $\tau_1 = 101$, $\tau_2 = 0$, and $\tau_3 = 11$.*

Proof. For τ_1, we want to find the smallest integer that maps to 312. Because it must consist of three digits, we search systematically starting at 100. Since $100 \to 321 \to 312$ and $101 \to 312$, $\tau_1 = 101$. For τ_2, we start with the smallest element in $\mathbb{N}$. Since $0 \to 110 \to 312$, $\tau_2 = 0$. For τ_3, we note that all single digit numbers must be in I_2. Since $10 \to 211 \to 312$ and $11 \to 202 \to 330 \to 312$, $\tau_3 = 11$. □

It is at this point where the problem becomes interesting. The previous strategy does not obviously extend to find τ_4. And because the numbers in the minimal height sequence will be getting very large, we introduce some notation to simplify the representation of these numbers.

The *exponential digit notation* for a positive integer n is $d_1^{p_1} d_2^{p_2} d_3^{p_3} \cdots d_j^{p_j}$ where $d_1 \in \mathbb{N}_{10} \setminus \{0\}, d_2, \ldots, d_j \in \mathbb{N}_{10}, p_1 \in \mathbb{Z}^+$, and $p_2, \ldots, p_j \in \mathbb{N}$, where $\mathbb{N}_{10} := \{0, 1, \ldots, 9\}$ is the set of the first ten nonnegative integers[3]. The notation $d_1^{p_1} d_2^{p_2} d_3^{p_3} \cdots d_j^{p_j}$ indicates that, for n, the digit d_1 is repeated p_1 times, d_2 is repeated p_2 times, and so on, up to d_j repeated p_j times. For convenience, we drop the exponent if it is one and drop the digit if the exponent is zero. For example, $10^4 9^2 7^0 6 = 10\,000\,996$ and, conveniently, $10^2 = 100$.

Let $\mathbb{L} := \mathbb{N} \times \mathbb{N} \setminus \{(0,0)\}$ be the set of lattice points in the plane with nonnegative integer coordinates excluding the origin. For $(a, b) \in \mathbb{L}$, we define $N(a, b)$ to be the smallest nonnegative integer consisting of a even digits and b odd digits. Then

$$N(a,b) = \begin{cases} 0, & \text{if } a = 1 \text{ and } b = 0; \\ 20^{a-1}, & \text{if } a > 1 \text{ and } b = 0; \\ 10^a 1^{b-1}, & \text{otherwise.} \end{cases}$$

Consequently, for any $n \in \mathbb{N}$, the definition implies we must have

$$n \geq N(\varepsilon_n, \omega_n).$$

It is sometimes useful to compare two numbers when written in this format, so we have the following lemma.

Lemma 3.2. *Let $(a_1, b_1), (a_2, b_2) \in \mathbb{L}$, then $N(a_1, b_1) < N(a_2, b_2)$ if and only if*

(I) $a_1 + b_1 < a_2 + b_2$ *or*
(II) $a_1 + b_1 = a_2 + b_2$ *and* $b_2 > b_1 > 0$ *or*
(III) $a_1 + b_1 = a_2 + b_2$, $b_1 > 0$ *and* $b_2 = 0$.

For each case, the conclusions are self evident once the conditions are described.

Proof. Condition (I) implies that the number of digits in $N(a_1, b_1)$ is less than the number of digits in $N(a_2, b_2)$.

Condition (II) implies that both have the same number of digits (with each being 0 or 1 only), but $N(a_2, b_2)$ must have more 1's than $N(a_1, b_1)$. Every occurrence of the digit 1 in $N(a_1, b_1)$ must have 1 in the corresponding position in $N(a_2, b_2)$ because there are more 1's in $N(a_2, b_2)$, then it must be larger than $N(a_1, b_1)$.

Condition (III) implies that $N(a_2, b_2)$ has 2 as a leading digit and $N(a_1, b_1)$ has 1 as a leading digit (and have the same number of digits). □

[3]We used Σ in §1 to represent the set of decimal digits and $\mathbb{N}_{10}$ for the set of integers from 0 to 9. The main difference is that the mathematical operations of addition and multiplication can be performed on $\mathbb{N}_{10}$, but are not formally defined on Σ.

Theorem 3.3. *The smallest nonnegative integer of height 4 through $\mathcal{S}$ is the twenty digit integer $\tau_4 = 10\,000\,000\,000\,111\,111\,111$.*

The outline of this proof is to verify the provided number is of the proper height and ensure that all smaller integers have smaller height.

