TOWARDS FORMULATING A TAGIURI GENERATING METHOD CONJECTURE

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Abstract. This paper continues the work on the Tagiuri Generating Method (TGM) for production of Fibonacci identities, recently introduced at the Caen Fibonacci conference. TGM starts with a trivial identity in products of Fibonacci numbers, for example, $F_{n+a}F_{n+b}F_{n+c} = F_{n+a}F_{n+b}F_{n+c} + F_{n+a}F_{n+b}F_{n+c} - F_{n+a}F_{n+b}F_{n+c}$. Using the Tagiuri identity, $F_{n+x}F_{n+y} = F_nF_{n+x+y} + (-1)^n F_xF_y$, TGM then makes substitutions on two-factor products in the start identity. TGM is capable of simply generating one-parameter families of identities. These identities are complex; in general, nothing further can be uniformly said about them. However, the histograms of the indices occurring in the family of identities have specific and interesting patterns. The purpose of this paper is to examine a new one-parameter family of identities that is rich enough to suggest a general conjecture about the histograms of arbitrary one-parameter families of identities arising from TGM. The one-parameter family studied also has interest in its own right.

1. Introduction

The Tagiuri Generating Method (TGM) was introduced in [1]. TGM allows generation of one-parameter families of Fibonacci identities that have interesting properties. The family of identities presented in [1] was not rich enough, and consequently, did not allow accurate formulation of a general conjecture.

The purpose of this paper is to present another one-parameter family of Fibonacci identities with sufficient richness to enable formulation of a general conjecture about TGM. In the next section, we review the basics of TGM. The definitions and notations presented in [1] are given with some minor changes whose purpose is to facilitate clarity and focus on generation of one-parameter families.

2. Review of TGM

The Tagiuri identity states [2, p. 114] that for integral n, x, and y

$$F_{n+x}F_{n+y} = F_nF_{n+x+y} + (-1)^n F_xF_y. \tag{2.1}$$

If a given product in an identity has the factor $F_{n+x}F_{n+y}$, then we can *apply* (2.1) by replacing the left side of (2.1) with the right side of (2.1). In the sequel, we may abuse language and say that we apply Tagiuri to the indices x and y, or we apply Tagiuri to the product at the index pair x, y.

For integer $q \geq 0$, TGM starts with formation of $2K(q) + 2$ identical products,

$$P(0) = P(1) = \cdots = P(2K(q) + 1) = \prod_{i=1}^{L(q)} F_{n+a_i}, \tag{2.2}$$

with the a_i parameters defined over the integers, $L(q)$ a function from non-negative integers to positive integers, and $K(q)$ a function from non-negative integers to non-negative integers.

We then define the *start identity* by

$$P(0) = P(1)+P(2)+\cdots+P(K(q))+P(K(q)+1)-\left(P(K(q)+2)+\cdots+P(2K(q)+1)\right), \quad (2.3)$$

with the parenthetical expression on the right side equaling zero if $K(q) = 0$.

Clearly, the start identity is true. It follows that all identities derived from the start identity are also true.

Definition 2.1. *The TGM notationally indicated by*

$$\langle L(q), K(q), Q_1, Q_2, \ldots, Q_{2K(q)+1}\rangle, \quad (2.4)$$

where for each k, $1 \leq k \leq 2K(q)+1$, Q_k is of the form $(i_{k,1}, j_{k,1}; i_{k,2}, j_{k,2}; \ldots; i_{k,m_k}, j_{k,m_k})$, for some m_k, with $1 \leq m_k \leq L(q)/2$, and where for each k, $1 \leq k \leq 2K(q)+1$, the sets $(i_{k,p}, j_{k,p})$, $1 \leq p \leq m_k$, are pairwise disjoint, refers to the process of applying (2.1) to $P(k)$ in (2.3) at each index pair $\{i_{k,p}, j_{k,p}\}$, $1 \leq p \leq m_k$, in Q_k for each k, $1 \leq k \leq 2K(q)+1$.

TGM, (2.4), is a *method* for producing identities; the actual identities produced are called Tagiuri Generated Identities (TGI).

The TGI are parameterized identities. To make their study easier, [1] applies substitutions to the TGI, which result in a Tagiuri Generated Identity With Substitutions (TGIWS). [1] uses the following two substitutions that are sufficient for this paper as well. When $L(q) = 2q$, we use

$$a_1 = -q,\ a_2 = -(q-1),\ \ldots,\ a_q = -1,\ a_{q+1} = 1,\ a_{q+2} = 2, \ldots,\ a_{2q} = q, \quad (2.5)$$

whereas when $L(q) = 2q+1$, we use

$$a_1 = -q,\ a_2 = -(q-1),\ \ldots,\ a_q = -1,\ a_{q+1} = 0,\ a_{q+2} = 1,\ \ldots,\ a_{2q+1} = q. \quad (2.6)$$

Throughout this paper, we use the acronym TGM in two senses. TGM can refer to the general Tagiuri Generation Method. When preceded by an article (e.g., a TGM), it refers to an application of Definition 2.1 with specific parameters. Similar comments apply to the acronyms TGI and TGIWS. Also, throughout the paper, these acronyms will be used to refer to single identities and families of identities.

