ON THE RELATION BETWEEN FIBONACCI AND LUCAS NUMBERS

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Abstract. In the present work we explore the relation between Fibonacci and Lucas numbers. We, also, extend our result it in the case of Tribonacci numbers and Lucas numbers of order 3.

1. Introduction

Let F_n $(n \geq 0)$ and L_n $(n \geq 0)$ be the Fibonacci and Lucas numbers defined as

$$F_0 = 0,\ F_1 = 1,\ F_n = F_{n-1} + F_{n-2},\ n \geq 2 \tag{1.1}$$

$$L_0 = 2,\ L_1 = 1,\ L_n = L_{n-1} + L_{n-2},\ n \geq 2. \tag{1.2}$$

Among the various properties of the sequences, a class of special interest has to do with the demonstration of identities which present the relation between the Fibonacci and the Lucas numbers. Benjamin and Quinn [1] employed colored tiling to prove the following identity

$$\sum_{i=0}^{n} 2^i L_i = 2^{n+1} F_{n+1}. \tag{1.3}$$

Two different proofs of (1.3) were published recently by Sury [11] and Kwong [6].

Marques [7] proved the identity

$$\sum_{i=0}^{n} 3^i (L_i + F_{i+1}) = 3^{n+1} F_{n+1}, \tag{1.4}$$

and a different proof of (1.4) was published by Martinlak [8].

Edgar [4], encompassing both (1.3)and (1.4), proved, for any $m \geq 2$, the following more general identity (see, also, [3] and [9])

$$\sum_{i=0}^{n} m^i (L_i + (m-2) F_{i+1}) = m^{n+1} F_{n+1}. \tag{1.5}$$

We shall, presently, prove a different identity involving Fibonacci and Lucas numbers, analogous to (1.5). That we do in Section 2. In Section 3, we explore the relation between Tribonacci and Lucas numbers of order 3, by extending the aforementioned relation.

2. An identity involving Fibonacci and Lucas numbers

We presently state and prove the following theorem.

We thank the participants of the 19^{th} International Fibonacci Conference for useful comments on an earlier version.

Theorem 2.1. *Let F_n be the Fibonacci numbers and L_n be the Lucas numbers. Then, for any $m \geq 2$,*

$$\sum_{i=0}^{n} (-1)^i m^{n-i} \ (L_{i+1} + (m-2) F_i) = (-1)^n F_{n+1}. \tag{2.1}$$

Proof. By (1.1) and (1.2)

$$L_n = F_{n-1} + F_{n+1}, \ n \geq 1,$$

or

$$L_{n+1} = F_n + F_{n+2} = 2F_n + F_{n+1}, \ n \geq 0.$$

Therefore,

$$\begin{aligned} \sum_{i=0}^{n} \ (-1)^i \quad & m^{n-i} \ (L_{i+1} + (m-2) F_i) \\ = \quad & m^n F_1 - m^{n-1}(mF_1 + F_2) + m^{n-2}(mF_2 + F_3) - \cdots + (-1)^n(mF_n + F_{n+1}) \\ = \quad & (-1)^n F_{n+1}, \end{aligned}$$

which was to be shown.

□

In the following section, we shall extend Theorem 2.1 in the case of Tribonacci and Lucas numbers of order 3.

3. An extension

Let T_n $(n \geq 0)$ be the Tribonacci numbers (see, for example, [5] and [10]) and V_n $(n \geq 0)$ be the Lucas numbers of order 3 (see, for example, [2]) defined as

$$T_0 = 0, \ T_1 = 1, \ T_2 = 1, \ T_n = T_{n-1} + T_{n-2} + T_{n-3}, \ n \geq 3 \tag{3.1}$$
$$V_0 = 3, \ V_1 = 1, \ V_2 = 3, \ V_n = V_{n-1} + V_{n-2} + V_{n-3}, \ n \geq 3. \tag{3.2}$$

We shall first prove the following lemma.

Lemma 3.1. *Let T_n $(n \geq 0)$ be the Tribonacci numbers and V_n $(n \geq 0)$ be the Lucas numbers of order 3, respectively. Set $T_{-1} = 0$. Then, for $n \geq 1$,*

$$V_n = T_n + 2T_{n-1} + 3T_{n-2}. \tag{3.3}$$

Proof. We shall prove (3.3) by strong induction. We first consider the base case. When $n = 1$, the LHS of (3.3) is $V_1 = 1$, and the RHS is $T_1 + 2T_0 + 3T_{-1} = 1$ (since $T_{-1} = 0$ by assumption), so (3.3) holds. When $n = 2$, the LHS of (3.3) is $V_2 = 3$, and the RHS is $T_2 + 2T_1 + 3T_0 = 1 + 2 = 3$, so (3.3) holds. When $n = 3$, the LHS of (3.3) is $V_3 = 7$, and the RHS is $T_3 + 2T_2 + 3T_1 = 2 + 2 + 3 = 7$, so (3.3) holds.