Proof. Since

$$10\,000\,000\,000\,111\,111\,111 = N(10,10) \to \langle 20,10,10\rangle \to 642 \to 330 \to 312, \tag{3.1}$$

$N(10,10)$ is of height 4.

Suppose $n \in \mathbb{N}$ is such that $n < N(10,10)$. Since

$$N(\varepsilon_n, \omega_n) \leq n < N(10,10),$$

by Lemma 3.2, either (I) $\delta_n = \varepsilon_n + \omega_n < 10 + 10 = 20$ or (II) $\delta_n = 20$ and $\omega_n < 10$. Condition (III) does not apply. For (I), $\delta_n < 20$ implies one or both of ε_n and ω_n consists of a single digit. Hence $\mathcal{S}(n)$ consists of five or fewer digits. If (II) applies, it forces ω_n and ε_n to consist of, respectively, one and two digits and so, $\mathcal{S}(n)$ consists of exactly five digits. For both (I) and (II), $\mathcal{S}(n)$ consists of five or fewer digits.

If $\mathcal{S}(n)$ consists of three or five digits, then Lemma 2.1 implies

$$\mathcal{S}^{\#}(\mathcal{S}(n)) \leq 2$$

and hence, $\mathcal{S}^{\#}(n) \leq 3$.

If $\mathcal{S}(n)$ consists of four digits, then there exists $d, a, b \in \mathbb{N}_{10}$ such that $\mathcal{S}(n) = \langle 1, d, a, b\rangle$ where $a + b = 10 + d$. It is not possible for all variables to be odd, nor is it possible for one to be odd and two to be even. Thus $\mathcal{S}^2(n)$ is 413 or 431. For both cases we have $\mathcal{S}^3(n) = 312$ and so, $\mathcal{S}^{\#}(n) = 3$. □

The following corollary is a direct consequence of Theorem 3.3 and will be used in the proofs of Theorems 4.1 and 5.2.

Corollary 3.4. *Any nonnegative integer consisting of 19 or fewer digits must converge through $\mathcal{S}$ to the stable point in less than four iterations.*

That is, $\delta_n \leq 19$ implies $\mathcal{S}^{\#}(n) \leq 3$.

4. Five Iterations

The next value in the minimal height sequence associated to $\mathcal{S}$ is quite monstrous consisting of over one million digits. Our answer in part comes from one of the terms in (3.1); we seek the smallest second-order preimage of 201 010.

Theorem 4.1. *The smallest nonnegative integer of height 5 through $\mathcal{S}$ is the* 1 100 110 *digit integer $\tau_5 = N(1\,000\,099, 100\,011)$.*

This proof parallels that of the proof of Theorem 3.3, except that the second part has more cases to consider.

Proof. Since

$$N(1\,000\,099, 100\,011) \to \langle 1\,100\,110, 1\,000\,099, 100\,011\rangle \to \langle 20,10,10\rangle \in I_3, \tag{4.1}$$

by (3.1), $N(1\,000\,099, 100\,011) \in I_5$.

Suppose $n < N(1\,000\,099, 100\,011)$. We show that n must reach the stable point in four or fewer iterations of $\mathcal{S}$.

If n is such that $\mathcal{S}(n)$ consists of less than twenty digits, then, by Corollary 3.4, $\mathcal{S}^{\#}(\mathcal{S}(n)) \le 3$ and so, $\mathcal{S}^{\#}(n) \le 4$.

Suppose n is such that $\mathcal{S}(n)$ consists of exactly twenty digits. We know that because n is bounded above by $N(1\,000\,099, 100\,011)$, which consists of $1\,100\,110$ digits, then $\delta_{\delta_n} \le 7$. It cannot be the case that $\delta_{\delta_n} \le 6$, for this would imply $\delta_{\mathcal{S}(n)} \le 18$. So $\delta_{\delta_n} = 7$. Since $\delta_{\mathcal{S}(n)} = 20$, one of δ_{ε_n} or δ_{ω_n} is 7 and the other 6. Without loss of generality, we presume $\delta_{\varepsilon_n} = 7$ and $\delta_{\omega_n} = 6$. These imply $\varepsilon_n \ge 1\,000\,000$ and $\omega_n \ge 100\,000$ and so, $\delta_n \ge 1\,100\,000$. Thus, n is such that $1\,100\,000 \le \delta_n \le 1\,100\,110$.

For each of the three cases below, we show $\varepsilon_{\mathcal{S}(n)} \ge 11$. This would imply that $\omega_{\mathcal{S}(n)} \le 9$ and so, $\mathcal{S}^2(n) = \langle 20, \varepsilon_{\mathcal{S}(n)}, \omega_{\mathcal{S}(n)} \rangle$ must consist of exactly five digits. Thus, Lemma 2.1 implies $\mathcal{S}^{\#}(\mathcal{S}^2(n)) = 2$ and hence, $\mathcal{S}^{\#}(n) = 4$.