Although each TGIWS is simply generated, it is not trivial. It is hard to make general statements uniformly about all TGIWS. To facilitate the study of TGIWS, [1] defines the *histogram* of an identity.

Definition 2.2. *The index histogram of a TGIWS is the numerical count of all indices occurring on the right side of that identity with the following conventions: i) all parenthetical expressions are assumed expanded; ii) like terms are not coalesced; iii) constants and $(-1)^n$ are ignored; iv) numerical coefficients of products are ignored; v) powers are counted with multiplicity; and vi) factors of the form F_x with x a constant expression (that is, independent of n) are assumed replaced by their numerical values.*

The examples in this and the next section illustrate the notations and conventions.

Example 2.3. *Let $L(q) = 2q$ and $K(q) = 0$. Then, when $q = 2$ the start identity,* (2.3)*, is*

$$P(0) = P(1),$$

with

$$P(0) = P(1) = \prod_{i=1}^{4} F_{n+a_i},$$

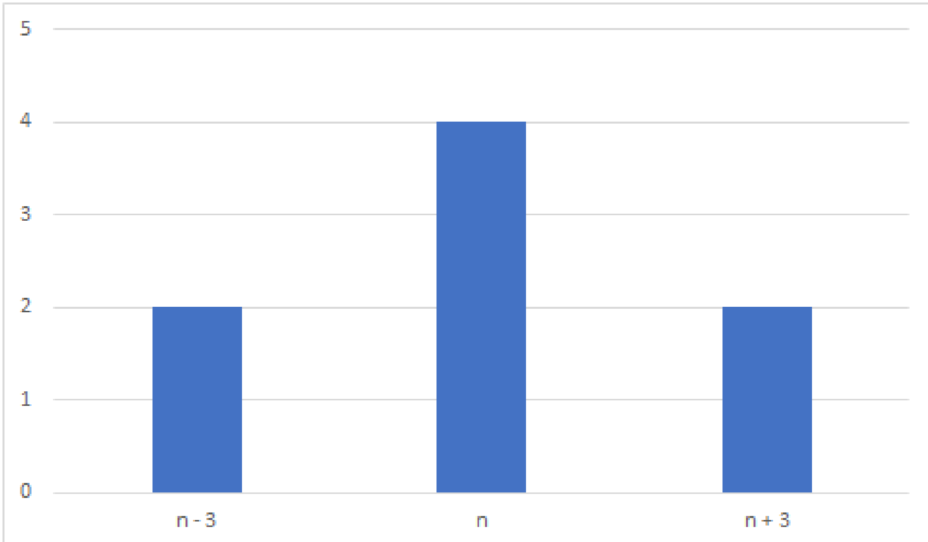

FIGURE 1. Index histogram for (2.9).

by (2.2)*. Using* (2.4)*, the TGM is indicated by*

$$\langle 4, 0, (1, 2; 3, 4)\rangle.$$

The interpretation of Definition 2.1 is that we apply Tagiuri (2.1) *to the pairs of indices* $(1, 2)$ *and* $(3, 4)$ *in* $P(1)$ *obtaining the TGI*

$$\begin{aligned} F_{n+a}F_{n+b}F_{n+c}F_{n+d} = F_n^2 F_{n+a+b}F_{n+c+d} + F_aF_bF_cF_d \\ + (1)^n F_nF_{n+a+b}F_cF_d + (-1)^n F_nF_aF_bF_{n+c+d}, \end{aligned} \tag{2.7}$$

where for convenience, $a = a_1$*,* $b = a_2$*,* $c = a_3$*, and* $d = a_4$*.*

Equation (2.5) *reduces to*

$$a_1 = a = -2, \qquad a_2 = b = -1, \qquad a_3 = c = 1, \qquad a_4 = d = 2. \tag{2.8}$$

Making the substitutions indicated by (2.8) *to* (2.7)*, we obtain the TGIWS*

$$F_{n-2}F_{n-1}F_{n+1}F_{n+2} = F_{n-3}F_n^2F_{n+3} - 1 + (-1)^n F_n(F_{n-3} - F_{n+3}). \tag{2.9}$$

Expanding the parenthetical expression on the right side of (2.9) *and using the conventions of Definition 2.2, we obtain the index histogram presented in Figure 1.*

The support of the histogram of (2.9) has three elements, $n - 3$, n, and $n + 3$ with weights 2, 4, and 2. The key point to focus on in this observation is that there are only two non-zero weights, 2 and 4.

That the histogram has only two weights is true for a general one-parameter family of identities to which (2.9) belongs. The next example studies this one-parameter family of identities.