We proceed to the induction step. Let $k \in \mathbb{N}$ with $k \geq 3$ be given and suppose (3.3) is true for $n = 1, 2, \ldots, k$. Then, we have to prove that (3.3) holds true for $n = k + 1$, i.e.

$$V_{k+1} = T_{k+1} + 2T_k + 3T_{k-1}. \tag{3.4}$$

In fact,

$$\begin{aligned}
V_{k+1} &= V_k + V_{k-1} + V_{k-2}, \text{ by (3.2)}, \\
&= T_k + 2T_{k-1} + 3T_{k-2} + T_{k-1} + 2T_{k-2} + 3T_{k-3} + T_{k-2} + 2T_{k-3} + 3T_{k-4}, \\
&\quad \text{by the induction hypothesis or by (3.3) } for\ n = k, k-1, k-2, \\
&= T_k + T_{k-1} + T_{k-2} + 2(T_{k-1} + T_{k-2} + T_{k-3}) + 3(T_{k-2} + T_{k-3} + T_{k-4}) \\
&= T_{k+1} + 2T_k + 3T_{k-1}, \text{by (3.1)},
\end{aligned}$$

which was to be proved.

By the strong induction principle, it follows that (3.3) is true for all $n \geq 1$. □

Having proved Lemma 3.1, we shall state and prove the following theorem.

Theorem 3.2. *Let $(T_n)_{n\geq 0}$ be the sequence of Tribonacci numbers and $(V_n)_{n\geq 0}$ be the sequence of Lucas numbers of order* 3*. Set $T_{-1} = 0$. Then, for any $m \geq 2$,*

$$\sum_{i=0}^{n} (-1)^i m^{n-i} \left(V_{i+1} + (m-2)\, T_i - 3T_{i-1} \right) = (-1)^n T_{n+1}. \tag{3.5}$$

Proof. We shall prove (3.5) by induction on n. When $n = 0$, the LHS of (3.5) is $V_1 = 1$ (since $T_{-1} = 0$ by assumption), and the RHS is $T_1 = 1$, so (3.5) holds.

We proceed to the induction step. Let $k \in \mathbb{N}$ with $k \geq 0$ be given and suppose (3.5) is true for $n = k$, i.e.

$$\sum_{i=0}^{k} (-1)^i m^{k-i} \left(V_{i+1} + (m-2)\, T_i - 3T_{i-1} \right) = (-1)^k T_{k+1}. \tag{3.6}$$

Then, we have to prove that (3.5) holds true for $n = k+1$, i.e.

$$\sum_{i=0}^{k+1} (-1)^i m^{k+1-i} \left(V_{i+1} + (m-2)\, T_i - 3T_{i-1} \right) = (-1)^{k+1} T_{k+2} \Leftrightarrow$$

$$\Leftrightarrow \sum_{i=0}^{k+1} (-1)^i m^{k-i} \left(V_{i+1} + (m-2)\, T_i - 3T_{i-1} \right) = \frac{(-1)^{k+1} T_{k+2}}{m}.$$

Making use of (3.6), we may rewrite the last equality as

$$(-1)^k T_{k+1} + \frac{(-1)^{k+1} \left(V_{k+2} + (m-2)\, T_{k+1} - 3T_k \right)}{m} = \frac{(-1)^{k+1} T_{k+2}}{m} \Leftrightarrow$$

$$\Leftrightarrow m(-1)^k T_{k+1} + (-1)^{k+1} \left(V_{k+2} + (m-2)\, T_{k+1} - 3T_k \right) = (-1)^{k+1} T_{k+2},$$

and, employing Lemma 3.1, the last equality can be written in the form

$$m(-1)^k T_{k+1} + (-1)^{k+1} \left(T_{k+2} + 2T_{k+1} + 3T_k + (m-2)\, T_{k+1} - 3T_k \right) = (-1)^{k+1} T_{k+2} \Leftrightarrow$$

$$\Leftrightarrow (-1)^k m T_{k+1} + (-1)^{k+1}\, T_{k+2} - (-1)^k m T_{k+1} = (-1)^{k+1} T_{k+2},$$

which holds true. This completes the induction and the proof of the theorem. □

References

[1] A. T. Benjamin and J. J. Quinn, Fibonacci and Lucas identities through colored tilings, *Util. Math.* **56** (1999), 137–142.

[2] C. A. Charalambides, Lucas numbers and polynomials of order k and the length of the longest circular success run, *Fibonacci Quart.* **29** (1991), 290–297.

[3] S. D. Dafnis, A. N. Philippou, and I. E. Livieris, An identity relating Fibonacci and Lucas numbers of order k. *TCDM 2018–2nd IMA Conference on Theoretical and Computational Discrete Mathematics*, 37–42, Electron. Notes Discrete Math., 70, Elsevier Sci. B. V., Amsterdam, 2018.

[4] T. Edgar, Extending some Fibonacci-Lucas relations, *Fibonacci Quart.* **54** (2016), 78–79.

[5] M. Feinberg, Fibonacci-Tribonacci, *Fibonacci Quart.* **1** (1963), 71–74.

[6] H. Kwong, An alternate proof of Sury's Fibonacci-Lucas relation, *Amer. Math. Monthly* **121** (2014), 514–514.

[7] D. Marques, A new Fibonacci-Lucas relation, *Amer. Math. Monthly* **122** (2015), 683–683.

[8] I. Martinjak, Two extensions of the Sury's identity, *Amer. Math. Monthly* **123** (2016), 919–919.

[9] A. N. Philippou and S. D. Dafnis, A simple proof of an identity generalizing Fibonacci-Lucas identities, *Fibonacci Quart.* **56** (2018), 334–336.

[10] A. N. Philippou and A. A. Muwafi, Waiting for the k-th consecutive success and the Fibonacci sequence of order k, *Fibonacci Quart.* **20** (1982), 28–32.

[11] B. Sury, A polynomial parent to a Fibonacci-Lucas relation, *Amer. Math. Monthly* **121** (2014), 236–236.

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