AN IDENTITY FOR INVERSE-CONJUGATE COMPOSITIONS

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Abstract. We prove a combinatorial identity between two classes of inverse-conjugate compositions, that is, integer compositions whose conjugates are given by a reversal of their sequences of parts. These are the set of inverse-conjugate compositions of $2n+3$ without 2's, and the set of inverse-conjugate compositions of $2n-1$ with parts not exceeding 3. Both sets are enumerated by $2F_n$, where F_n is the nth Fibonacci number.

1. Introduction

A composition of a positive integer n is a representation of n as a sequence of positive integers $(c_1, \dots, c_k)$ that sum to n. The terms c_i are called parts, while n is the *weight*, of the composition. For example, there are eight compositions of $n = 4$, namely

$$(4), (1,3), (2,2), (3,1), (1,1,2), (1,2,1), (2,1,1), (1,1,1,1).$$

The *conjugate of a composition* $C = (c_1, \dots, c_k)$ may be obtained by drawing its *zig-zag graph.* The latter is constructed by depicting each part c_i by a row of c_i dots such that the first dot on a row is aligned with the last dot on the previous row. The conjugate of a composition C will be denoted by C'.

For example, the zig-zag graph of $C = (5,3,1,3,1)$ is shown in Figure 1.

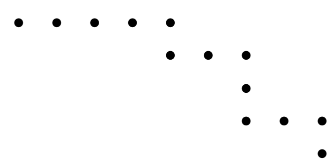

Figure 1

The conjugate is the composition corresponding to the columns of the graph, from left to right. Thus the conjugate of C, from Figure 1, is $C' = (1,1,1,1,2,1,3,1,2)$.

The *inverse* of a composition C, denoted by $\overline{C}$, is the composition obtained by reversing the sequence of the parts of C. A composition C is called *inverse-conjugate* if it satisfies $C' = \overline{C}$. For example, $C = (5,3,1,3,1)$ (as above) is not inverse conjugate since $C' \neq (1,3,1,3,5) = \overline{C}$; but it can be readily verified that $(3,1,1,2,4,1,1)$ is an inverse-conjugate composition of 13.

Let $IC(N, \hat{2})$ be the number of inverse-conjugate compositions of N without 2's, and let $IC_k(N)$ be the number of inverse-conjugate compositions of N with parts less than or equal to k, $k > 0$.

The purpose of this paper is to provide a bijective proof of the following identity.

Theorem 1.1. *Given an integer $n > 1$, we have*

$$IC(2n+3, \hat{2}) = IC_3(2n-1). \tag{1.1}$$

Both numbers are equal to $2F_n$, where F_n is the nth Fibonacci number defined by

$$F_1 = F_2 = 1,\ F_n = F_{n-1} + F_{n-2},\ n > 2.$$

Example 1. If $n = 4$, then $IC(11, \hat{2}) = IC_3(7) = 2F_4 = 6$, where the corresponding sets of compositions are given by

$$\begin{aligned} IC(11, \hat{2}) &: (1,1,1,1,1,6), (6,1,1,1,1,1), (1,1,1,3,1,4), (3,1,1,4,1,1), \\ &\quad (1,1,4,1,1,3), (4,1,3,1,1,1); \\ IC_3(7) &: (1,1,2,3), (2,1,3,1), (1,3,1,2), (3,2,1,1), (1,2,2,2), (2,2,2,1). \end{aligned}$$

The statement $IC_3(2n-1) = 2F_n$ is a special case of a general theorem proved in [3] using recurrence relations, and the equality $IC(2n+3, \hat{2}) = 2F_n$ may be deduced from a result established in [7].

However, to the best of our knowledge there has not been a direct association of the enumeration functions, $IC_3(2n-1)$ and $IC(2n+3, \hat{2})$, until now. Since both functions enumerate special classes of inverse-conjugate compositions, a purely bijective proof of their equality is expected to highlight some of the rich structure of these compositions.

We will prove (1.1) by building a bridge between the enumerated sets by means of the following known result about compositions into odd parts (see, for example, [1, 4]).