Example 2.4. *We study the one-parameter family of TGIWS obtained from the TGM*

$$\langle 2q, 0, (1, 2; 3, 4; \dots ; 2q - 1, q)\rangle, \qquad q = 2, 3, \dots, \tag{2.10}$$

by applying (2.1) *and* (2.5). *The Main Theorem of* [1] *is that for any* $q \geq 2$, *the index histogram of the corresponding TGIWS when restricted to its support excluding the point* n, *has one value. If we divide each index count by the total number of indices, counting multiplicity, the resulting distribution would be a mixture of a uniform distribution and a point mass. A consequence of this is that for each* $q \geq 2$, *the histogram weights have exactly two non-zero values, with one weight associated with only one point. The numerical counts of these two values are presented in* [1].

Such a result motivates asking about characteristics of index-histograms of other one-parameter families of TGIWS. A simplistic approach might suggest that each index-histogram has one value except for c points, where c is a constant that does not depend on the parameter q. This however, is not true. The goal of this paper is to study a one-parameter family of TGIWS whose histogram characteristics seem to capture the general situation. The description of this one-parameter family is the subject of the next section.

3. The One-Parameter Family Studied in This Paper

This paper studies the one-parameter family of TGIWS obtained from the TGM

$$\langle 2q+1, q, (1,2), (2,3), (3,4), \ldots, (2q, 2q+1), (2q+1, 1) \rangle, \qquad q = 1, 2, 3, \ldots, \tag{3.1}$$

by applying (2.1) and (2.6). The examples clarify the notation and setup.

Example 3.1. *Let* $L(q) = 2q+1$ *and* $K(q) = q$. *When* $q = 1$, *the start identity,* (2.3), *is*

$$P(0) = P(1) + P(2) - P(3),$$

with the four identical products defined by (2.2),

$$P(0) = \cdots = P(3) = \prod_{i=1}^{3} F_{n+a_i}.$$

Applying (2.1) *to the start identity yields the TGI*

$$F_{n+a_1}F_{n+a_2}F_{n+a_3} = F_nF_{n+a_1+a_2}F_{n+a_3} + (-1)^nF_{a_1}F_{a_2}F_{n+a_3} + F_nF_{n+a_2+a_3}F_{n+a_1} + (-1)^nF_{a_2}F_{a_3}F_{n+a_1} - F_nF_{n+a_3+a_1}F_{n+a_2} - (-1)^nF_{a_3}F_{a_1}F_{n+a_2}.$$

Applying (2.6) *yields the following TGIWS,*

$$F_{n-1}F_nF_{n+1} = F_{n-1}F_nF_{n+1} + F_{n-1}F_nF_{n+1} - (-1)^nF_n - F_n^3. \tag{3.2}$$

Simplifying equation (3.2) by canceling F_n from both sides of the equation and collecting like terms, reduces to the Cassini identity [2, p. 74]. This should not surprise us because the Tagiuri identity is the two-parameter generalization of the Catalan identity [2, p. 83], which is the one-parameter generalization of the Cassini identity.

The appropriate tables, graphs, and theorem connected with this one-parameter family of TGIWS are presented in the next section.

4. Histograms, Tables, and the Main Theorem

Table 1 presents the numerical counts underlying the histograms for the TGIWS arising from application of (2.1) and (2.6) to (3.1) for $1 \leq q \leq 5$. Figures 2 and 3 display the associated connected graphs for $q = 5$ and $q = 25$.

The interpretation of Table 1 should be clear. For example, when $q = 1$, the TGIWS obtained from (3.1) by applying (2.1) and (2.6) is (3.2), whose right side has six occurrences

Index	n	$n \pm 1$	$n \pm 2$	$n \pm 3$	$n \pm 4$	$n \pm 5$	$n \pm 6$	$n \pm 7$	$n \pm 8$	$n \pm 9$
$q = 1$	6	2								
$q = 2$	12	6	4	1						
$q = 3$	18	10	8	9	0	1				
$q = 4$	24	14	12	13	12	1	0	1		
$q = 5$	30	18	16	17	16	17	0	1	0	0
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
q	$6q$	$4q-2$	$4q-4$	$4q-3$	$4q-4$	$4q-3\ldots$	0	1	0	1

Table 1. Tabular presentation of the underlying numerical counts for the index-histograms of the TGIWS family arising from application of (2.1) and (2.6) to (3.1) for $q = 1, \ldots, 5$. Note, 1) the formulas in the row beginning q and the column labeled $n \pm j$ apply to the row beginning $q = k$ only when $j \leq k$; 2) the tail of alternating zeroes and ones starts at index $n \pm (q + 1)$ and ends at index $n \pm (2q - 1)$, as described below in Theorem 4.1(e).

