FRACTAL BEHAVIOR OF THE FIBONOMIAL TRIANGLE MODULO PRIME p, WHERE THE RANK OF APPARITION OF p IS $p+1$

MICHAEL DEBELLEVUE AND EKATERINA KRYUCHKOVA

Abstract. Pascal's triangle is known to exhibit fractal behavior modulo prime numbers. We tackle the analogous notion in the Fibonomial triangle modulo prime p with the rank of apparition $p^* = p+1$, proving that these objects form a structure similar to the Sierpinski Gasket. Within a large triangle of p^*p^{m+1} many rows, in the i^{th} triangle from the top and the j^{th} triangle from the left, $\binom{n+ip^*p^m}{k+jp^*p^m}_F$ is divisible by p if and only if $\binom{n}{k}_F$ is divisible by p. This proves the existence of the recurring triangles of zeroes that are the principal component of the Sierpinski Gasket. The exact congruence classes follow the relationship $\binom{n+ip^*p^m}{k+jp^*p^m}_F \equiv_p (-1)^{ik-nj}\binom{i}{j}\binom{n}{k}_F$, where $0 \leq n, k < p^*p^m$.

1. Introduction

Pascal's triangle is known to exhibit fractal behavior modulo prime numbers. This can be proven by using Lucas' Theorem:

Theorem 1.1. *Write n and k in base p with digits $n_0, n_1, \dots, n_m$ and $k_0, k_1, \dots, k_m$. Then,*

$$\binom{n}{k} \equiv_p \binom{n_0}{k_0}\binom{n_1}{k_1}\cdots\binom{n_m}{k_m}.$$

Consider the Fibonacci numbers as defined by $F_0 = 0$, $F_1 = 1$, and $F_n = F_{n-1} + F_{n-2}$ for $n \geq 2$. The Fibonomial triangle is formed using the Fibotorial $!_F$ function in place of the factorial function, where $n!_F = F_nF_{n-1}F_{n-2}\cdots F_1$. Then the Fibonomial coefficient $\binom{n}{k}_F$ is defined as $\frac{n!_F}{(n-k)!_F k!_F}$, where $\binom{n}{0}_F$ is defined to be 1 for $n \geq 0$, as with binomial coefficients. The Fibonomial triangle appears to exhibit a fractal structure, but Lucas' Theorem does not directly apply to Fibonomial coefficients [8]. Instead, we prove an analogue of Lucas' Theorem for divisibility by a particular class of primes p in section 3 and address exact congruence classes in section 4.

2. Background

We define p^* to be the rank of apparition of p in the Fibonacci sequence. The rank of apparition is the index of the first Fibonacci number divisible by p.

The Fibonacci sequence exhibits a number of interesting properties that will be used throughout this paper, among them the divisibility property, regular divisibility by a prime, and the shifting property. The following lemmas can be found in a variety of sources, including [10].

Lemma 2.1. (Lucas [6]) *For positive integers n and m, $gcd(F_n, F_m) = F_{gcd(n,m)}$. If $n \mid m$ then $gcd(n,m) = n$, so $gcd(F_n, F_m) = F_n$, and so $F_n \mid F_m$.*

Lemma 2.2. *For positive integer i and prime p, $p \mid F_{ip^*}$*

The periodic nature of the Fibonacci sequence modulo p follows.

Lemma 2.3. *For positive integers n and m, $F_{n+m} = F_m F_{n+1} + F_{m-1} F_n$.*

By a result of Sagan and Savage [7], the Fibonomial coefficients have a combinatorial interpretation. It follows that $\binom{n}{k}_F$ is a nonnegative integer.

The Fibonomial coefficients conform to a recurrence relation analogous to the recurrence relation on binomial coefficients:

Lemma 2.4. *For positive integers n and k,*

$$\binom{n}{k}_F = F_{n-k+1}\binom{n-1}{k-1}_F + F_{k-1}\binom{n-1}{k}_F.$$

Like the binomial coefficients, the Fibonomial coefficients possess a number of useful properties, among them the negation property and the iterative property:

Lemma 2.5. (Gould [2]) *For $n, k \in \mathbb{Z}$,*

$$\binom{n}{k}_F = \binom{n}{n-k}_F.$$

Lemma 2.6. (Gould [2]) *For $a, b, c \in \mathbb{Z}$,*

$$\binom{a}{b}_F \binom{b}{c}_F = \binom{a}{c}_F \binom{a-c}{a-b}_F.$$

It is commonly known that the Fibonacci sequence modulo an integer is periodic. The period modulo p is called the Pisano period and is denoted $\pi(p)$. A related notion is the Pisano semiperiod, defined as the period of the modulo p Fibonacci sequence up to a sign.