Proposition 1.2. *The number of compositions of n into odd parts is F_n.*

We will collect relevant properties of inverse-conjugate compositions in Section 2. Then in Section 3 we demonstrate that $\frac{1}{2}IC(2n+3, \hat{2}) = F_n = \frac{1}{2}IC_3(2n-1)$; thus (1.1) would follow.

2. Properties of Inverse-Conjugate Compositions

We recall few properties of inverse-conjugate compositions that will be used in the next section.

I. **Alternative Conjugation Rule**. It will be convenient to abbreviate compositions by representing a maximal string of 1's of length x by 1^x, where two adjacent big parts (i.e., parts > 1) are assumed to be separated by 1^0. Then the general composition has the following two forms up to inversion.

(1) $C = (1^{a_1}, b_1, 1^{a_2}, b_2, \dots),\ a_1 > 0, a_i \geq 0, i > 1,\ b_i \geq 2\ \forall\, i$;

(2) $C = (b_1, 1^{a_1}, b_2, 1^{a_2}, \dots),\ a_i \geq 0, b_i \geq 2$.

The conjugate, in each case, is given by the rule:

(1′) $C' = (a_1 + 1, 1^{b_1 - 2}, a_2 + 2, 1^{b_2 - 2}, \dots)$.

(2′) $C' = (1^{b_1 - 1}, a_1 + 2, 1^{b_2 - 2}, a_2 + 2, \dots)$.

For example, the conjugate of $(1,1,1,5,3,1,2) = (1^3, 5, 1^0, 3, 1, 2)$ is given by $(4, 1^3, 2, 1, 3, 1)$, that is, $(1,1,1,5,3,1,2)' = (4, 1^3, 2, 1, 3, 1)$.

II. **The Shape of an Inverse-Conjugate Composition** (see [6, 8]). The following lemma may be verified by means of the foregoing conjugation rule.

Lemma 2.1. *An inverse-conjugate composition C (or its inverse) has the form:*

$$C = (1^{b_r - 1}, b_1, 1^{b_{r-1} - 2}, b_2, 1^{b_{r-2} - 2}, b_3, \dots, b_{r-1}, 1^{b_1 - 2}, b_r),\ b_i \geq 2. \tag{2.1}$$

The following properties follow at once from Lemma 2.1.

(a) The weight of C is an odd integer.

(b) The composition C is completely determined by the sequence of big parts. Every non-empty sequence of integers > 1 generates two inverse-conjugate compositions such that one composition has first part equal to 1 and the other has a first part > 1. The respective conversion transformations are defined by

$$f_1 : \ (b_1, \ldots, b_r) \mapsto (1^{b_r - 1}, b_1, 1^{b_{r-1}-2}, b_2, \ldots, b_{r-1}, 1^{b_1 - 2}, b_r);$$
$$f_2 : \ (b_1, \ldots, b_r) \mapsto (b_1, 1^{b_r - 2}, b_2, 1^{b_{r-1}-2}, \ldots, 1^{b_2 - 2}, b_r, 1^{b_1 - 1}).$$

(c) The number of compositions enumerated by $IC(N, \hat{2})$ with first part 1 is equal to the number of compositions enumerated by $IC(N, \hat{2})$ with first part > 1. Similarly for $IC_3(N)$.
This property is a consequence of the definitions of the enumeration functions, and the transformations f_1 and f_2 which preserve big parts. It is illustrated in Example 1, whereby the compositions are displayed in pairs according to the generating sequences of big parts.

3. Bijective Proof of Theorem 1.1

If $T(n)$ is an enumeration function for compositions, then the set of objects counted by $T(n)$ will be denoted by $\{T(n)\}$. For example, $IC(N, \hat{2}) = |\{IC(N, \hat{2})\}|$.

We will also use the following notations:

$T(n)_1$	is the number of compositions counted by $T(n)$ with first part 1
$C_{odd}(n)$	is the number of compositions of n into odd parts
$C_{>1}(n)$	is the number of compositions of n into parts > 1
$C_{(1,2)}(n)$	is the number of compositions of n into 1's and 2's with first and last parts 1
$CC_k(n)$	is the number of compositions E of n when parts of E and E' are $\leq k$.