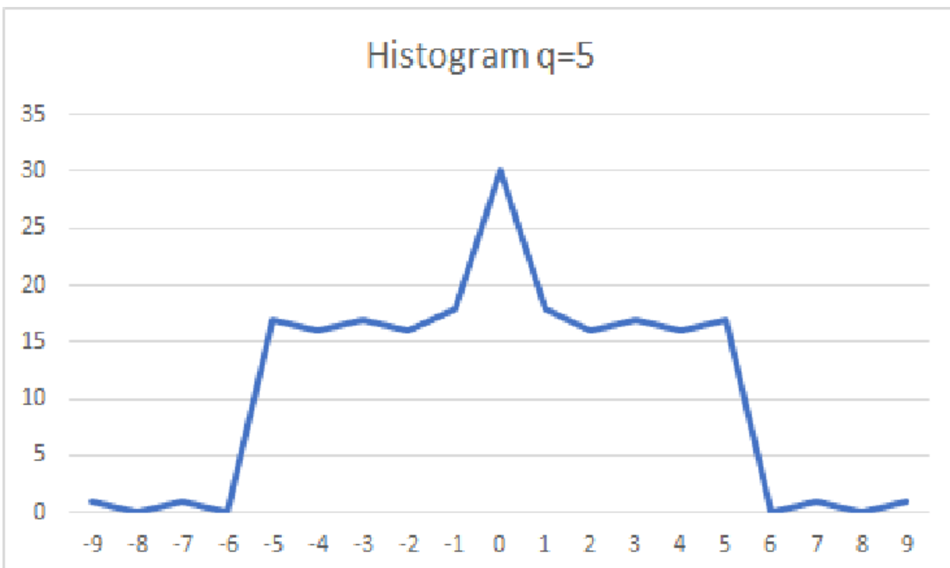

Figure 2. Index-histogram for the TGIWS arising from application of (2.1) and (2.6) to (3.1) with $q = 5$.

of F_n and two occurrences each of F_{n+1} and F_{n-1}, consistent with the row beginning $q = 1$ in the table.

Table 1 exhibits certain patterns that are summarized in the row beginning q. This motivates the following theorem, which is the main theorem of this paper. The five cases of the theorem will be proven in a later section of the paper.

Theorem 4.1. *Let q be a positive integer. Let $L(q) = 2q + 1$ and $K(q) = q$. Consider, the family of TGIWS obtained from* (3.1) *by applying* (2.1) *and* (2.6)*.*

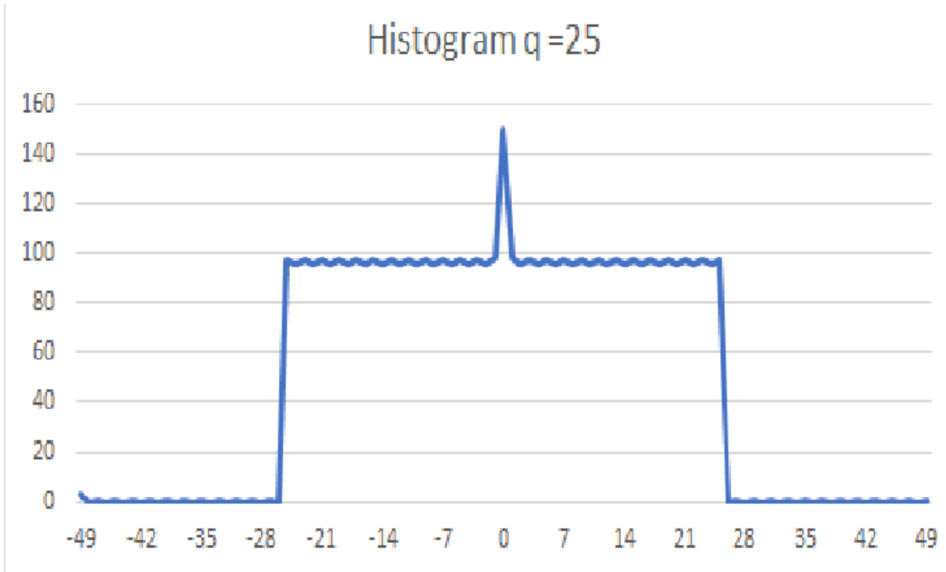

FIGURE 3. Index-histogram for the TGIWS arising from application of (2.1) and (2.6) to (3.1) with $q = 25$.

a) *For all* $q \geq 1$*, the weight (that is, number of occurrences) of index* n *is* $6q$*;*

b) *For all* $q \geq 1$*, the weight of indices* $n-1$ *and* $n+1$ *are each* $4q-2$*;*

c) *For all* $q \geq 2$*, if* e *is even with* $2 \leq |e| \leq q$*, then the weight of index* $n+e$ *is* $4q-4$*;*

d) *For all* $q \geq 3$*, if* o *is odd with* $3 \leq |o| \leq q$*, then the weight of index* $n+o$ *is* $4q-3$*;*

e) *For* $q \geq 2$*, if* O *is odd with* $q+1 \leq |O| \leq 2q-1$*, then the weight of index* $n+O$ *is 1.*

Corollary 4.2. *For given* $q \geq 1$*, the total number of indices, counting multiplicity, occurring in the TGIWS obtained from* (3.1) *by applying* (2.1) *and* (2.6)*, is* $8q^2+2$*.*

Proof. The cases $q = 1,\ 2$ can be verified manually using Table 1. For the rest of the proof, we assume $q \geq 3$.

The proof depends on the parity of q. We first assume q is even.

By the main theorem, there are five cases to consider.