Southwick proved an analogue of Lucas' theorem in the case $p = 5$ using a theorem by Hu and Sun [9, 4]. Southwick requested a proof using only a prior theorem by Knuth and Wilf [5]. This method is utilized in Section 3.

3. Divisibility

By a result of Harris [3], when $p^* = p + 1$, $\pi(p) \mid 2p^*$, and p^* is the Pisano semiperiod. We rely on the notion of the semiperiod and assume for this paper that p is an odd prime and $p^* = p + 1$.

For a nonnegative integer x, $\nu_p(x)$ denotes the p-adic valuation of x, i.e. the highest power of p dividing x.

We use a result of Knuth and Wilf [5], adapted for the Fibonacci sequence.

Theorem 3.1. (Knuth and Wilf) *The highest power of an odd prime p that divides the Fibonomial coefficient $\binom{n}{k}_F$ is the number of carries that occur to the left of the radix point when k/p^* is added to $(n-k)/p^*$ in p-ary notation, plus the p-adic valuation $\nu_p(F_{p^*}) = 1$ if a carry occurs across the radix point.*

We require that p not be a Wall-Sun-Sun prime for $\nu_p(F_{p^*}) = 1$ to hold.

Since we are interested in divisibility, we only require that the p-adic valuation is at least one, so it suffices to show that a carry occurs.

As in [1] and [8], we consider the base $\mathcal{F}_{p^*} = (1, p^*, p^*p, p^*p^2, \dots)$. So $n = n_0 + n_1 p^* + n_2 p^* p + \cdots + n_m p^* p^{m-1} = (n_0, n_1, n_2, \dots, n_m)_{\mathcal{F}_{p^*}}$. In this base, division by p^* results in a number $(n_1, n_2, n_3, \dots, n_m)_p$, with fractional part $\frac{n_0}{p^*}$ only, which simplifies the counting of the carries.

Generalizing Southwick's proof in [8], we prove the following:

Theorem 3.2. *Given that* $p^* = p + 1$ *and integers* $n, k > 0$,

$$p \mid \binom{n}{k}_F \iff p \mid \binom{n_0}{k_0}_F \binom{n_1}{k_1}_F \cdots \binom{n_m}{k_m}_F .$$

Visually this corresponds to the recurring triangles of zeroes in the Fibonomial triangle mod p. *This is illustrated in Figure 1.*

Proof. By Theorem 3.1, $p \mid \binom{n}{k}_F$ if and only if a carry occurs in the addition of $(\frac{k}{p^*})$ and $(\frac{n-k}{p^*})$ in base p. The first carry occurs across the radix point or to the left of the radix point. Let $q = n - k = (q_0, q_1, \dots, q_m)_{\mathcal{F}_{p^*}}$

(1) First consider the conditions necessary for the carry across the radix point.

If $n_0 \geq k_0$, then $q_0 = n_0 - k_0$, $k_0 + (q_0) = n_0 < p^*$. In this case, there will be no carry.

Alternatively, if $k_0 > n_0$, then a borrow occurs, so $q_0 = n_0 - k_0 + p^*$. The addition of $\frac{k_0}{p^*}$ and $\frac{q_0}{p*}$ produces:

$$\frac{k_0 + n_0 - k_0 + p^*}{p^*} = \frac{n_0 + p^*}{p^*} \geq 1.$$

Thus a carry occurs across the radix point if and only if $k_0 > n_0$.

(2) If a carry across the radix point does not occur, then let the first carry occur in the $(j+1)^{st}$ digit, that is, in the addition of k_j with q_j (note that the $(j+1)^{st}$ digit of n in base $\mathcal{F}_{p^*}$ is $n_j p^* p^{j-1}$). The division by p^* moves the digits to the right by one, so the carry occurs at the j^{th} digit in base p.