Case 1. Suppose $1\,100\,000 \le \delta_n < 1\,100\,100$. Then, there exist $d_1, d_2 \in \mathbb{N}_{10}$ such that $\delta_n = \langle 11\,000, d_2, d_1 \rangle$ and, consequently, there exist $a_1, a_2, b_1, b_2 \in \mathbb{N}_{10}$ such that $\varepsilon_n = \langle 10\,000, a_2, a_1 \rangle$ and $\omega_n = \langle 1\,000, b_2, b_1 \rangle$ with $\langle a_2, a_1 \rangle + \langle b_2, b_1 \rangle = \langle d_2, d_1 \rangle$. Hence,

$$\mathcal{S}(n) = \langle 11\,000, d_2, d_1, 10\,000, a_2, a_1, 1\,000, b_2, b_1 \rangle$$

contains ten zeros without regard to the unknown digits. But one of a_1, b_1, and d_1 must be even (they can't all be odd), indicating that $\varepsilon_{\mathcal{S}(n)} \ge 11$.

Case 2. Suppose $1\,100\,100 \le \delta_n < 1\,100\,110$. Then there exists $d \in \mathbb{N}_{10}$ such that $\delta_n = \langle 110\,010, d \rangle$. There are two subcases to consider. The first subcase is if $\varepsilon_n = \langle 1\,000, c, 0, a \rangle$ and $\omega_n = \langle 100, c', 0, b \rangle$, where $a, b \in \mathbb{N}_{10}$, $c \in \{0, 1\}$, $c' = 1 - c$, and $a + b = d$. Then

$$\mathcal{S}(n) = \langle 110\,010, d, 1\,000, c, 0, a, 100, c', 0, b \rangle$$

contains ten zeros without regard to the unknown digits. Since c and c' must be of different parity, then $\varepsilon_{\mathcal{S}(n)} \ge 11$. The second subcase is if

$$\varepsilon_n = \langle 10\,000, a_2, a_1 \rangle \text{ and } \omega_n = \langle 1\,000, b_2, b_1 \rangle,$$

where $a_1, a_2, b_1, b_2 \in \mathbb{N}_{10}$ and $\langle a_2, a_1 \rangle + \langle b_2, b_1 \rangle = \langle 10, d \rangle$. Then

$$\mathcal{S}(n) = \langle 110\,010, d, 10\,000, a_2, a_1, 1\,000, b_2, b_1 \rangle$$

contains ten zeros without regard to the unknown digits. A similar argument as given in Case 1 will imply $\varepsilon_{\mathcal{S}(n)} \ge 11$.

Case 3. Suppose $\delta_n = 1\,100\,110$. With the condition $\omega_n \ge 100\,000$, we have $\varepsilon_n \le 1\,000\,110$. Because $n < N(1\,000\,099, 100\,011)$ and both of these integers have the same number of digits, we must have that n contains at least $1\,000\,100$ zero digits (that is, $\varepsilon_n \ge 1\,000\,100$). There are two subcases to consider: when $1\,000\,100 < \varepsilon_n < 1\,000\,110$ and when ε_n takes on either of the bounds of the first subcase. The first subcase implies there exists $a \in \mathbb{N}_{10}$ such that $\varepsilon_n = \langle 100\,010, a \rangle$ and $\omega_n = \langle 10\,000, a' \rangle$ with $a' = 10 - a$ so that

$$\mathcal{S}(n) = \langle 1\,100\,110, 100\,010, a, 10\,000, a' \rangle,$$

which implies $\varepsilon_{\mathcal{S}(n)} \ge 11$. The second subcase implies $\varepsilon_n = \langle 10\,001, c, 0 \rangle$ and $\omega_n = \langle 1\,000, c', 0 \rangle$, where $c \in \{0, 1\}$ and $c' = 1 - c$. Then

$$\mathcal{S}(n) = \langle 1\,100\,110, 10\,001, c, 0, 1\,000, c', 0 \rangle$$

and so, $\varepsilon_{\mathcal{S}(n)} \ge 11$.

Thus, all three cases imply that $n \notin I_5$. $\square$

5. Six or More Iterations

Theorem 5.2 gives a recursive definition of all the remaining values in the minimal height sequence of $\mathcal{S}$.

However, before presenting, we need to establish a lemma that is utilized for one case in the proof of the theorem. The purpose of the lemma is to establish a bound on the number of even digits in $\mathcal{S}(n)$ given a bound on the number of even digits of n.