- There is one index of the form n, and that index has weight $6q$.
- There are two indices of the form $n \pm 1$, each with weight $4q-2$.
- There are q indices of the form $n+e$ with e even and with $2 \leq |e| \leq q$, each with weight $4q-4$.
- There are $q-2$ indices of the form $n+o$ with o odd and with $3 \leq |o| \leq q$, each with weight $4q-3$.
- There are q indices of the form $n+O$ with O odd and with $q+1 \leq |O| \leq 2q-1$, each with weight 1.

It immediately follows that the total number of indices equals $6q \times 1 + (4q-2) \times 2 + (4q-4) \times q + (4q-3) \times (q-2) + 1 \times q = 8q^2+2$, as was to be shown.

The proof for the q is odd case is similar to the q is even case and hence is omitted. □

5. The Conjecture

The purpose of this section is to formulate a conjecture covering any one-parameter family of identities generated by a TGM and substitutions similar to (2.5) and (2.6).

The conjecture stated in [1], based on the one-parameter family of TGIWS obtained from TGM (2.10) with application of (2.1) and (2.5), is that the index-histogram of each TGIWS when restricted to its support, possibly excluding c points, where c is a constant not depending on q, has one value.

A glance at Figures 2 and 3 shows that this conjecture does not hold for the one-parameter family of TGIWS studied in this paper, obtained from TGM (3.1) with application of (2.1) and (2.6). More precisely, Theorem 4.1 shows that when q is even, there are q indices with weight 1, q indices with weight $4q-4$, and $q-2$ indices with weight $4q-3$. Similar observations hold for the q is odd case. Consequently, the assertion that the index-histogram of each member of the family of TGIWS, when restricted to its support, has one value except for c points, where c does not depend on q, is false.

To obtain a broader perspective, we can compare Theorem 4.1 of this paper with the Main Theorem of [1].

- The Main Theorem of [1] studies the one-parameter family of TGIWS generated by (2.10) with substitution (2.5). The Main Theorem asserts that for each member of this family, depending on the parity of the parameter q, the only weights that occur in index-histograms are $q2^{q-1}$, 2^{q-1}, $(q+1)2^{q-1}$, and 0. Thus, each member of this family has at most four weights occurring in its index-histogram.
- Theorem 4.1 of this paper studies the one-parameter family of TGIWS generated by applying (2.1) and (2.6) to (3.1). Theorem 4.1 shows that the *only* weights occurring in the index-histograms are $6q$, $4q-2$, $4q-4$, $4q-3$, 1, and 0. Thus, each member of this family has at most six weights occurring in its index-histogram.

Notice, that 4 and 6 are constants, that is, they do not depend on the parameter q. This motivates the following conjecture.

Conjecture 5.1. *For each one-parameter family of TGIWS, arising from a TGI with a substitution like* (2.5) *or* (2.6)*, there is a constant c, independent of q, such that the number of distinct weights in the index-histogram of any member of that TGIWS, is bounded by c.*

It seems reasonable that this conjecture is true for *all* one-parameter families of TGIWS or for a large portion of them. It also seems reasonable that the techniques in this paper and [1] can be used to develop a proof in the near future.

6. Proof of the Main Theorem

In this section, we prove the Main Theorem. We first review the setup. We begin with the start identity, (2.3), which has $2q+1$ summands, each a product of the form (2.2). To each of the $2q+1$ products we first apply to the related pair of indices listed in (3.1) the Tagiuri identity (2.1); after all applications of Tagiuri are made, we apply the substitutions (2.6). Then, following the requirements of Definition 2.2, we count, for each x, the total number of occurrences of F_{n+x}. The resulting collection of indices and counts gives us the histogram for the identity.

Note that for each $x \in \{-q, -(q-1), \ldots, q\}$, there is exactly one pair (x, y) in (3.1). As we prove each of the five assertions of the Main Theorem, it suffices to consider for each x,

$-q \leq x \leq q$, the effect on index count of applying (2.1) and (2.6). To facilitate this counting, we first consider the following four cases of application of (2.6) and (2.1) to (3.1); these four cases cover each x in $\{-q, \ldots, q\}$ exactly once.

Applying (2.1) and (2.6) with $x = a_q = -1$ and $y = a_{q+1} = 0$ yields,

$$\prod_{i=1}^{2q+1} F_{n+a_i} = P_1, \text{ with } P_1 = \prod_{i=-q}^{q} F_{n+i}. \tag{6.1}$$

Applying (2.1) and (2.6) with $x = a_{q+1} = 0$ and $y = a_{q+2} = 1$, yields

$$\prod_{i=1}^{2q+1} F_{n+a_i} = P_2, \text{ with } P_2 = \prod_{i=-q}^{q} F_{n+i}. \tag{6.2}$$

Applying (2.1) and (2.6) with $x = a_{2q+1} = q$ and $y = a_1 = -q$, yields

$$\prod_{i=1}^{2q+1} F_{n+a_i} = T_3 P_3, \text{ with } T_3 = \left(F_n^2 + (-1)^n F_q F_{-q}\right) \text{ and } P_3 = \prod_{-(q-1)}^{q-1} F_{n+i}. \tag{6.3}$$

Note, since $q \geq 1$, F_q and F_{-q} in (6.3) are non-zero integers.