If $n_j \geq k_j$, then $k_j + q_j = k_j + (n_j - k_j) < p$ since we assume there was no previous carry.

If $k_j > n_j$, the subtraction $n_j - k_j$ results in a borrow, so $q_j = n_j - k_j + p$, and so

$$k_j + q_j = k_j + (n_j - k_j + p) = n_j + p \geq p.$$

This case is the only case in which a carry occurs.

Therefore if a carry occurs in the j^{th} position, then $n_j < k_j$, and so $p \mid \binom{n_j}{k_j}_F$ since $\binom{n_j}{k_j} = 0$, and so $p \mid \binom{n_0}{k_0}_F \binom{n_1}{k_1}_F \cdots \binom{n_m}{k_m}_F$. Using the above result and Theorem 3.1, we conclude that if $p \mid \binom{n}{k}_F$ then $p \mid \binom{n_0}{k_0}_F \binom{n_1}{k_1}_F \cdots \binom{n_m}{k_m}_F$.

For the reverse direction, we note that all these steps and Theorem 3.1 are reversible.

Note that for $n < k$ the statement follows trivially.

Therefore, $p \mid \binom{n}{k}_F \Leftrightarrow p \mid \binom{n_0}{k_0}_F \binom{n_1}{k_1}_F \cdots \binom{n_m}{k_m}_F$. □

Corollary 3.3. *Given* $0 \leq m$ *and* $0 \leq n$, $k < p^* p^m$, *for all* $i, j \in \mathbb{Z}$ *such that* $0 \leq j < i < p$,

$$p \mid \binom{n + i p^* p^m}{k + j p^* p^m}_F \iff p \mid \binom{n}{k}_F .$$

Proof. By Theorem 3.2, since $p \nmid \binom{i}{j}_F$,

$$p \mid \binom{n}{k}_F \iff p \mid \binom{n_0}{k_0}_F \binom{n_1}{k_1}_F \cdots \binom{n_m}{k_m}_F \iff p \mid \binom{n_0}{k_0}_F \binom{n_1}{k_1}_F \cdots \binom{n_m}{k_m}_F \binom{i}{j}_F$$
$$\iff p \mid \binom{n + ip^*p^m}{k + jp^*p^m}_F.$$

☐

4. Exact Nonzero Congruence Classes

We begin with a number of necessary Lemmas.

Lemma 4.1. *If* $\frac{a}{b}, \frac{a}{c}, \frac{b}{c} \in \mathbb{Z}$, *with* $\frac{a}{c} \equiv_p a'$ *and* $\frac{b}{c} \equiv_p b' \not\equiv_p 0$, *then* $\frac{a}{b} \equiv_p a'(b')^{-1}$.

Proof. Since $c \neq 0$, we can multiply the fraction $\frac{a}{b}$ by $\frac{1/c}{1/c}$. Since the resulting fraction is an integer, it can be reduced modulo p to $a'(b')^{-1}$. Note that $(b')^{-1}$ exists because p is prime and $b' \not\equiv_p 0$. ☐

Lemma 4.2. *For* $0 \leq n < p^*p^m$, $F_{n+p^*p^m} \equiv_p -F_n$.

Proof. Since $p^* = \frac{1}{2}\pi(p)$ is the semiperiod, $F_{n+p^*} \equiv_p -F_n$.
Then, since $(p^m - 1)$ is even and $\pi(p) = 2p^*$, $F_{n+p^*} \equiv_p -F_n$ implies $F_{n+p^*p^m} \equiv_p -F_n$.

☐

Lemma 4.3. *For* $i > 0$,

$$\frac{F_{ip^*p^m}}{F_{p^*p^m}} \equiv_p i(-1)^{i-1}.$$

Proof. We prove this by induction.

First, let $i = 1$. Then, the statement follows trivially.