Lemma 5.1. *Suppose $n \in \mathbb{N}$ is such that $\delta_n = 10^{u+v+2}$, where $u, v \in \mathbb{N}$. If*

$$\langle a_{u+1}, \ldots, a_1, b_{v+1}, \ldots, b_1 \rangle \le \varepsilon_n \le \langle a_{u+1}, \ldots, a_1, c_{v+1}, \ldots, c_1 \rangle, \tag{5.1}$$

where $a_1, a_2, \ldots, a_{u+1}, b_1, b_2, \ldots, b_v, c_1, c_2, \ldots, c_v \in \mathbb{N}_{10}$, $b_{v+1} \in \mathbb{E}_{10} := \{0, 2, 4, 6, 8\}$ (the set of nonnegative integers less than 10) with $a_{u+1} > 0$ and $c_{v+1} = 1 + b_{v+1}$, then

$$2u + 2v + 3 \le \varepsilon_{\mathcal{S}(n)} \le 2u + 3v + 4.$$

Proof. The inequality (5.1) implies there exist $d_1, d_2, \ldots, d_v, d_{v+1}, e_1, e_2, \ldots, e_v, e_{v+1} \in \mathbb{N}_{10}$ such that

$$\varepsilon_n = \langle a_{u+1}, \ldots, a_1, d_{v+1}, \ldots, d_1 \rangle \text{ and } \omega_n = \langle a'_{u+1}, \ldots, a'_1, e_{v+1}, \ldots, e_1 \rangle$$

where, for each $j \in \{1, \ldots, u+1\}$, $a'_j = 9 - a_j$ and

$$\langle d_{v+1}, \ldots, d_1 \rangle + \langle e_{v+1}, e_v, \ldots, e_1 \rangle = 10^{v+1}, \tag{5.2}$$

with all choices subject to the constraint

$$\langle b_{v+1}, \ldots, b_1 \rangle \le \langle d_{v+1}, d_v, \ldots, d_1 \rangle \le \langle c_{v+1}, \ldots, c_1 \rangle. \tag{5.3}$$

Thus,

$$\mathcal{S}(n) = \langle 10^{u+v+2}, a_{u+1}, \ldots, a_1, d_{v+1}, \ldots, d_1, a'_{u+1}, \ldots, a'_1, e_{v+1}, \ldots, e_1 \rangle.$$

From (5.2), there exists $i \in \{1, \ldots, v+1\}$ such that $d_i + e_i = 10$. Then the following three conditions are true.

(A) if $i \ne 1$, then

$$d_1 = d_2 = \ldots = d_{i-1} = e_1 = e_2 = \ldots = e_{i-1} = 0,$$

(B) the parity of d_i and e_i are the same, and

(C) if $i \ne v+1$, then, for any j such that $i < j \le v+1$, the parity of d_j and e_j are different.

Counting the number of even digits in $\mathcal{S}(n)$ for each of the three cases gives $\varepsilon_{\mathcal{S}(n)} = 2u + 2v + i + 2 + 2$, if $d_i \in \mathbb{E}^+_{10} := \{2, 4, 6, 8\}$ (the set of positive even integers less than 10) and $\varepsilon_{\mathcal{S}(n)} = 2u + 2v + i + 2$, otherwise. The minimum occurs when $i = 1$ and $d_i \notin \mathbb{E}^+_{10}$. The maximum does not occur when $i = v + 1$ and $d_i \in \mathbb{E}^+_{10}$ because it would violate (5.1), but rather when $i = v$ and $d_i \in \mathbb{E}^+_{10}$. □

Theorem 5.2. *For $k \ge 6$, the smallest nonnegative integer of height k through $\mathcal{S}$ is defined recursively as $\tau_k = N(89^{u_k} 80^{v_k}, 10^{u_k} 20^{v_k})$, where*

$$u_k = 33^{u_{k-1}} 20^{v_{k-1}} - 2 \text{ and } v_k = 13^{v_{k-1}}$$

for $k \ge 7$ with initial conditions $u_6 = 100\,097$ and $v_6 = 266\,604$.

A useful property that will be utilized in the proof and is a direct consequence of the definition of the minimal height value (2.2) is that if $n \in \mathbb{N}$, $k \in \mathbb{Z}^+$, and $\mathcal{S}(n) < \tau_{k-1}$, then $n \notin I_k$.

