SOME GENERALIZATIONS OF A FORMULA OF REZNICK

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Abstract. In 2008, Reznick published a formula for the statistical behavior of Stern's sequence modulo m. We reprove this result and, using it, prove similar results for other sequences.

1. Introduction

For a given integer sequence (x_n), we define its *distribution modulo* m as the numbers

$$P(a,m) := \lim_{N\to\infty} \frac{1}{N} \left|\{n \le N : x_n \equiv a \mod m\}\right|. \tag{1.1}$$

These limits, of course, do not necessarily exist in which case we say that (x_n) has no distribution modulo m. A good reference for this topic is the last chapter of [9]. It is easy to see that in the special case where (x_n) is periodic these numbers do exist (for all m). For example, for a fixed m, $(F_n, F_{n+1}) \mod m$ has only finitely many possible values and so is eventually periodic for all m (in fact periodic since the map $(a,b) \mapsto (b, a+b)$ is invertible). More is known about the distribution of the Fibonacci sequence: Niederreiter [12] has shown that if m is a power of 5 then the distribution is uniform (i.e., $P(a,m) = P(b,m)$ for all a,b) modulo m; Kuipers and Shiu [10] have shown the converse.

For non-periodic sequences, other techniques must be used. In 2006, Reznick [18] showed that Stern's sequence, defined by $a_1 = 1$, $a_{2n} = a_n$, $a_{2n+1} = a_n + a_{n+1}$, has distribution

$$P(a,m) = \frac{1}{m} \prod_{p|m} \frac{p^2}{p^2-1} \prod_{p|(a,m)} \frac{p-1}{p}. \tag{1.2}$$

We shall give a new proof of this fact [Theorem 3.5] using Markov chains. This technique was mentioned, but not used, in [18].

A consequence of (1.2) is

$$P(i, m^k) = \frac{P(i,m)}{m^{k-1}}. \tag{1.3}$$

From this we prove that, for all m, $\left\lfloor \frac{a_n}{m} \right\rfloor$ is uniformly distributed modulo m^k for all k [Corollary 3.6].

In [15], an analogue (b_n) of Stern's sequence, using $x \oplus y = x + y + \sqrt{1+4xy}$ instead of $x + y$ in its definition, was introduced:

$$b_1 = 0, b_{2n} = b_n, b_{2n+1} = b_n \oplus b_{n+1} = b_n + b_{n+1} + \sqrt{1 + 4b_nb_{n+1}}.$$

Using the identity (Theorem 3.6 of [15])

$$b_k = a_{2^{j+1}-k} \cdot a_{k-2^j}, (2^j \le k \le 2^{j+1}), \tag{1.4}$$

it follows that (b_n) has distribution

$$P(i,m) = \frac{1}{m}\prod_{p|m}\frac{p}{p+1}\prod_{p|(i,m)} 2 = \frac{2^{\omega((i,m))}}{\psi(m)} \tag{1.5}$$

where $\omega(m)$ is the number of distinct prime divisors of m and $\psi(m)$ is Dedekind's psi-function [Theorem 4.2]. For this sequence, (1.3) holds and thus $\lfloor \frac{b_n}{m} \rfloor$ is uniformly distributed modulo m^k for all k [Corollary 4.3].

It is easy to see that

$$P(i,m) = \frac{1}{m}\prod_{p|(i,m)}\frac{f(p)}{p^{-1}} \cdot \prod_{\substack{p|m \\ p\nmid(i,m)}}\frac{1-f(p)}{1-p^{-1}} \tag{1.6}$$

where $f(p) = \dfrac{1}{p+1}$ and $f(p) = \dfrac{2}{p+1}$ for the distributions of (a_n) and (b_n) respectively. The arguments for Corollaries 3.6 and 4.3 carry over to any sequence with distribution of the form (1.6). Two questions come to mind: What sequences have distribution of the form (1.6)? What functions $f(p)$ are "represented" by a sequence with distribution (1.6)?

By equation (1.4), it turns out that the values of (b_n) are those attained by the quadratic form $Q(x,y) = xy$ over all the pairs of relatively prime non-negative integers x, y. This points to our next result: for a primitive integral quadratic form $Q(x,y)$ with discriminant Δ, when $\gcd(m,\Delta) = 1$, the sequence $(Q(a_n, a_{n+1}))$ has distribution of the form (1.6) where

$$f(p) = \frac{1+\left(\frac{\Delta}{p}\right)}{p+1} \tag{1.7}$$

(here (Δ/p) is the usual Legendre symbol when p is odd, and is specially defined when $p = 2$) [Theorem 5.4]. Hence $x_n := Q(a_n, a_{n+1})$ satisfies, for all m relatively prime to Δ, $\lfloor x_n/m \rfloor$ is uniform mod m^k for all k [Corollary 5.5].

Lastly, we consider the sequence (R_n) where R_n is the number of ways to represent n as a sum of distinct Fibonacci numbers. This sequence, though similar to Stern's sequence, does not share a distribution of form (1.6). In this case, we show that $P(0,m) = 1$ for all m [Theorem 6.3].

I thank my colleague Naveen Somasunderam for suggesting the problem "What is the distribution of $a_n \mod m$?", and Keith Conrad for answering, on MathOverflow, two questions that helped complete a proof of Theorem 5.4.

2. Preliminaries

To show the limits

$$P(a,m) := \lim_{N\to\infty}\frac{1}{N}\left|\{n \le N : x_n \equiv a \mod m\}\right| \tag{2.1}$$

exist, we rely on the following lemma. Suppose $c_k \in \{0,1\}$ for all k and define, for $m < n$,

$$A(m,n) := \frac{1}{n-m}\sum_{k=m+1}^{n} c_k. \tag{2.2}$$

Lemma 2.1. *If L is a number such that for all $\epsilon > 0$ there exists some j such that $A(2^j k, 2^j(k+1))$ is within ϵ of L for any k, then $\lim_{N\to\infty} A(0,N)$ exists and equals L.*

Proof. For any m, n with $m < n$, $A(2^j m, 2^j n)$ is the average of several values of the form $A(2^j k, 2^j(k+1))$ and so is itself within ϵ of L. If $N = 2^j m + i$, $0 < i < 2^j$, then

$$A(0, N) = \frac{2^j m}{2^j m + i} A(0, 2^j m) + \frac{i}{2^j m + i} A(2^j m, 2^j m + i). \tag{2.3}$$

On the right, the first term is between $\frac{m}{m+1}(L - \epsilon)$ and $L + \epsilon$ while the second term is between 0 and $\frac{1}{m}$. For N large enough, $A(0, N)$ is within 2ϵ of L. The result follows. □

3. Stern's sequence modulo m

3.1. **Stern's diatomic array and sequence.** Stern's diatomic array, sometimes thought of as "Pascal's triangle with memory", begins thus:

1																1
1								2								1
1				3				2				3				1
1		4		3		5		2		5		3		4		1
1	5	4	7	3	8	5	7	2	7	5	8	3	7	4	5	1
.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.