Now, assume the inductive hypothesis:

$$\frac{F_{ip^*p^m}}{F_{p^*p^m}} \equiv_p (-1)^{i-1} i.$$

Consider

$$\frac{F_{(1+i)p^*p^m}}{F_{p^*p^m}} = \frac{F_{p^*p^m+ip^*p^m}}{F_{p^*p^m}}.$$

We apply the shifting property of the Fibonacci sequence to obtain:

$$\frac{F_{p^*p^m+ip^*p^m}}{F_{p^*p^m}} = \frac{F_{p^*p^m}F_{ip^*p^m+1} + F_{p^*p^m-1}F_{ip^*p^m}}{F_{p^*p^m}}.$$

Then we simplify by canceling like terms on the left and applying the induction hypothesis on the right:

$$F_{ip^*p^m+1} + F_{p^*p^m-1}(-1)^{i-1}(i) \equiv_p (-1)^i + (-1)^i(i) \equiv_p (-1)^{(i+1)-1}(i+1).$$

☐

Lemma 4.4. *For $i > 0$,*

$$\binom{ip^*p^m}{p^*p^m}_F \equiv_p i.$$

Proof. By definition of the Fibonomial coefficient,

$$\binom{ip^*p^m}{p^*p^m}_F = \frac{F_{ip^*p^m}F_{ip^*p^m-1}\cdots F_{(i-1)p^*p^m}F_{(i-1)p^*p^m-1}\cdots F_1}{(F_{(i-1)p^*p^m}F_{(i-1)p^*p^m-1}\cdots F_1)F_{p^*p^m}F_{p^*p^m-1}\cdots F_1}$$

Canceling like terms gives

$$\frac{F_{ip^*p^m}F_{ip^*p^m-1}\cdots F_{ip^*p^m-(p^*p^m-1)}}{F_{p^*p^m}F_{p^*p^m-1}\cdots F_1}$$

The terms in the above expression take three forms, which we represent separately for clarity. Note that all reduction modulo p happens term-wise, and thus the result is an integer.

(1) We first consider terms of the form $F_{ip^*p^m-a}$, where $p^* \nmid a$. For each of these terms, we identify a corresponding term in the denominator:

$$\frac{F_{ip^*p^m-a}}{F_{p^*p^m-a}}.$$

Altogether, these terms take the form

$$\prod_{\substack{a=1\\ p^*\nmid a}}^{p^*p^m-1} \frac{F_{ip^*p^m-a}}{F_p^*p^m-a}.$$

We apply Lemma 4.2 to the top so that we can cancel the top and bottom. Since there are $p^*p^m - 1 - (p^m - 1) = p^{m+1}$ many such terms, the result after applying Lemma 4.2 to each is $(-1)^{(i-1)(p^{m+1})} \equiv_p (-1)^{i-1}$, because p^{m+1} is odd, as p is assumed to be an odd prime.

(2) Next we consider terms of the form $F_{(ip^m-a)p^*}$:

$$\prod_{a=1}^{p^m-1} \frac{F_{(ip^m-a)p^*}}{F_{(p^m-a)p^*}}.$$

By Lemma 4.1 and Lemma 4.3,

$$\left(\prod_{a=1}^{p^m-1} \frac{F_{(ip^m-a)p^*}}{F_{(p^m-a)p^*}}\right)\left(\frac{\frac{1}{F_{p^*}}}{\frac{1}{F_{p^*}}}\right)^{p^m-1} \equiv_p \prod_{a=1}^{p^m-1} \frac{(-1)^{ip^m-a-1}(ip^m-a)}{(-1)^{p^m-a-1}(p^m-a)} \equiv_p \prod_{a=1}^{p^m-1} \frac{(-1)^{(i-1)p^m}(-a)}{(-a)}.$$

Note that in the modular group we use division notation to represent multiplication by an inverse.

Since p^m is odd and $p^m - 1$ is even,

$$\prod_{a=1}^{p^m-1} \frac{(-1)^{(i-1)p^m}(-a)}{(-a)} \equiv_p (-1)^{(i-1)(p^m-1)} \equiv_p 1.$$

(3) The only remaining term is the quotient $\frac{F_{ip^*p^m}}{F_{p^*p^m}} \equiv_p i(-1)^{i-1}$, by Lemma 4.3.