3.1. **The Bijection** $\{IC(2n+3, \hat{2})_1\} \to \{C_{odd}(n)\}$

First, define the map $h : \{IC(2n+3, \hat{2})_1\} \to \{C_{>1}(n+1)\}$ by

$$h : (1^{b_r - 1}, b_1, 1^{b_{r-1}-2}, \cdots, 1^{b_1 - 2}, b_r) \mapsto (b_1 - 1, b_2 - 1, \ldots, b_r - 1), \ b_i > 2.$$

Second, define the (conjugation) map $g : \{IC_{>1}(n+1)\} \to \{C_{(1,2)}(n)\}$ by $g : E \mapsto E'$.

Third, define the map $u : \{C_{(1,2)}(n)\} \to \{C_{odd}(n)\}$ as follows: u acts on $E \in \{C_{(1,2)}(n)\}$ by deleting the last part, and replacing every maximal string of the form $1, 2, \ldots, 2$ with the sum of its parts.

Clearly the maps h, g and u are reversible, and so are bijections.

Hence the bijection $\alpha : \{IC(2n+3, \hat{2})_1\} \to \{C_{odd}(n)\}$ may be specified by

$$\alpha = ugh \qquad \text{and} \qquad \alpha^{-1} = h^{-1} g^{-1} u^{-1}. \tag{3.1}$$

Example 2. Let $n = 14$ and consider $C = (1^3, 3, 1^2, 3, 1^4, 6, 1, 4, 1, 4) \in \{IC(31, \hat{2})_1\}$. Then

$$\begin{aligned} \alpha(C) = ugh(C) &= ug((2, 2, 5, 3, 3)) \\ &= u((2, 2, 5, 3, 3)') = u((1, 2, 2, 1^3, 2, 1, 2, 1^2)) \\ &= u(((1, 2, 2), 1, 1, (1, 2), (1, 2), 1^2)) \\ &= (5, 1, 1, 3, 3, 1) \ \in \ \{C_{odd}(14)\}. \end{aligned}$$

3.2. **The Bijection** $\{IC_3(2n-1)_1\} \to \{C_{odd}(n)\}$

First, define the map $w : \{IC_3(2n-1)_1\} \to \{CC_3(n)_1\}$. We observe that w is a restriction, to compositions with parts ≤ 3, of the classical MacMahon bijection between inverse-conjugate compositions of $2n - 1$ and compositions of n ([5] but see [3]). The full bijection preserves part-sizes. Let $(1, c_1, c_2, \ldots) \in \{CC_3(n)_1\}$. Then, using '|' to denote concatenation, we have

$$w^{-1} : (1, c_1, \ldots, c_k, 1) \mapsto (1, c_1, \ldots, c_k, 1)|\overline{(1, c_1, \ldots, c_k, 1)'}$$

and

$$w^{-1} : (1, c_1, \ldots, c_r) \mapsto (1, c_1, \ldots, c_r)|\overline{(1, c_1, \ldots, c_r - 1)'},\ c_r > 1.$$

Conversely w may be defined by splitting any $E \in \{IC_3(2n-1)_1\}$ into two sub-compositions whose weights differ by 1. As examples we have

$$w((1, 2, 1, 3, 1, 3, 2)) = w((1, 2, 1, 3)|(1, 3, 2)) = (1, 2, 1, 3);$$
$$w^{-1}((1, 2, 3, 1)) = (1, 2, 3)|\overline{(1, 2, 3, 1)'} = (1, 2, 3)|(2, 1, 2, 2) = (1, 2, 3, 2, 1, 2, 2).$$

Second, define the map $v : \{CC_3(n)_1\} \to \{C_{odd}(n)\}$. Yuhong Guo [2] has proved that v is a bijection. Since Guo's proof contains some nontrivial transformations relative to our original identity, we reproduce it below.

Note that any $C \in \{CC_3(n)_1\}$ may contain at most two initial 1's. Thus if C has one initial 1, then $C \mapsto v(C) \in \{C_{odd}(n)\}$; otherwise the first part of the conjugate C' is 3 and $C \mapsto v(C') \in \{C_{odd}(n)\}$. Conversely if $R \in \{C_{odd}(n)\}$, then $R \mapsto v^{-1}(R) \in \{CC_3(n)_1\}$ or $R \mapsto v^{-1}(R)$ with $v^{-1}(R)' \in \{CC_3(n)_1\}$, depending on whether the first part of R is 1 or > 1, respectively.