It is defined recursively: Start with row 1 1. Then, given the nth row, define the next one by copying the numbers on the nth row but inserting, in each gap, the sum of the two numbers above.

The numbers in the diatomic array, read like a book (but deleting the right-most column of 1s), form what is known as Stern's diatomic sequence which begins (for $n = 1, 2, \ldots$):

$$a_n = 1, 1, 2, 1, 3, 2, 3, 1, 4, 3, 5, 2, 5, 3, 4, 1, 5, 4, 7, 3, 8, 5, 7, 2, 7, 5, 8, 3, 7, 4, 5, 1, 6, \ldots \tag{3.1}$$

It is defined recursively by

$$a_1 = 1, \qquad a_{2n} = a_n, \qquad a_{2n+1} = a_n + a_{n+1}. \tag{3.2}$$

See [13] and its references, and sequence A002478 of [16], for information about this exceptional, and exceptionally well studied, sequence.

A key result for us is a well-know result (e.g., Theorem 5.1 of [13]).

Proposition 3.1. *Every ordered pair of relatively prime positive integers appears exactly once in the sequence* (a_n, a_{n+1}).

3.2. **Calkin-Wilf and Stern-Brocot Trees.** We introduce a tree that first appeared in "Recounting the Rationals" [3] by Calkin and Wilf. See also Section 2.2 of [14]. Starting with $\frac{1}{1}$, we repeatedly apply the two maps $L : \frac{a}{b} \mapsto \frac{a}{a+b}$ and $R : \frac{a}{b} \mapsto \frac{a+b}{b}$.

It is easy to see that if $r_n := a_n / a_{n+1}$, then for all n

$$L : r_n \mapsto r_{2n} \text{ and } R : r_n \mapsto r_{2n+1}. \tag{3.3}$$

It follows, by Proposition 3.1, that the sequence (r_n) is an enumeration of the positive rationals and that every positive rational appears exactly once on the Calkin-Wilf tree.

We may assign an "address" to each node of a binary rooted tree with a word in $\{L, R\}^*$ via the obvious interpretation. For example, in the CW tree, 5/2 is at location LRR and 1/1 is at location addressed by the "empty word" $*$. We define a new rooted binary tree: to a node with address ω, assign the value on the CW tree with address ω', the reverse of the word ω. For example, at node RRL, we assign the value 5/2. This gives a new tree which we will call the Stern-Brocot tree (equivalent to the Stern-Brocot tree defined in [7], which can be easily checked).

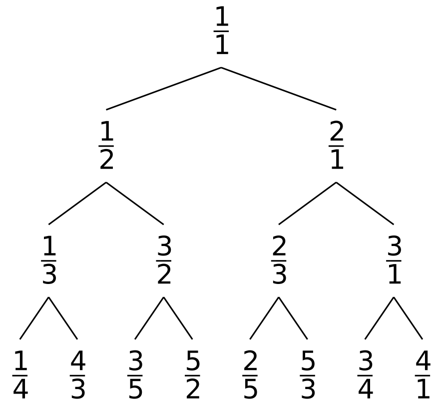

FIGURE 1. Calkin-Wilf Tree (or CW Tree).

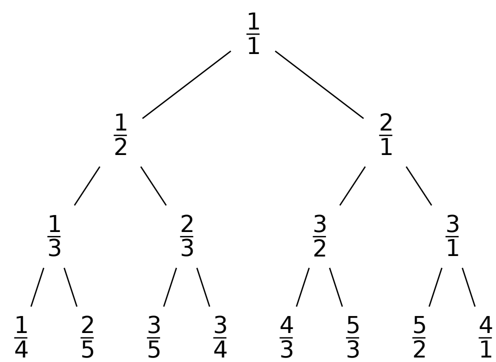

FIGURE 2. Stern-Brocot Tree (or SB Tree).

Conflating an address on the CW tree with the value assigned to it, note that if $\omega_1 < \omega_2$ then $\omega_1 L < \omega_2 L$ and $\omega_1 R < \omega_2 R$ and so, for all ω, $\omega_1\omega < \omega_2\omega$. Since $\omega_1 L < 1 < \omega_2 R$ for all ω_1, ω_2, it follows that, on the SB tree, $\omega L \omega_1 < \omega R \omega_2$ for all $\omega_1, \omega_2, \omega$. Therefore, the nth row of the SB tree is a permutation (in fact, involution) of the nth row of the CW tree that orders those entries in increasing order.

We may define L' and R' for the SB tree in terms of Stern's sequence as follows:

$$L' : \frac{a_m}{a_n} \mapsto \frac{a_{2m-1}}{a_{2n+1}} \text{ and } R' : \frac{a_m}{a_n} \mapsto \frac{a_{2m+1}}{a_{2n-1}}. \tag{3.4}$$

The nth row of the SB tree is then $\frac{a_{2k+1}}{a_{2^n-2k-1}}$ for $k = 0, \dots, 2^n - 1$ and therefore the combined first n rows of the SB tree gives, when written in increasing order, $\frac{a_k}{a_{2^n-k}}$ for $k = 1, \dots, 2^{n+1}$.

Proposition 3.2. *The combined first $n-1$ rows of the CW tree are the same as for the SB tree and give, in terms of Stern's sequence,*

$$\left\{ \frac{a_k}{a_{2^n-k}} : k = 1, \dots, 2^n \right\} = \left\{ \frac{a_k}{a_{k+1}} : k = 1, \dots, 2^n \right\}. \tag{3.5}$$

3.3. **The CW tree mod m and its Markov chain.** To get the "CW tree mod m", we replace each fraction $\frac{a}{b}$ in the CW tree with the ordered pair $(a \mod m, b \mod m)$. Since the entries of the CW tree include all of the positive rationals (in lowest terms), the CW tree mod m has entries in

$$S_m := \{(i,j) \in \overline{m}^2 : \gcd(i,j,m) = 1\}. \tag{3.6}$$

The cardinality of S_m is thus $J_2(m)$, one of the Jordan totient functions. By a known product formula (see for example Exercise 1.5.2 of [11]), we have

Proposition 3.3. *For all m, $|S_m| = J_2(m) = m^2 \prod_{p|m} \left(1 - \frac{1}{p^2}\right)$.*

Consider assigning probability $\frac{1}{2}$ to each downward edge of the CW tree mod m. This creates a Markov chain with state space S_m; here is when $m = 3$:

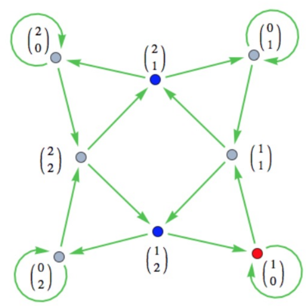

FIGURE 3. Markov chain for S_3 (with (a,b) replaced by $\binom{a}{b}$); from [8].