From the three cases above,

$$\binom{ip^*p^m}{p^*p^m}_F \equiv_p i(-1)^{2(i-1)} \equiv_p i$$

□

Lemma 4.5. *For* $0 \leq i, j < p$,

$$\binom{ip^*p^m}{jp^*p^m}_F \equiv_p \binom{i}{j}.$$

Proof. We prove this using induction. For a base case, let $i = 0$. Then if $j = 0$,

$$\binom{0}{0}_F \equiv_p 1 \equiv_p \binom{0}{0}.$$

If $j > 0$, then

$$\binom{0}{jp^*p^m}_F \equiv_p 0 \equiv_p \binom{0}{j}.$$

Now assume $\binom{ip^*p^m}{jp^*p^m}_F \equiv_p \binom{i}{j}$. Then we apply Lemmas 2.5 and 2.6.
We let $a = (i+1)p^*p^m$, $b = (i+1-j)p^*p^m$, and $c = p^*p^m$, thus yielding the following:

$$\binom{(i+1)p^*p^m}{jp^*p^m}_F \binom{(i+1-j)p^*p^m}{p^*p^m}_F = \binom{(i+1)p^*p^m}{p^*p^m}_F \binom{ip^*p^m}{jp^*p^m}_F$$

Applying the induction hypothesis and Lemma 4.4 gives

$$\binom{(i+1)p^*p^m}{jp^*p^m}_F (i+1-j) \equiv_p (i+1)\binom{i}{j}.$$

We then multiply both sides by $(i+1-j)^{-1}$ to obtain

$$\binom{(i+1)p^*p^m}{jp^*p^m}_F \equiv_p \frac{(i+1)}{(i+1-j)}\binom{i}{j}.$$

Equivalently,

$$\binom{(i+1)p^*p^m}{jp^*p^m}_F \equiv_p \binom{i+1}{j},$$

as desired.

□

We now proceed with our main theorem for the exact congruence classes of the Fibonomial triangle modulo p. For a visual representation of the relation using $p = 7$, see Figure 1.

Theorem 4.6. *For* $0 < n < p^*p^m$, $0 \leq k < p^*p^m$, $0 \leq i, j < p, 0 \leq m$,

$$\binom{n + ip^*p^m}{k + jp^*p^m}_F \equiv_p (-1)^{ik-nj}\binom{i}{j}\binom{n}{k}_F.$$

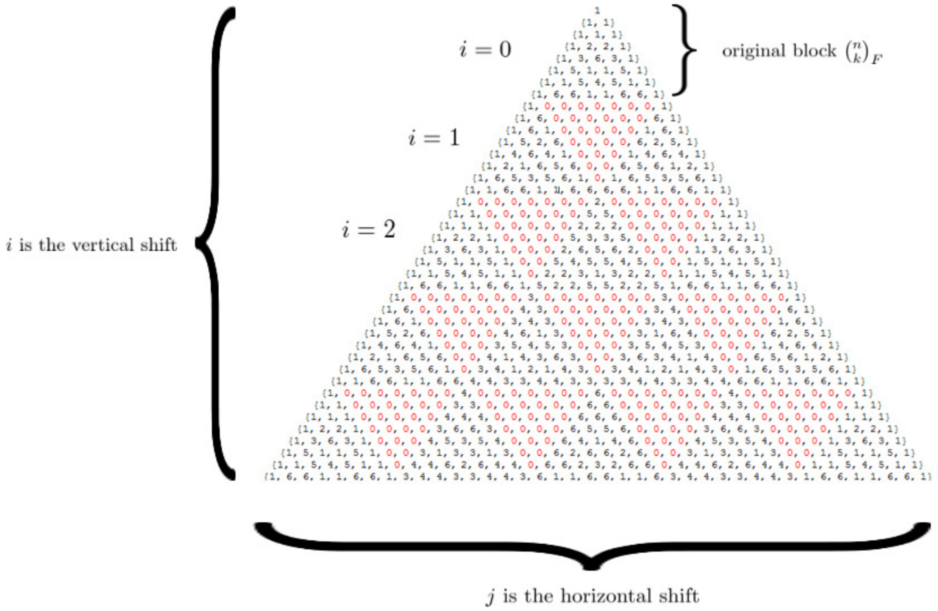

Figure 1. Exact congruence classes modulo 7.

Proof. We proceed by induction.

First let $n = k = 0$. Then the statement follows directly from Lemma 4.5.