For reference, the key identities for $k \geq 7$, which can be easily verified, are

$$\begin{aligned}2(u_k + v_k) + 4 &= 6^{u_{k-1}+v_{k-1}+2},\\ 2u_k + 3v_k + 5 &= 66^{u_{k-1}}80^{v_{k-1}}, \text{ and}\\ 3(u_k + v_k) + 7 &= 10^{u_{k-1}+v_{k-1}+2}.\end{aligned}$$

These identities will be used freely within the proof.

Proof. We first establish that $N(89^{u_k}80^{v_k}, 10^{u_k}20^{v_k}) \in I_k$ for each $k \geq 6$.

Starting with the initial case of $k = 6$, three applications of $\mathcal{S}$ gives

$$\begin{aligned}N(89^{u_6}80^{v_6}, 10^{u_6}20^{v_6}) &\to \left\langle 10^{u_6+v_6+2}, 89^{u_6}80^{v_6}, 10^{u_6}20^{v_6}\right\rangle\\ &\to \langle 3(u_6 + v_6) + 7, 2u_6 + 3v_6 + 5, u_6 + 2\rangle \qquad (5.4)\\ &= \langle 1\,100\,110, 1\,000\,011, 100\,099\rangle\\ &\to \langle 20, 10, 10\rangle \in I_3,\end{aligned}$$

utilizing (4.1). Then, $N(89^{u_6}80^{v_6}, 10^{u_6}20^{v_6}) \in I_6$.

For all other cases of $k \geq 7$, two applications of $\mathcal{S}$ give

$$\begin{aligned}N(89^{u_k}80^{v_k}, 10^{u_k}20^{v_k}) &\to \left\langle 10^{u_k+v_k+2}, 89^{u_k}80^{v_k}, 10^{u_k}20^{v_k}\right\rangle\\ &\to \langle 3(u_k + v_k) + 7, 2u_k + 3v_k + 5, u_k + 2\rangle\\ &= \left\langle 10^{u_{k-1}+v_{k-1}+2}, 66^{u_{k-1}}80^{v_{k-1}}, 33^{u_{k-1}}20^{v_{k-1}}\right\rangle.\end{aligned}$$

If $k \neq 8$, then for each j such that $6 < j \leq k - 1$, one application of $\mathcal{S}$ gives

$$\begin{aligned}\left\langle 10^{u_j+v_j+2}, 66^{u_j}80^{v_j}, 33^{u_j}20^{v_j}\right\rangle &\to \langle 3(u_j + v_j) + 7, 2u_j + 3v_j + 5, u_j + 2\rangle\\ &= \left\langle 10^{u_{j-1}+v_{j-1}+2}, 66^{u_{j-1}}80^{v_{j-1}}, 33^{u_{j-1}}20^{v_{j-1}}\right\rangle.\end{aligned}$$

Repeating the previous step until $j = 7$ will produce the last expression in terms of u_6 and v_6. One more application of $\mathcal{S}$ gives

$$\left\langle 10^{u_6+v_6+2}, 66^{u_6}80^{v_6}, 33^{u_6}20^{v_6}\right\rangle \to \langle 3(u_6 + v_6) + 7, 2u_6 + 3v_6 + 5, u_6 + 2\rangle \in I_4$$

via (5.4). Overall, starting at $N(89^{u_k}80^{v_k}, 10^{u_k}20^{v_k})$, $k - 4$ applications of $\mathcal{S}$ were applied. Thus, $\mathcal{S}^{k-4}(N(89^{u_k}80^{v_k}, 10^{u_k}20^{v_k})) \in I_4$ and so,

$$N(89^{u_k}80^{v_k}, 10^{u_k}20^{v_k}) \in I_k.$$

Next, we show that for each $k \geq 6$, $N(89^{u_k}80^{v_k}, 10^{u_k}20^{v_k})$ is the smallest integer in I_k. Take any $n < N(89^{u_k}80^{v_k}, 10^{u_k}20^{v_k})$ and show $n \notin I_k$. There are five cases to consider. For a given k, we use strong induction and presume that

$$\tau_j = N(89^{u_j}80^{v_j}, 10^{u_j}20^{v_j}) \qquad (5.5)$$

for all $6 \leq j < k$ (noting that τ_0 through τ_5 have already been established). For all but one of the following cases, we establish $k = 6$ separately.