It turns out that every state is equally likely in the long run, independent of the starting state. This is true in general.

Lemma 3.4. *For every starting state, the distribution of $(a_n, a_{n+1}) \mod m$ is uniform on S_m.*

Proof. Fix m. By Proposition 2.1, there is a sequence of steps in the Markov chain that goes from $(1,1)$ to any particular (a,b). Note that, modulo m, the map $L : (a,b) \mapsto (a, a+b)$ is invertible ($L^{-1} : (a,b) \mapsto (a, b-a)$) and its iterates eventually return to (a,b). Hence $L^{-1} = L^k$ for some k. The same result holds for $R : (a,b) \mapsto (a+b, b)$. Hence, there is a sequence of steps that takes (a,b) to $(1,1)$ and then onto any (c,d) of our choosing. Therefore, it is possible to get from any state to any other state: the Markov chain is irreducible.

Since $L((0,b)) = (0,b)$, the chain is non-periodic. It follows, by the "Fundamental Theorem of Markov Chains" (see, for example, [2]), that there exists a unique stationary distribution. Since L and R are invertible, every state has a 2 arrows out and 2 arrows in and so the uniform distribution is stationary. By uniqueness, it is the unique stationary distribution. That is, no matter what starting point, as n approaches ∞, the distribution becomes uniform. This implies that for any node on the CW tree, the distribution modulo m of the 2^n descendants uniformly approaches the uniform distribution. □

Example. For the case when $m = 3$,

$$\frac{0}{1}, \frac{0}{2}, \frac{1}{0}, \frac{1}{1}, \frac{1}{2}, \frac{2}{0}, \frac{2}{1}, \frac{2}{2} \tag{3.7}$$

become, in the long run, equally likely. The distribution of Stern's sequence modulo 3 is then $P(0,3) = 1/4$ and $P(1,3) = P(2,3) = 3/8$.

Theorem 3.5. *The distribution of Stern's sequence modulo m is*

$$P(i,m) = \frac{1}{m}\prod_{p|m}\frac{p^2}{p^2-1}\prod_{p|(i,m)}\frac{p-1}{p}. \tag{3.8}$$

Proof. By Lemma 3.4, the distribution of $a_n \mod m$ satisfies

$$P(i,m) = \left|\{(k,j) \in S_m : k = i\}\right| / |S_m|. \tag{3.9}$$

Let $g := \gcd(i,m)$. Since

$$\begin{aligned}
|\{(k,j) \in S_m : k = i\}| &= |\{j \in \overline{m} : \gcd(j,g) = 1\}| \\
&= \left|\left\{\bigcup_{k=1}^{m/g}\{j \in \{kg+1,\ldots,kg+g\} : \gcd(j,g) = 1\right\}\right| \\
&= \frac{m}{g}\cdot\{j \in \overline{g} : \gcd(j,g) = 1\}| = \frac{m\phi(g)}{g} = m\prod_{p|g}\frac{p-1}{p},
\end{aligned}$$

the result follows by Proposition 3.3. □

3.4. **A consequence.** Although $a_n \mod m$ is not uniformly distributed, it is, when rounded down one "digit".

Corollary 3.6. *For all m, $\left\lfloor \frac{a_n}{m}\right\rfloor$ is distributed uniformly modulo m^k for all k.*

Proof. Note that, by Theorem 3.5, $P(i,m^k) = P(i,m)/m^{k-1}$.

Since $\left\lfloor \frac{a_n}{m}\right\rfloor \equiv i \pmod{m^k}$ if and only if $a_n \equiv (mi+j) \pmod{m^{k+1}}$ for some $j \in \overline{m}$, the distribution of $\left\lfloor \frac{a_n}{m}\right\rfloor \mod m^k$ is

$$P(i,m^k) = \sum_{i=1}^{m} P(mj+i, m^{k+1}) = \sum_{i=1}^{m} P(i,m)/m^k = \frac{1}{m^k}. \tag{3.10}$$

□

Corollary 3.7. *For any prime p, $\left\lfloor \frac{a_n}{p}\right\rfloor$ is equidistributed in the p-adic integers $\mathbb{Z}_p$.*

Example. Here are values of $\lfloor a_n/10\rfloor \mod 100$, $n \geq 5\times 10^5$:

19, 9, 90, 61, 71, 95, 23, 76, 52, 90, 37, 23, 85, 4, 18, 51, 33, 46, 13, 94, 80, 8, 27, 75, 47, 56,

8, 70, 61, 36, 75, 89, 14, 93, 79, 45, 65, 82, 16, 68, 51, 91, 40, 28, 88, 14, 26, 63, 37, 72, 35,

32, 97, 56, 58, 18, 60, 4, 44, 27, 83, 90, 6, 30, 23, 25, 1, 80, 78, 12, 34, 89, 55, 98, 43, 31, ...

3.5. **The Chinese Remainder Theorem.** We note that for any function $F : S_m \to \mathbb{Z}$, the sequence $F(a_n, a_{n+1}) \mod m$ has a distribution. Under rather mild conditions, the distribution has a multiplicative property.

The Chinese Remainder Theorem states that if $m \perp n$ (i.e., m and n are relatively prime) then

$$\mathbb{Z}/(m)\times\mathbb{Z}/(n) \cong \mathbb{Z}/(mn). \tag{3.11}$$

If $*$ denotes this isomorphism so that, for example when $m = 2, n = 3$, $0*0 = 0, 1*1 = 1, 0*2 = 2, 1*0 = 3, 0*1 = 4$, and $1*2 = 5$, we have an induced isomorphism

$$S_m \times S_n \cong S_{mn}, [((i,j),(u,v)) \mapsto (i*u, j*v)]. \tag{3.12}$$

We say that a function $F : \mathbb{Z} \times \mathbb{Z} \to \mathbb{Z}$ is *normal* if, for all m,

$$F(x, y) \equiv F(x \mod m, y \mod m) \pmod{m}. \tag{3.13}$$

Every polynomial in $\mathbb{Z}[x, y]$ is normal.