Applying (2.1) and (2.6) with any $x, -q \leq x \leq q$, such that
$x \notin \{a_q = -1,\ a_{q+1} = 0,\ a_{2q+1} = q\}$ yields

$$\prod_{i=1}^{2q+1} F_{n+a_i} = T_4 P_4, \text{ with } T_4 = \left(F_n F_{n+2x+1} + (-1)^n F_x F_{x+1}\right) \text{ and } P_4 = \prod_{\substack{-q \leq i \leq q \\ i \notin \{x, x+1\}}} F_{n+i}. \tag{6.4}$$

The letters T and P in (6.3)–(6.4) refer to the two-summand parenthetical expression and product, respectively. Similarly, in (6.1)–(6.2), P refers to the product.

In the sequel, an expression of the form "there are s occurrences of F_z in $T_i P_i, i \in \{3, 4\}$" will, by convention, refer to the number of occurrences of F_z after parenthetical expansion of $T_i P_i$ into a sum of two products.

In (6.4), since $x \neq q$, it follows from (3.1) that $n+y = n+x+1$. Since $x \notin \{-1, 0\}$, it follows that F_x and F_{x+1} are non-zero integers. The reason for considering (6.1) and (6.2) separately from (6.3) and (6.4) is because in (6.1) and (6.2), either $F_x = 0$ or $F_{x+1} = 0$, so that the application of (2.1) produces one product, P_1 or P_2, whereas contrastively, the application of (2.1) in (6.3) and (6.4) produces, $T_3 P_3$ or $T_4 P_4$, which each consists of a sum of two products.

Finally, to avoid vacuous cases, we assume throughout the proof that $q \geq 3$. There is no harm in this restriction because the cases $q = 1$ and $q = 2$ can be managed manually. Table 1 contains these cases and, as can be seen, they exhibit the same patterns as the cases $q \geq 3$, except that in case $q = 1$, there are no indices with weights 1, $4q - 3$, or $4q - 4$, and similarly, in case $q = 2$, there are no indices with weight $4q - 3$.

Proof of (a). As outlined in the introduction of this section, we let x take on the $2q + 1$ values, $-q \leq x \leq q$, and after determining which of the four cases connected with (6.1)-(6.4) it belongs to, we count the number of occurrences of F_n in the corresponding product on the right side.

- When $x = -1$ and $y = 0$, P_1 in (6.1) has one occurrence of F_n.
- When $x = 0$ and $y = 1$, P_2 in (6.2) has one occurrence of F_n.
- When $x = q$ and $y = -q$, $T_3 P_3$ in (6.3) has four occurrences of F_n. To see this, note that P_3 has two occurrences of F_n. Therefore, $F_n^2 P_3$ has three occurrences of F_n, whereas $(-1)^n F_q F_{-q} P_3$ has one occurrence of F_n.

- Fix an $x \notin \{-1, 0, q\}$. Then T_4P_4 has three occurrences of F_n. To see this, note that P_4 in (6.4) has one occurrence of F_n. Therefore, $F_nF_{n+2x+1}P_4$ has two occurrences of F_n^2, and similarly, $(-1)^nF_xF_{x+1}P_4$ has one occurrence of F_n. Since we excluded three possibilities of x from $\{-q, -(q-1), \dots, 0, 1, \dots, q\}$, it follows there are $2q+1-3 = 2q-2$ possible $x \notin \{-1, 0, q\}$. Therefore, there is a total of $3 \times (2q-2) = 6q-6$ occurrences of F_n.

Aggregating the contributions arising from (6.1)–(6.4), we have a total contribution of $1+1+4+6q-6 = 6q$ distinct occurrences of F_n over all summands. This proves assertion (a). □

Proof of (b). There are two cases to consider, $x = 1$ and $x = -1$. We prove the case $x = 1$. The proof of the case $x = -1$ is almost identical and hence omitted.

As outlined in the introduction of this section, we let x take on the $2q+1$ values, $-q \le x \le q$, and we consider the contribution of the four cases arising from (6.1)–(6.4).

- When $x = -1$ and $y = 0$, P_1 in (6.1) has one occurrence of F_{n+1}.
- When $x = 0$ and $y = 1$, P_2 in (6.2) has one occurrence of F_{n+1}.
- When $x = q$ and $y = -q$, T_3P_3 in (6.3) contributes two occurrences of F_{n+1}, since P_3 has one occurrence of F_{n+1} and therefore, T_3P_3 has two occurrences of F_{n+1}.
- Equation (6.4) excludes consideration of the cases $x \in \{0, -1, q\}$, because they have already been dealt with in (6.1)–(6.3). We also must exclude the case $x = 1$, since P_4 does not contain indices in the set $\{n+x, n+x+1\}$. Accordingly, fix an $x \notin \{-1, 0, 1, q\}$. T_4P_4 in (6.4) has two occurrences of F_{n+1}. Since we excluded from $\{-q, -(q-1), \dots, 0, 1, \dots, q\}$ the four cases of $x \in \{-1, 0, 1, q\}$, it follows that there are $2q+1-4 = 2q-3$ possible $x \notin \{-1, 0, 1, q\}$. Thus, the total number of occurrences of index $n+1$ is $2 \times (2q-3) = 4q-6$.