Case 1. Suppose n is such that $\delta_n < 10^{u_k+v_k+2}$. Since $\mathcal{S}(n) = \langle \delta_n, \varepsilon_n, \omega_n\rangle$, the number of digits in each imply $\delta_{\mathcal{S}(n)} = \delta_{\delta_n} + \delta_{\varepsilon_n} + \delta_{\omega_n}$. Since neither ε_n nor ω_n can exceed δ_n,

$$\delta_{\mathcal{S}(n)} = \delta_{\delta_n} + \delta_{\varepsilon_n} + \delta_{\omega_n} \leq 3\delta_{\delta_n} \leq 3(u_k + v_k + 2). \qquad (5.6)$$

If $k = 6$, this previous inequality implies

$$\delta_{\mathcal{S}(n)} \leq 3(u_6 + v_6 + 2) = 1\,100\,109 < \delta_{\tau_5}.$$

Thus, $\mathcal{S}(n) < \tau_5$ and so, $\mathcal{S}(n) \notin I_5$ implies $n \notin I_6$. If $k \geq 7$, we need only use (5.5) for $j = k-1$. The inequality (5.6) implies

$$\delta_{\mathcal{S}(n)} \leq 3(u_k + v_k + 2) = 9^{u_{k-1}+v_{k-1}+2} < 10^{u_{k-1}+v_{k-1}+2} = \delta_{\tau_{k-1}}.$$

Thus, $\mathcal{S}(n) < \tau_{k-1}$ implies $n \notin I_k$. An illustration of this case is given in Figure 1. The thick line on the right in the illustration indicates the set of all integers satisfying $\tau_{k-1} < \delta_n < 10^{u_k+v_k+2}$. The thick line on the left is the image set of these values through $\mathcal{S}$ (although this set need not be contiguous).

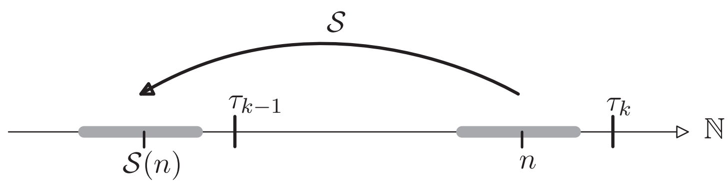

Figure 1. Number line representation of mapping n through $\mathcal{S}$ for Case 1 in the proof of Theorem 5.2. Not drawn to scale.

For Cases 2 through 5, we assume n consists of the same number of digits as $N(89^{u_k}80^{v_k}, 10^{u_k}20^{v_k})$.

Case 2. Suppose n is such that $\delta_n = \varepsilon_n = 10^{u_k+v_k+2}$. Then, $\omega_n = 0$ and so, $n = 20^{u_k+v_k+1}$. This contradicts the assumption that $n < N(89^{u_k}80^{v_k}, 10^{u_k}20^{v_k})$, since both have the same number of digits and the leading digit of $N(89^{u_k}80^{v_k}, 10^{u_k}20^{v_k})$ is 1. Thus, no such n can exist.

Case 3. Suppose n is such that $\delta_n = 10^{u_k+v_k+2}$ and

$$90^{u_k}00^{v_k} < \varepsilon_n < 100^{u_k}00^{v_k}.$$

Then, $0 < \omega_n < 10^{u_k}00^{v_k}$. Hence, $\delta_{\varepsilon_n} = u_k + v_k + 2$ and $\delta_{\omega_n} < u_k + v_k + 2$. Then,

$$\delta_{\mathcal{S}(n)} = \delta_{\delta_n} + \delta_{\varepsilon_n} + \delta_{\omega_n} < 3(u_k + v_k) + 7. \tag{5.7}$$

If $k = 6$, this previous inequality implies

$$\delta_{\mathcal{S}(n)} < 3(u_6 + v_6) + 7 = 1\,100\,110 = \delta_{\tau_5}.$$

Thus, $\mathcal{S}(n) < \tau_5$ and so, $n \notin I_6$. If $k \geq 7$, we use (5.5) for $j = k-1$ and apply (5.7) to get

$$\delta_{\mathcal{S}(n)} < 3(u_k + v_k) + 7 = 10^{u_{k-1}+v_{k-1}+2} = \delta_{\tau_{k-1}}.$$

Thus, $\mathcal{S}(n) < \tau_{k-1}$ implies $n \notin I_k$. Figure 1 also represents this case.