Lemma 3.8. *For a normal function F, the sequence $F(a_n, a_{n+1})$ has distribution satisfying*

$$P(i, m)P(j, n) = P(i * j, mn). \tag{3.14}$$

Proof. With isomorphism $*$ of (3.12), since

$$x * y \equiv x \pmod{m} \text{ and } x * y \equiv y \pmod{n},$$

$$F(i * u, j * v) \equiv F(i, j) \pmod{m} \text{ and } F(i * u, j * v) \equiv F(u, v) \pmod{n}$$

and so

$$F(i * j, u * v) \equiv F(i, j) * F(u, v) \pmod{mn}. \tag{3.15}$$

Hence, the isomorphism of (3.12) is a bijection between ordered pairs $((i, j), (u, v))$ of solutions of $F \equiv a \pmod{m}$ and $F \equiv b \pmod{n}$ and solutions $(i * u, j * v)$ of $F \equiv a * b \pmod{mn}$. □

4. An analogue of Stern's sequence

4.1. The distribution of $a_n a_{n+1}$. As in the proof of Theorem 3.5, we'll use Lemma 3.4 and a counting argument.

Dedekind's psi function is defined by $\psi(n) = n \prod_{p|n} \left(1 + \frac{1}{p}\right)$ and satisfies

$$\psi(n) = \phi(n)/J_2(n) \tag{4.1}$$

where, as was noted in Proposition 3.3,

$$J_2(n) = n^2 \prod \left(1 - \frac{1}{p^2}\right) \tag{4.2}$$

is one of the Jordan totient functions that is also the cardinality of S_n.

Lemma 4.1. *The distribution of $(a_n a_{n+1})$ satisfies, for prime powers p^n,*

$$P(i, p^n) = \begin{cases} \frac{2}{\psi(p^n)} & \text{if } p|i, \\ \frac{1}{\psi(p^n)} & \text{if } p \nmid i. \end{cases} \tag{4.3}$$

Proof. Consider S_{p^ν} with each entry (i, j) replaced by the product ij. For example, applying this process to S_8 yields the array

$$S_8' := \begin{bmatrix} & 0 & & 0 & & 0 & & 0 \\ 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ & 2 & & 6 & & 2 & & 6 \\ 0 & 3 & 6 & 1 & 4 & 7 & 2 & 5 \\ & 4 & & 4 & & 4 & & 4 \\ 0 & 5 & 2 & 7 & 4 & 1 & 6 & 3 \\ & 6 & & 2 & & 6 & & 2 \\ 0 & 7 & 6 & 5 & 4 & 3 & 2 & 1 \end{bmatrix}. \tag{4.4}$$

For each unit i (i.e., $\gcd(i, p) = 1$), the corresponding row $\{ij : j \in \overline{p^\nu}\}$ is a permutation of $\overline{p^\nu}$, the corresponding column $\{ji : j \in \overline{p^\nu}\}$ is a permutation of $\overline{p^\nu}$, and the whole of S_{p^ν}' is the union of these unit rows and unit columns.

A particular unit u appears once in each unit row and, since u must be a product of units, it can only occur at an intersection of a unit row and unit column. Hence u occurs $\phi(p^\nu)$ times in S'_{p^ν}.

A particular non-unit v appears once in each unit row and once in each unit column (but never at the intersection of a row and a column), and so v must occur $2\phi(p^\nu)$ times in S'_{p^ν}. The result follows. □

Let $\omega(n)$ denote the number of distinct prime divisors of n and (i,m) denote the gcd of i,m. Lemmas 3.8 and 4.1 yield the following theorem.

Theorem 4.2. *The distribution of* $(a_n a_{n+1})$ *is*

$$P(i,m) = \frac{2^{\omega((i,m))}}{\psi(m)}. \tag{4.5}$$

An interesting rephrasing of the Riemann hypothesis is based on one involving Robin's inequality and ψ – see [17].

Conjecture 4.3. *For* $N_k :=$ *the product of the first* k *primes,*

$$P(1,N_k) < \frac{\pi^2}{6e^\gamma N_k \log\log N_k} \text{ for all } k > 2. \tag{4.6}$$

4.2. **The sequence** (b_n)**.** For non-negative real numbers a, b, let

$$a \oplus b = a + b + \sqrt{4ab+1}. \tag{4.7}$$

We may form a "diatomic array", as for Stern's sequence, but using $\oplus$ instead of ordinary addition:

$$\begin{array}{ccccccccccccccccc}
0 & & & & & & & & & & & & & & & & 0 \\
0 & & & & & & & & 1 & & & & & & & & 0 \\
0 & & & & 2 & & & & 1 & & & & 2 & & & & 0 \\
0 & & 3 & & 2 & & 6 & & 1 & & 6 & & 2 & & 3 & & 0 \\
0 & 4 & 3 & 10 & 2 & 15 & 6 & 12 & 1 & 12 & 6 & 15 & 2 & 10 & 3 & 4 & 0 \\
. & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & .
\end{array}.$$

An analogue of Stern's sequence is

$$b_1 = 0, \qquad b_{2n} = b_n, \qquad b_{2n+1} = b_n \oplus b_{n+1} \tag{4.8}$$

The sequence begins

$$0, 0, 1, 0, 2, 1, 2, 0, 3, 2, 6, 1, 6, 2, 3, 0, 4, 3, 10, 2, 15, 6, 12, 1, 12, 6, 15, ... \tag{4.9}$$

Although this is an integer sequence (A272569 of [16]), it is hardly clear why it does not take on irrational values. Its connection with Stern's sequence, from Theorem 3.6 of [15], explains why and we state it with the following proposition.

Proposition 4.4. *If* $2^j \le k \le 2^{j+1}$*, then* $b_k = a_{2^{j+1}-k} a_{k-2^j}$.

4.3. **The distribution of** (b_n)**.** It follows, by replacing k by $2^j + k$ in Proposition 4.4, that if $0 \le k \le 2^j$ then

$$b_{2^j+k} = a_{2^j-k} a_k. \tag{4.10}$$

By Proposition 3.2, it follows that the sequence $b_{2^j}, b_{2^j+1}, \ldots, b_{2^{j+1}-1}$ is an involution of the sequence $a_{2^j}a_{2^j+1}, a_{2^j+1}a_{2^j+2}, \ldots a_{2^{j+1}-1}a_{2^{j+1}}$. Hence, the distribution of (b_n) is that of $(a_n a_{n+1})$ (with the important *caveat* that (b_n) has some distribution).

A difficulty we encounter in this case is that the SB tree mod m does not represent a Markov chain: on the SB tree, $\frac{3}{4} \mapsto \frac{5}{7}, \frac{4}{5}$ and $\frac{3}{1} \mapsto \frac{5}{2}, \frac{4}{1}$ and so, mod 3, $\frac{0}{1} \mapsto \frac{2}{1}, \frac{1}{2}$ in the first case, but $\frac{0}{1} \mapsto \frac{1}{1}, \frac{2}{2}$ in the second. We may not then proceed as we did for the CW tree.