Aggregating the contributions of all four cases, we have a total contribution of $1+1+2+4q-6 = 4q-2$ distinct occurrences of F_{n+1} over all summands. This proves assertion (b). □

Proof of (c). Fix even e with $2 \le |e| \le q$. As outlined in the introduction of this section, we let x take on the $2q+1$ values, $-q \le x \le q$, and after determining which of the four cases, (6.1)–(6.4), it is connected with, count the number of occurrences of F_{n+e} in the corresponding product on the right side of (6.1)–(6.4). Although the result is the same, the counting is different for the following three cases, (c1) $2 \le |e| \le q-1$, (c2) $e = q$, and (c3) $e = -q$. Note, that if q is not even, we need not consider cases (c2) and (c3).

We first consider case (c1), $2 \le |e| \le q-1$.

- When $x = -1$ and $y = 0$, P_1 in (6.1) has one occurrence of F_{n+e}.
- When $x = 0$ and $y = 1$, P_2 in (6.1) has one occurrence of F_{n+e}.
- When $x = q$ and $y = -q$, T_3P_3 of (6.3) contributes two occurrences of F_{n+e}. To see this, note that P_3 has one occurrence of F_{n+e} and therefore, T_3P_3 has two occurrences of F_{n+e}.
- Equation (6.4) requires us to exclude considerations of x satisfying $x \in \{-1, 0, q\}$. We must also exclude $x \in \{e-1, e\}$, since the product on the right side of (6.4) does not contain indices in the set $\{n+x, n+x+1\}$. Accordingly, fix an $x \notin \{-1, 0, e-1, e, q\}$. T_4P_4 in (6.4) has two occurrences of F_{n+e}. Since we have excluded five cases of x, we have contributions from $2q+1-5 = 2q-4$ cases of x. It follows that the total number of occurrences of index $n+e$ is $2 \times (2q-4) = 4q-8$.

Aggregating the contributions of all four cases, we have a total contribution of $1+1+2+4q-8 = 4q-4$ distinct occurrences of F_{n+e} over all summands. This proves assertion (c) for the case (c1).

We next consider the case (c2), $e = q$ with e even.

- When $x = -1$ and $y = 0$, P_1 in (6.1) has one occurrence of F_{n+e}.
- When $x = 0$ and $y = 1$, P_2 in (6.2) has one occurrence of F_{n+e}.
- When $x = q$ and $y = -q$, P_3 in (6.3) contributes no occurrences of $F_{n+e} = F_{n+q}$.
- Equation (6.4) requires us to exclude considerations of x satisfying $x \in \{-1, 0, q\}$. We must also exclude $x \in \{e-1 = q-1, e = q\}$, since P_4 in (6.4) does not contain indices in the set $\{n+x, n+x+1\}$. Accordingly, fix an $x \notin \{-1, 0, q, q-1\}$. T_4P_4 in (6.4) has two occurrences of F_{n+e}. Since we have excluded four cases of x, we have contributions from $2q+1-4 = 2q-3$ cases of x. It follows that the total number of occurrences of index $n+e$ is $2 \times (2q-3) = 4q-6$.

Aggregating the contributions of all four cases, we have a total contribution of $1+1+0+4q-6 = 4q-4$ distinct occurrences of F_{n+e} over all summands. This proves assertion (c) for the case (c2).

The case (c3), $x = -q$, is treated similarly and hence is omitted. This completes the proof of assertion (c). □

Proof of (d). Fix odd o with $3 \leq |o| \leq q$. As outlined in the introduction of this section, we let x take on the $2q+1$ values, $-q \leq x \leq q$, and we consider the contribution of the four cases arising from (6.1)–(6.4).

Note that the proof of assertion (d) is similar to the proof of assertion (c) except that there is an additional occurrence of $n+o$ from (6.4) arising when $2x+1 = o$. We again must consider three cases, (d1) $3 \leq |o| \leq q-1$, (d2) $o = q$, and (d3) $o = -q$. Note, that if q is not odd, we need not consider cases (d2) and (d3).

We first consider case (d1).