Case 4. Suppose n is such that $\delta_n = 10^{u_k+v_k+2}$ and $\varepsilon_n = 90^{u_k}00^{v_k}$. Then, $\omega_n = 10^{u_k}00^{v_k}$. Apply $\mathcal{S}$ twice to n to get

$$n \to \langle 10^{u_k+v_k+2}, 90^{u_k+v_k+1}, 10^{u_k+v_k+1}\rangle \to \langle 3(u_k+v_k)+7, 3(u_k+v_k)+4, 3\rangle. \tag{5.8}$$

If $k = 6$, the previous calculation implies

$$\mathcal{S}^2(n) = \langle 3(u_6+v_6)+7, 3(u_6+v_6)+4, 3\rangle = \langle 1\,100\,110, 1\,100\,107, 3\rangle \to \langle 15, 6, 9\rangle \in I_2,$$

and so, $n \in I_5$. If $k \geq 7$, we use (5.5) for $j = k-2$ and map n twice through $\mathcal{S}$ via (5.8) to get

$$\mathcal{S}^2(n) = \langle 3(u_k+v_k)+7, 3(u_k+v_k)+4, 3\rangle = \langle 10^{u_{k-1}+v_{k-1}+2}, 9^{u_{k-1}+v_{k-1}+1}7, 3\rangle.$$

Hence, $\delta_{\mathcal{S}^2(n)} = 2(u_{k-1}+v_{k-1}) + 6 < 3(u_{k-1}+v_{k-1}) + 7 = \delta_{\tau_{k-2}}$. Thus, $\mathcal{S}^2(n) < \tau_{k-2}$ implies $n \notin I_k$. An illustration of this case is given in Figure 2.

As in Figure 1, the initial set and the image sets are represented by the thicker lines on $\mathbb{N}$ (but they need to be contiguous).

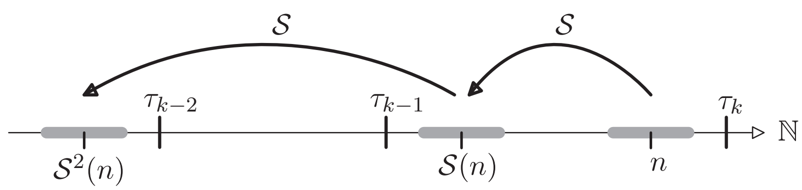

Figure 2. Number line representation of mapping n through $\mathcal{S}$ twice for Case 4 in the proof of Theorem 5.2.

Case 5. Suppose n is such that $\delta_n = 10^{u_k+v_k+2}$ and

$$89^{u_k}80^{v_k} < \varepsilon_n < 90^{u_k}00^{v_k}. \tag{5.9}$$

Then, $10^{u_k}00^{v_k} < \omega_n < 10^{u_k}20^{v_k}$.

If $k = 6$, apply Lemma 5.1 to (5.9) to get

$$733\,405 = 2u_6 + 2v_6 + 3 \leq \varepsilon_{\mathcal{S}(n)} \leq 2u_6 + 3v_6 + 4 = 1\,000\,010. \tag{5.10}$$

If $733\,405 \leq \varepsilon_{\mathcal{S}(n)} < 1\,000\,000$, then the number of digits in $\mathcal{S}^2(n)$ is 19 (because $\delta_{\mathcal{S}(n)} = 1\,100\,110$ and $\delta_{\varepsilon_{\mathcal{S}(n)}} = \delta_{\omega_{\mathcal{S}(n)}} = 6$). By Corollary 3.4, $\mathcal{S}^{\#}(\mathcal{S}^2(n)) \leq 3$ and so, $\mathcal{S}^{\#}(n) \leq 5$. If $1\,000\,000 \leq \varepsilon_{\mathcal{S}(n)} \leq 1\,000\,010$, then $\delta_{\mathcal{S}^2(n)} = 20$ and $\varepsilon_{\mathcal{S}^2(n)} \geq 11$. Hence, $\mathcal{S}^3(n)$ consists of exactly 5 digits. By Lemma 2.1, $\mathcal{S}^{\#}(\mathcal{S}^3(n)) = 2$ implies that $\mathcal{S}^{\#}(n) = 5$. For both subcases, we have $n \notin I_6$.

If $k \geq 7$, apply Lemma 5.1 to (5.9) to get

$$6^{u_{k-1}+v_{k-1}+1}5 = 2u_k + 2v_k + 3 \leq \varepsilon_{\mathcal{S}(n)} \leq 2u_k + 3v_k + 4 = 66^{u_{k-1}}79^{v_{k-1}}. \tag{5.11}$$

For each j such that $6 < j < k$, apply Lemma 5.1 to (5.11) to get

$$6^{u_{j-1}+v_{j-1}+1}5 = 2u_j + 2v_j + 3 \leq \varepsilon_{\mathcal{S}^{k-j+1}(n)} \leq 2u_j + 3v_j + 4 = 66^{u_{j-1}}79^{v_{j-1}}. \tag{5.12}$$

Continuing to apply Lemma 5.1 to (5.12) until $j = 7$, when we have

$$6^{u_6+v_6+1}5 = 2u_7 + 2v_7 + 3 \leq \varepsilon_{\mathcal{S}^{k-6}(n)} \leq 2u_7 + 3v_7 + 4 = 66^{u_6}79^{v_6}.$$

One final application of Lemma 5.1 gives

$$2u_6 + 2v_6 + 3 \leq \varepsilon_{\mathcal{S}^{k-7}(n)} \leq 2u_6 + 3v_6 + 4,$$

which is the same bounds given in (5.10). A similar analysis will yield $\mathcal{S}^{\#}(\mathcal{S}^{k-8}(n)) \leq 5$, which simplifies to $\mathcal{S}^{\#}(n) \leq k-3$ and so, $n \in I_{k-1}$.