Let $A_0 := \left(\begin{smallmatrix} 1 & 0 \\ 1 & 1 \end{smallmatrix}\right)$ and $A_1 := \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)$. It is a fact that A_0 and A_1 generate $SL_2(\mathbb{Z})$ and therefore, modulo m, generate $SL_2(\mathbb{Z}/(m))$ – see [5]. If $\omega := \omega_n \omega_{n-1} \dots \omega_0$ is a word in $\{0,1\}^*$, let

$$[\omega] := \sum_{i=0}^{n} \omega_i \cdot 2^i \text{ and } A_\omega = A_{\omega_n} \cdot A_{\omega_{n-1}} \cdots A_{\omega_0}. \tag{4.11}$$

For example, $[01101] = 1 + 4 + 8 = 13$ and $A_{01101} = \left(\begin{smallmatrix} 3 & 5 \\ 4 & 7 \end{smallmatrix}\right)$. Note that the determinant of any A_ω is 1.

The following is easy to prove (and is left as an exercise for the reader). See [13] for a similar result.

Proposition 4.5. *For $\omega_j \dots \omega_0$, and $n = [\omega]$,*

$$A_\omega = \begin{pmatrix} a_{n+1} & a_n \\ a_{2^j-n-1} & a_{2^j-n} \end{pmatrix}. \tag{4.12}$$

Let M^* denote the anti-transpose of M and $M(x)$ be the Möbius transformation defined by M (i.e., $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)^* = \left(\begin{smallmatrix} d & b \\ c & a \end{smallmatrix}\right)$ and $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)(x) = \frac{ax+b}{cx+d}$). Note that if the words ω of length n are ordered lexicographically, then the nth row of the SB tree coincides with $A_\omega(1)$ and the nth row of the CW tree coincides with $A^*_\omega(1)$. This of course illustrates a common ground for the SB and CW trees.

Consider the tree formed by $A_\omega \mapsto A_{\omega 0}, A_{\omega 1}$. For each address (i.e. word) $\alpha \in \{L, R\}^*$, substitute 0 for L and 1 for R to get a word $w(\alpha)$ in $\{0,1\}^*$ in Figure 4.

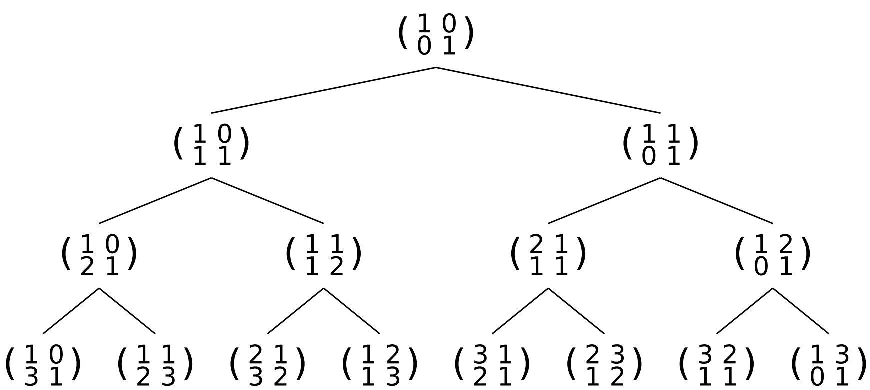

FIGURE 4. The path $A_* \to A_0 \to A_{01} \to A_{011}$ goes from $\left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$ to $\left(\begin{smallmatrix} 1 & 2 \\ 1 & 3 \end{smallmatrix}\right)$.

Lemma 4.6. *If $\frac{a}{b}$ is at address α on the SB tree, then*

$$A_{w(\alpha)}(1) = \frac{a}{b}. \tag{4.13}$$

Let $G_m := SL_2(\mathbb{Z}/(m))$. Considering the elements of S_m as column vectors, the elements of G_m act on S_m by matrix multiplication on the left. For a fixed a, and every $\left(\begin{smallmatrix} a \\ b \end{smallmatrix}\right) \in S_m$, it is clear that A_0 fixes a and permutes the second coordinates. Hence A_0 permutes S_m. Similarly, A_1 permutes S_m as well and, since G_m is generated by A_0, A_1 (since $SL_2(\mathbb{Z})$ is – see [5]), every element of G_m permutes the elements of S_m. Therefore, if a finite sequence $v_1, v_2, \dots, v_k$ in

S_m is nearly uniform (e.g., every $P(a,m)$ is within ϵ of $1/m$ for all a), then so is the sequence $Mv_1, Mv_2, \ldots, Mv_k$.

A *subtree* of the SB-tree is the tree containing all vertices with addresses of the form $\omega_0\omega$ for some fixed ω_0 (as ω varies through $\{L,R\}^*$).

Consequently, modulo m, as $j \to \infty$, the jth row in any SB subtree approaches the uniform distribution uniformly over all ω_0. Hence, by Lemma 2.1, we have the following theorem.

Theorem 4.7. *The distribution of* (b_n) *is*

$$P(i,m) = \frac{2^{\omega((i,m))}}{\psi(m)}. \tag{4.14}$$

This distribution satisfies equation (1.3) and thus, as in Corollary 3.6, we have the following.

Corollary 4.8. *For all* m *and* $k > 0$*, the sequences* $\left\lfloor \frac{b_n}{m} \right\rfloor$ *and* $\left\lfloor \frac{a_n a_{n+1}}{m} \right\rfloor$ *are uniformly distributed modulo* m^k*.*

4.4. **Generalizations.** It turns out that the two distributions defined above satisfy, for an appropriate $f(p)$,

$$P(i,m) = \frac{1}{m} \prod_{p|(i,m)} \frac{f(p)}{p^{-1}} \cdot \prod_{\substack{p|m \\ p\nmid(i,m)}} \frac{1-f(p)}{1-p^{-1}}. \tag{4.15}$$

In particular, the distribution of (a_n) arises when $f(p) = \frac{1}{p+1}$ and the distribution of (b_n) arises when $f(p) = \frac{2}{p+1}$.

Every distribution described by equation (4.15) satisfies

$$P(0,p) = f(p), P(i,p) = P(j,p) \text{ whenever } p \nmid i,j \tag{4.16}$$

as well as the conclusion of Lemma 3.8.

It is worth noting that the uniform distribution is when $f(p) = \frac{1}{p}$ (and a sequence that has that distribution is, of course, (n)). An interesting question is: "for what functions $f(p)$ is there a sequence with a distribution given by (4.15)"? For example, $f(p) = 1$ gives

$$P(i,m) = \begin{cases} d/m \text{ if } d|i \\ 0 \text{ otherwise} \end{cases} \tag{4.17}$$

where d is the largest square-free divisor of m. The sequences $(n!)$ and (R_n) (the latter of which is studied later in this paper) have that distribution for square-free m only. Is there a sequence with this distribution for all m?