- When $x = -1$ and $y = 0$, P_1 in(6.1) has one occurrence of F_{n+o}.
- When $x = 0$ and $y = 1$, P_2 in (6.2) has one occurrence of F_{n+o}.
- When $x = q$ and $y = -q$, T_3P_3 in (6.3) contributes two occurrences of F_{n+o}. To see this, note that P_3 has one occurrence of F_{n+o} and therefore, T_3P_3 has two occurrences of F_{n+o}.
- Equation (6.4) requires us to exclude x satisfying $x \in \{-1, 0, q\}$. We must also exclude $x \in \{o-1, o\}$, since the product on the right side of (6.4) does not contain indices in the set $\{n+x, n+x+1\}$. Thus, when dealing with (6.4), we must exclude five cases of x, namely, those satisfying $x \in \{-1, 0, q, o-1, o\}$. We now consider two cases of $x \notin \{-1, 0, q, o-1, o\}$.
- We first deal with the case $x = x_0$, where x_0 is the unique solution to the equation $2x+1 = o$, with $-q \leq x \leq q$, and with $x \notin \{-1, 0, q, o-1, o\}$. Since o is odd with $3 \leq |o| \leq q$, such a unique solution always exists. When $x = x_0$, T_4P_4 in (6.4) has three occurrences of F_{n+o}, since $F_{n+2x_0+1}P_4 = F_{n+o}P_4$ has a factor of F^2_{n+o}, whereas $(-1)^n F_{x_0}F_{x_0+1}P_4$ has a factor of F_{n+o}.
- Next, consider $x \notin \{-1, 0, q, o-1, o, x_0\}$. T_4P_4 in (6.4) has two occurrences of F_{n+o}. Since we have excluded six cases of x, we have contributions from $2q+1-6 = 2q-5$ cases of x. These $2q-5$ cases contribute $2 \times (2q-5) = 4q-10$ cases of index $n+o$.

Aggregating the contributions of all five cases, we have a total contribution of $1+1+2+3+4q-10 = 4q-3$ distinct occurrences of F_{n+o} over all summands. This proves assertion (d) for the case (d1).

The proofs for the cases (d2) and (d3) are similar. We prove case (d3) and omit case (d2).

We consider case (d3) with $o = -q$ and q odd.

- When $x = -1$ and $y = 0$, P_1 in (6.1) has one occurrence of F_{n+o}.
- When $x = 0$ and $y = 1$, P_2 in (6.1) has one occurrence of F_{n+o}.
- When $x = q$ and $y = -q$, T_3P_3 in (6.3) contributes no occurrences of F_{n+o}.
- Equation (6.4) requires us to exclude x satisfying $x \in \{-1, 0, q\}$. We must also exclude $x \in \{o-1, o\}$, since the product on the right side of (6.4) does not contain indices in the set $\{n+x, n+x+1\}$. Thus, when dealing with (6.4), we must exclude four cases of x, namely, those satisfying $x \in \{o = -q, -1, 0, q\}$. We now consider two cases of $x \notin \{o = -q, -1, 0, q\}$.
- We first deal with the case $x = x_0$, where x_0 is the unique solution to the equation $2x + 1 = o = -q$, with $-q \leq x \leq q$, and with $x \notin \{o = -q, -1, 0, q\}$. Since o is odd with $3 \leq |o| \leq q$, such a unique solution always exists. When $x = x_0$, T_4P_4 in (6.4) has three occurrences of F_{n+o}, since $F_{n+2x_0+1}P_4 = F_{n+o}P_4$ has a factor of F^2_{n+o}, whereas $(-1)^n F_{x_0}F_{x_0+1}P_4$ has one factor of F_{n+o}.
- Next, consider $x \notin \{-q, -1, 0, q, x_0\}$. T_4P_4 in (6.4) has two occurrences of F_{n+o}. Since we have excluded five cases of x, we have contributions from $2q + 1 - 5 = 2q - 4$ cases of x. These $2q - 4$ cases contribute $2 \times (2q - 4) = 4q - 8$ cases of index $n + o$.

Aggregating the contributions of all five cases, we have a total contribution of $1 + 1 + 0 + 3 + 4q - 8 = 4q - 3$ distinct occurrences of F_{n+o} over all summands. This proves assertion (d) for the case (d3). This completes the proof of assertion (d). □

Proof of (e). Since $-q \leq i \leq q$ and $-q \leq x \leq q$, it follows that only the cases described in (6.4) can contribute an index of the form $n + O$ with $|O| > q$. Clearly, for each odd O with $q + 1 \leq O \leq 2q - 1$, there is a unique x with $1 \leq x \leq q - 1$ such that $2x + 1 = O$. A similar argument applies for each odd O with $-(2q-1) \leq O \leq -(q+1)$. This proves assertion (e). □

This completes the proof of the Main Theorem.

7. Conclusion

This paper has proven the Main Theorem, completely describing the index-histogram arising from (3.1). This paper also offered a general conjecture about index-histograms arising from TGM. We believe this is a good open problem that can be solved in the near future.

References

[1] R. Hendel, *Proof and generalization of the Cassini-Catalan-Tagiuri-Gould identities*, in *Proceedings of the Seventeenth International Conference on Fibonacci Numbers and Their Applications*, P. Anderson, C. Ballot, and T. Komatsu, (eds.), The Fibonacci Quarterly, **55.5** (2017), 76–85.

[2] T. Koshy, *Fibonacci and Lucas Numbers with Applications*, John Wiley and Sons, New York, NY, 2001.

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