An illustration of this case is given in Figure 3.

□

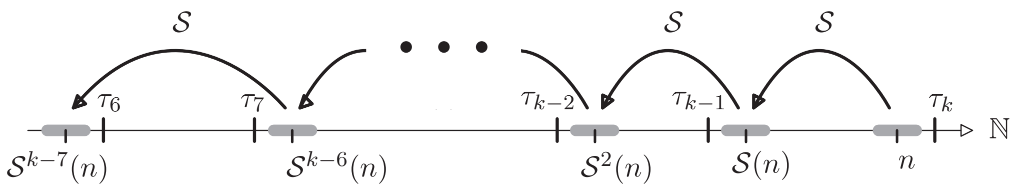

Figure 3. Number line representation of iterating n through $\mathcal{S}$ for Case 5 in the proof of Theorem 5.2.

$\boldsymbol{k}$	$\boldsymbol{\tau_k}$	# of digits
0	312	3
1	$N(1,2) = 101$	3
2	$N(1,0) = 0$	1
3	$N(0,2) = 11$	2
4	$N(10,10) = 10\,000\,000\,000\,111\,111\,111$	20
5	$N(1\,000\,099, 100\,011)$	1 100 110
6	$N(89^{u_6}80^{v_6}, 10^{u_6}20^{v_6}), u_6 = 100\,097, v_6 = 266\,604$	$10^{366\,703}$
7	$N(89^{u_7}80^{v_7}, 10^{u_7}20^{v_7}), u_7 = 3^{u_6-1}20^{v_6} - 2, v_7 = 13^{v_6}$	$10^{3^{366\,701}}$
$\vdots$	$\vdots$	$\vdots$
k	$N(89^{u_k}80^{v_k}, 10^{u_k}20^{v_k}), u_k = 3^{u_{k-1}-1}20^{v_{k-1}} - 2, v_k = 13^{v_{k-1}}$	$10^{3^{u_{k-1}+v_{k-1}}}$
$\vdots$	$\vdots$	$\vdots$

Table 1. Minimal height sequence associated to $\mathcal{S}$.

6. Summary, Modifications, and Further Research

A summary of the results are given in Table 1.

A natural extension of the Sisyphus function would be to count more than just even and odd digits. See [2] for the first discussion of this. For $j \geq 2$, define the *jth order Sisyphus function*, $\mathcal{S}_j : \mathbb{N} \to \mathbb{Z}^+$, to be the number obtained by writing the integer constructed from left to right of the total number of digits of a nonnegative integer n followed by the number of digits of n congruent to $0 \pmod{j}$, the number of digits of n congruent to $1 \pmod{j}$, and so on up to the total number of digits of n congruent to $j-1 \pmod{j}$. For example, $\mathcal{S}_3(123\,589) = \langle 6,2,1,3 \rangle = 6\,213$ and $\mathcal{S}_2$ is the same function defined in §2. For each $j = 3,4,\ldots,9$, determine the stable points, nontrivial cycles and the minimal height sequence of $\mathcal{S}_j$. ($\mathcal{S}_3$ has one cycle and no stable points.)

We can also use other bases to represent the domain of $\mathcal{S}$ instead of base 10. How would binary and/or hexadecimal representation change the conclusions? Determine the stable points, nontrivial cycles and the minimal height sequence of $\mathcal{S}$. For example, in binary, there is one cycle (of length 2) and no stable points.

Another extension is the *Sisyphean digital sum*, $\mathcal{S}_\Sigma : \mathbb{Z}^+ \to \mathbb{Z}^+$ which takes any nonnegative integer n and writing an integer whose digits are constructed by concatenating from left to right the sum of all the digits of n, the sum of the even digits of n, and the sum of the odd digits of n. For example, $\mathcal{S}_\Sigma(123\,589) = \langle 28, 10, 18\rangle = 281\,018$. Determine the stable points, nontrivial cycles and the minimal height sequence of $\mathcal{S}_\Sigma$.

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