5. Quadratic Forms

A binary primitive integral quadratic form is a function of the form

$$Q(x,y) := Ax^2 + Bxy + Cy^2 \tag{5.1}$$

where A, B, C are relatively prime integers. Its *discriminant* is the quantity

$$\Delta := B^2 - 4AC. \tag{5.2}$$

For example, $Q(x,y) = xy$ has $[A,B,C] = [0,1,0]$ and thus $\Delta = 1$.

5.1. **The distribution of $Q(a_n, a_{n+1})$ modulo 2.** Let $c_n := Q(a_n, a_{n+1})$. This sequence has a distribution; we now find a formula for it. Note that, modulo 8, $\Delta \in \{0, 1, 4, 5\}$.

Lemma 5.1. *The sequence $c_n := Q(a_n, a_{n+1})$ has distribution satisfying*

$$P(0,2) = \begin{cases} 0 & \text{if } \Delta \equiv 5 \pmod 8 \\ 1/3 & \text{if } \Delta \equiv 0 \text{ or } 4 \pmod 8 \\ 2/3 & \text{if } \Delta \equiv 1 \pmod 8. \end{cases} \tag{5.3}$$

Proof. Let $a, b, c \in \{0, 1\}$ be defined by $A = 2\alpha + a$, $B = 2\beta + b$, $C = 2\gamma + c$ for some α, β, γ. Since Q is primitive, at least one of a, b, c is odd. Then, modulo 8,

$$\Delta = 4(\beta(\beta + b) - ac) + b^2. \tag{5.4}$$

Because consecutive values of a_n are relatively prime, it follows that, modulo 2, $(a_n, a_{n+1}) \in \{(0, 1), (1, 1), (1, 0)\}$. Further, since (a_n) is periodic (with period 3) modulo 2, the values of (a_n, a_{n+1}) cycle through $\{(0, 1), (1, 1), (1, 0)\}$ and therefore $c_n \mod 2$ cycles through $c, a+b+c, a$.

If $\Delta \mod 8$ is 0 or 4, then b is even, and thus exactly one of $c, a + b + c, a$ is even. If $\Delta \mod 8$ is odd, then $b = 1$ and, modulo 8, $\Delta = 1 + 4ac$. If $\Delta \equiv 1 \pmod 8$, then exactly two of $c, a + b + c, a$ are even while if $\Delta \equiv 5 \pmod 8$ then none of $c, a + b + c, a$ are even. The result follows. □

The Legendre symbol is defined, for an odd prime p and integer n, to be

$$\left(\frac{n}{p}\right) = \begin{cases} 1 & \text{if } n \equiv x^2 \pmod p \text{ for some } x \\ -1 & \text{if } n \not\equiv x^2 \pmod p \text{ for all } x \\ 0 & \text{if } p|n. \end{cases} \tag{5.5}$$

It is generally left undefined for $p = 2$ and, for odd non-prime m, a Jacobi symbol is defined.

We define

$$\left(\frac{\Delta}{2}\right) = \begin{cases} 1 & \text{if } \Delta \equiv 1 \pmod 8 \\ -1 & \text{if } \Delta \equiv 5 \pmod 8 \\ 0 & \text{if } 2|\Delta \end{cases} \tag{5.6}$$

and so Lemma 5.1 states that $P(i, 2)$ satisfies (4.15) where

$$f(p) = \frac{1 + \left(\frac{\Delta}{p}\right)}{1 + p}. \tag{5.7}$$

We note that for Δ an odd prime, our $\left(\frac{\Delta}{2}\right)$ equals $\left(\frac{2}{\Delta}\right)$ (its values often called "the second supplement of the law of quadratic reciprocity").

5.2. **The distribution of $Q(a_n, a_{n+1})$ modulo p.**

Lemma 5.2. *The number of solutions in S_p of $Q(x, y) = 0$ is*

$$(p - 1)\left(1 + \left(\frac{\Delta}{p}\right)\right). \tag{5.8}$$

If $p \nmid \Delta$, then the number of solutions in S_p of the equation $Q(x, y) = u$ for any unit $u \in F^\times$ is

$$p - \left(\frac{\Delta}{p}\right). \tag{5.9}$$

Proof. The case of $p = 2$ was covered earlier. Let p be an odd prime and let $F := \mathbb{Z}/(p)$, the field with p elements. If $A = C = 0$ then that is equivalent to $Q(x, y) = xy$ dealt with in the section on (b_n). We may assume $A \neq 0$ since, otherwise, we can always switch x and y.

The equation

$$Ax^2 + Bux + Cu^2 = 0 \tag{5.10}$$

can then, by completing the square, be written as

$$\frac{(2Ax + Bu)^2}{u^2} = \Delta. \tag{5.11}$$

If $\left(\frac{\Delta}{p}\right) = 1$, then $\Delta = v^2$ for some unit v and so, for every unit u and choice of sign for v, there is a solution x of (5.11). Hence there are $2(p-1)$ solutions altogether. If $\left(\frac{\Delta}{p}\right) = -1$, then $\Delta \neq x^2$ for any x and so (5.11) has no solutions. Lastly, if $\left(\frac{\Delta}{p}\right) = 0$ then $\Delta = 0$ in $\mathbb{Z}_p$ and so there is one solution x to (5.11) for each unit u and so there are $p - 1$ solutions to (5.11) altogether. Equation (5.8) summarizes these three cases.

Suppose now that $p \nmid \Delta$ so that $\left(\frac{\Delta}{p}\right) = \pm 1$. We first seek the number of solutions of $Q = u$ where u is a unit and so, since u is arbitrary, we may take $A = 1$; we thus consider $x^2 + Bxy + Cy^2 = u$ where u is a unit.

Set $R = F[t]/(t^2 + Bt + C)$, a finite ring. The norm map $N_{R/F} : R \to F$ is multiplicative, and using the basis $\{1, t\}$

$$N_{R/F}(-x + yt) = \det \begin{pmatrix} -x & -Cy \\ y & -x - By \end{pmatrix} = Q(x, y). \tag{5.12}$$

Therefore the equation $x^2 + Bxy + Cy^2 = u$ is the same as $N_{R/F}(-x + yt) = u$ for $x, y \in F$. On units the norm map $N_{R/F} : R^\times \to F^\times$ is a group homomorphism, so as with all homomorphisms between finite groups, all values are taken on an equal number of times. Thus it remains to show the norm map $N_{R/F} : R^\times \to F^\times$ is surjective.