We now describe the function v.

If $2 \notin C \in \{CC_3(n)_1\}$, then $v(C) = C \in \{C_{odd}(n)\}$. Otherwise let A be the composition obtained from C by replacing every instance of the string "1,2" by "1,1,1". Then replace each maximal string of the form $1, 2, 2, \ldots$ or $3, 2, 2, \ldots$ in A by the sum of its parts. Set the resulting composition equal to $v(C)$. Clearly $v(C) \in \{C_{odd}(n)\}$.

Conversely we obtain v^{-1} using the following algorithm. Consider any $R \in \{C_{odd}(n)\}$.
(i) Replace every string of r ones, $r \geq 3$, by $1, 2, 1, 2, \ldots$ from left to right, to produce a composition B which has at most two 1's immediately before an odd part.
(ii) Let $d > 1$ be an odd part of B. If the string "$1, 1, d$" occurs, then replace d with its partition of the form "$1, 2, \ldots, 2$", otherwise replace d with its partition of the form $3, 2, \ldots, 2$, to obtain a composition S.
(iii) Replace every occurrence of the string "$1, 1, 1$" in S by "$1, 2$" to obtain $v^{-1}(R)$.

This shows that v is a bijection.

For example, $v((1, 1, 2, 2, 3, 2, 3, 1)) = (7, 1, 1, 3, 1, 1, 1)$ is obtained as follows

$$(1, 1, 2, 2, 3, 2, 3, 1) \to (3, 2, 2, 1, 2, 2, 1, 2) \to (3, 2, 2, 1, 1, 1, 2, 1, 1, 1) \to (7, 1, 1, 3, 1, 1, 1);$$

and conversely, $v^{-1}((7, 1, 1, 3, 1, 1, 1)) = (1, 1, 2, 2, 3, 2, 3, 1)$ is obtained as follows

$$(7, 1, 1, 3, 1, 1, 1) \to (7, 1, 1, 3, 1, 2) \to (3, 2, 2, 1, 1, 1, 2, 1, 2) \to (3, 2, 2, 1, 2, 2, 1, 2)$$
$$\to (1, 1, 2, 2, 3, 2, 3, 1).$$

Hence the bijection $\beta : \{IC_3(2n-1)_1\} \to \{C_{odd}(n)\}$ may be specified by

$$\beta = vw \qquad \text{and} \qquad \beta^{-1} = w^{-1}v^{-1}. \tag{3.2}$$

Lastly, we deduce from earlier remarks, with property (c) in Section 2, the following bijection proving Theorem 1.1.

$$\Theta : \left\{IC(2n+3, \hat{2})_1\right\} \to \{IC_3(2n-1)_1\},$$

where

$$\Theta = \beta^{-1}\alpha = w^{-1}v^{-1}ugh \tag{3.3}$$

Example 3. In Example 2 we found that $C = (1^3, 3, 1^2, 3, 1^4, 6, 1, 4, 1, 4) \in \{IC(31, \hat{2})_1\}$ gives $\alpha(C) = (5, 1, 1, 3, 3, 1) \in \{C_{odd}(14)\}$. So under Θ we obtain

$$\begin{aligned}\Theta(C) = \beta^{-1}\alpha(C) = \cdots = \beta^{-1}((5,1,1,3,3,1)) &= w^{-1}v^{-1}((5,1,1,3,3,1)) \\ &= w^{-1}((3,2,1,1,1,2,3,1) \to (3,2,1,2,2,3,1) \to (1,1,2,3,2,2,1,2)) \\ &= w^{-1}((1,1,2,3,2,2,1,2)) \\ &= (1,1,2,3,2,2,1,2)|\overline{(1,1,2,3,2,2,1,1)'} \\ &= (1,1,2,3,2,2,1,2)|(3,2,2,1,2,3) \\ &= (1,1,2,3,2,2,1,2,3,2,2,1,2,3) \ \in \ \{IC_3(27)_1\}.\end{aligned}$$

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