When $n = 0$, $k > 0$, by Theorem 3.3, since $p | \binom{0}{k}_F$,

$$\binom{ip^*p^m}{k + jp^*p^m}_F \equiv_p 0 \equiv_p (-1)^{ik-0j} \binom{i}{j} \binom{0}{k}.$$

Let $n > 0$, $k \geq 0$. We assume

$$\binom{n-1+ip^*p^m}{k+jp^*p^m}_F \equiv_p (-1)^{ik-(n-1)j} \binom{i}{j} \binom{n-1}{k}_F$$

for all k.

Using the recurrence relation for Fibonomial coefficients,

$$\begin{aligned}
\binom{n+ip^*p^m}{k+jp^*p^m}_F &\equiv_p F_{n+(i-j)p^*p^m-k+1} \binom{n-1+ip^*p^m}{k-1+jp^*p^m}_F + F_{k-1+jp^*p^m} \binom{n-1+ip^*p^m}{k+jp^*p^m}_F \\
&\equiv_p (-1)^{i-j} F_{n-k+1} (-1)^{i(k-1)-(n-1)j} \binom{i}{j} \binom{n-1}{k-1}_F \\
&\quad + (-1)^j F_{k-1} (-1)^{i(k)-(n-1)j} \binom{i}{j} \binom{n-1}{k}_F \\
&\equiv_p (-1)^{ik-nj} \binom{i}{j} \left[F_{n-k+1} \binom{n-1}{k-1}_F + F_{k-1} \binom{n-1}{k}_F \right] \\
&\equiv_p (-1)^{ik-nj} \binom{i}{j} \binom{n}{k}_F .
\end{aligned}$$

This completes the proof.

□

5. Further Directions

In Theorem 4.6, note that

$$ik - nj = \det \begin{pmatrix} i & n \\ j & k \end{pmatrix}.$$

This may be a coincidence but, alternatively, it might indicate the existence of a more general relation for different types of primes.

Theorem 4.6 can be generalized to other primes by proving variants of the prerequisite lemmas. For example, in the case $p = 5$, $5^* = 5$, and $F_{n+5} \equiv_5 3F_n$ [9, 8].

However, for some primes, problems arise. In the case $p = 11$, $p^* = 10$. In this case, one would need a base other than base $\mathcal{F}_{p^*}$, because the divisibility theorem cannot be proven in base $\mathcal{F}_{p^*}$ [8]. The form of such a base remains to be investigated.

References

[1] X. Chen and B. Sagan, *The fractal nature of the Fibonomial triangle*, (2013), arXiv:1306.2377.

[2] M. Dziemianczuk, *Generalization of Fibonomial coefficients*, (2009), arXiv:0908.3248.

[3] T. Harris, *Notes on the Pisano semiperiod*, `http://www.tkmharris.net/maths/`.

[4] H. Hu and Z.-W. Sun, *An extension of Lucas' theorem*, Proceedings of the American Mathematical Society, **129.12** (2001), 3471–3478.

[5] D. E. Knuth and H. S. Wilf, *The power of a prime that divides a generalized binomial coefficient*, J. Rein Angew. Math., **396** (1989), 212–219.

[6] E. Lucas, *Théorie des fonctions numériques simplement périodiques*, American Journal of Mathematics, **1.4** (1878), 289–321.

[7] B. E. Sagan and C. D. Savage, *Combinatorial interpretations of binomial coefficient analogues related to Lucas sequences*, Integers, **10.6** (2010), 697–703.

[8] J. Southwick, *A conjecture concerning the Fibonomial triangle*, (2016), arXiv:1604.04775.

[9] J. T. Southwick, *Divisibility conditions for Fibonomial coefficients*, Ph. D. thesis, Wake Forest University, 2016.

[10] H. J. Wilcox, *Fibonacci sequences of period n in groups*, The Fibonacci Quarterly, **24.4** (1986), 356–361.

MSC2010: 11B65, 11B39

Department of Mathematics, University of Nebraska-Lincoln, Lincoln, NE 68588-0130

E-mail address: `michael.debellevue@gmail.com`

Center for Applied Mathematics, Cornell University, Ithaca, NY 14853

E-mail address: `ek672@cornell.edu`