Case 1: $t^2 + Bt + C$ is irreducible in $F[t]$. Then R is a field so $R^\times$ is cyclic and the norm map $N_{R/F} : R^\times \to F^\times$ on the nonzero elements of finite fields is onto (if $|F| = q$ then $|R| = q^2$ and a generator of $R^\times$ is mapped to a generator of $F^\times$). This corresponds to $\left(\frac{\Delta}{p}\right) = -1$ and, in this case, $|R^\times| = p^2 - 1$ and thus the number of solutions is $|\ker(N_{R/F})| = p + 1$ for each unit.

Case 2: t^2+Bt+C is reducible in $F[t]$. Write it as $(t-r)(t-s)$. Since $B^2-4C = (r-s)^2$, from $B^2-4C \neq 0$ we have $r \neq s$. Then $R \simeq F[t]/(t-r) \times F[t]/(t-r)$ and in the basis $\{(1, 0), (0, 1)\}$, the norm mapping has the formula $N_{R/F}(x, y) = xy$ which maps $R^\times = F^\times \times F^\times$ onto $F^\times$. This corresponds to $\left(\frac{\Delta}{p}\right) = 1$ and, in this case, $|R^\times| = (p-1)^2$ and thus the number of solutions is $|\ker(N_{R/F})| = p - 1$ for each unit. □

5.3. **The distribution of $Q(a_n, a_{n+1})$ where $m \perp \Delta$.** In [6], Theorem 2.1, Conrad proves the following multi-dimensional Hensel's lemma.

Lemma 5.3. *If $|f(\mathbf{a})|_p < 1$ and $||(\nabla f)(\mathbf{a})||_p = 1$, then there exists some $\boldsymbol{\alpha} \in \mathbb{Z}_p^2$ such that $f(\boldsymbol{\alpha}) = 0$ and $\boldsymbol{\alpha} \equiv \mathbf{a} \pmod{p}$.*

Suppose $p \nmid \Delta$. Fix $z \in \mathbb{Z}_p$ and let $F(x,y) = Ax^2 + Bxy + Cy^2 - (1+pz)$. If $||\nabla F(x,y)||_p < 1$ and $\gcd(x,y,p) = 1$ then p divides both $2Ax + By$ and $Bx + 2Cy$. This implies

$$\begin{pmatrix} 2A & B \\ B & 2C \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \equiv \begin{pmatrix} 0 \\ 0 \end{pmatrix} \pmod{p} \tag{5.13}$$

and thus the determinant of the matrix, $B^2 - 4AC$, is divisible by p – a contradiction. Hence $||\nabla F(x,y)||_p = 1$ whenever $\gcd(x,y,p) = 1$.

Hence, for any solution $\mathbf{a} \in S_p$ of $Q(\mathbf{a}) = x$ ($x \in F$), there are $p^{\nu-1}$ solutions of $Q(\boldsymbol{\alpha}) \equiv x \pmod{p}$ in $\mathbb{Z}/(p^\nu)$. We have the following lemma.

Lemma 5.4. *If p is an odd prime and $p \nmid \Delta$, then*

$$P(i,p^\nu) = \begin{cases} \left(1 + \left(\frac{\Delta}{p}\right)\right)/\psi(p^\nu) & \text{if } p|i \\ \left(p - \left(\frac{\Delta}{p}\right)\right)/(\psi(p^\nu)(p-1)) & \text{if } p \nmid i. \end{cases} \tag{5.14}$$

By Lemma 5.3 and the multiplicative property of $\psi(n)$, we have the following theorem.

Theorem 5.5. *For a binary primitive integral quadratic form Q with discriminant Δ, if $m \perp \Delta$ then*

$$P(i,m) = \frac{1}{m} \prod_{p|(i,m)} \frac{f(p)}{p^{-1}} \cdot \prod_{\substack{p|m \\ p\nmid(i,m)}} \frac{1-f(p)}{1-p^{-1}} \tag{5.15}$$

where

$$f(p) = \frac{1 + \left(\frac{\Delta}{p}\right)}{p+1}. \tag{5.16}$$

Corollary 5.6. *For a binary primitive integral quadratic form Q, let $x_n := Q(a_n, a_{n+1})$. For all m relatively prime to Δ, $\lfloor \frac{x_n}{m} \rfloor$ is uniformly distributed modulo m^k for all k.*

5.4. **Further directions.** Curiously, the distribution of the primitive values of Q modulo m depends only on the discriminant of Q (as long as this discriminant and m are relatively prime). Since non-equivalent quadratic forms take on different values, it seems inevitable that their distributions will differ modulo m when $\gcd(m,\Delta) \neq 1$.

As for the section on (b_n), the sequence

$$c_n := Q(a_{2^j+n}, a_{2^{j+1}-n}) \text{ for } 2^j \le n \le 2^{j+1} \tag{5.17}$$

has the same distribution as $Q(a_n, a_{n+1})$. Furthermore, (c_n) will obey the Stern-like recursion

$$c_{2n} = c_n, c_{2n+1} = c_n \oplus c_{n+1} \tag{5.18}$$

if $\oplus$ is defined by

$$x \oplus y = x + y + \sqrt{4xy + \Delta} \tag{5.19}$$

or, equivalently, when $|ad - bc| = 1$ then

$$Q(a+b, c+d) = Q(a,c) \oplus Q(b,d). \tag{5.20}$$

6. Fibonacci Representations

Every integer can be represented in at least one way as a sum of distinct Fibonacci numbers – see [4].

Let R_n denote the number of ways to represent n as a sum of distinct Fibonacci numbers. Its generating function thus satisfies

$$\sum_{n=0}^{\infty} R_n x^n = \prod_{i=2}^{\infty} \left(1 + x^{F_n}\right). \tag{6.1}$$

It also has a recursive definition

$$R_n = \sum_{\sigma(i) \in \{n, n-1\}} R_i \tag{6.2}$$

where $\sigma(n) := \left\lfloor n\phi + \frac{1}{\phi} \right\rfloor$ is the "Fibonacci shift" (called ρ in [15]). This recursion can be implemented in Maple; here's for the shifted sequence $r(n) = R_{n-1}$:

r := proc(n) option remember; if n < 2 then 1; elif sigma(n + 1) - sigma(n) = 2 then r(sigma(n) - n); else r(2*n - 2 - sigma(n - 1)) + r(2*n - 1 - sigma(n - 1)); end if; end proc

The first few terms of R_n are (for $n = 0, 1, 2, \ldots$):

$$1, 1, 1, 2, 1, 2, 2, 1, 3, 2, 2, 3, 1, 3, 3, 2, 4, 2, 3, 3, 1, 4, 3, 3, 5, 2, 4, 4, 2, 5, 3, 3, 4, 1, 4, 4, 3, 6, \ldots \tag{6.3}$$

6.1. Words.

For a word $\omega \in \{0, 1\}^*$, let

$$R(\omega) := R_{[\omega]} \text{ where } [\omega_k \omega_{k-1} \ldots \omega_0] := \sum_{i=0}^{k} \omega_i F_{i+2}. \tag{6.4}$$

We define the set of "Zeckendorf words" as

$$\mathbf{Z} := 1\{0, 01\}^* \tag{6.5}$$

and recall that for every positive integer n, there is a unique $\omega \in \mathbf{Z}$ such that $[\omega] = n$. We define the set of "blockhead words" to be

$$\mathbf{B} := 1\{00, 01\}^*00 \tag{6.6}$$

and define

$$\mathbf{\Lambda} := \{0, 010, 01010, \ldots\} = 0\{10\}^*. \tag{6.7}$$

Lemma 6.1. *For all $\Omega \in \mathbf{B}$ and $\omega \in \mathbf{Z} \cup \mathbf{\Lambda}$,*

$$R(\Omega\omega) = R(\Omega)R(\omega). \tag{6.8}$$

Proof. For $\Omega \in \mathbf{B}$, if ρ is a Fibonacci representation of $[\Omega]$ then it must end in 00 or 11. For $\omega \in \mathbf{Z} \cup \mathbf{\Lambda}$, if ρ' is a Fibonacci representation of $[\omega]$ then it begins with 10 or 01 (and no other representation starts with 00). Therefore, every representation of $[\Omega\omega]$ is a concatenation of two words that represent $[\Omega]$ and $[\omega]$ respectively. $\square$

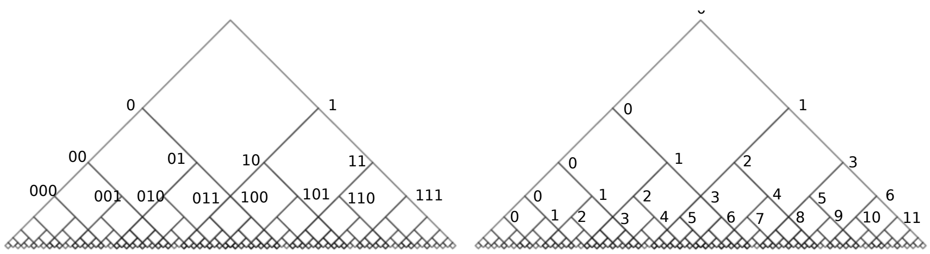

FIGURE 5. Vertices labeled by path, and by number.

6.2. **Fibonacci triangle.** In Figure 5, a Fibonacci hyperbolic graph (see [13, 15]). with vertices labeled with words in $\{0,1\}^*$, sometimes in multiple ways. On the right are numerical values assigned in the obvious way. The fact that $R_3 = 2$ is illustrated by the two words 011, 100 on the left and 3 on the right.

Next, we label each square with the Zeckendorf word of its top vertex; Figure 6 is of the subtree headed by the word representing 3. There, the blockhead words are in boldface and they are at the head of "blocks" of the form $\Omega\mathbf{Z} \cup \Omega\mathbf{\Lambda}$.

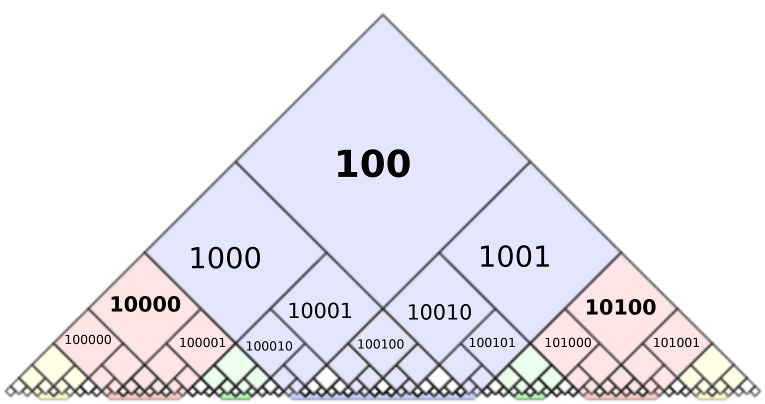

FIGURE 6. Sub-triangle labeled with Zeckendorf representations.

Following the construction of Pascal's triangle, start with box on top of the Fibonacci triangle labeled 1 and then fill out according to the rule: for each square of side length $C\phi^{-n+1}$, take the sum of the numbers of all adjacent squares of side length $C\phi^{-n}$. The subtriangle corresponding to the one in Figure 6 are illustrated in Figure 7. The numbers in Figure 8 are, of course, just $R(\omega)$ for each ω in Figure 7.

A block headed by Ω is characterized in Figure 7, via Lemma 6.1, by having every number in it a multiple of $R(\Omega)$.

6.3. **The function $g(n)$.** Let $g(n)$ denote the sequence

$$1, 3, 4, 8, 9, 11, 12, 21, 22, 24, \ldots \tag{6.9}$$

defined by, for $\epsilon_i \in \{0,1\}$,

$$g : \sum_{i=0}^{k} \epsilon_i 2^i \longmapsto \sum_{i=0}^{k} \epsilon_i F_{2i+2}. \tag{6.10}$$

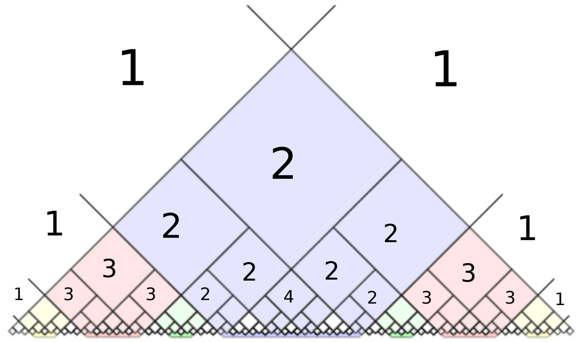

FIGURE 7. A triangle labeled by R_n.

The set $\{g(n)\}$ is the set of all numbers represented as a sum of distinct even-indexed Fibonacci numbers (OEIS sequence A054204 [16]). This function also satisfies the recursive definition:

$$g(1) = 1, g(2n) = g(n) + \sigma(g(n)), g(2n+1) = g(2n) + 1 \tag{6.11}$$

with σ, the Fibonacci shift, defined above.

The following is an analogue of Theorem 4.1 of [13]

$$R_n = \sum_{\sigma(i)+j=n} \mathbb{I}_{g(\mathbb{N})}(i) \cdot \mathbb{I}_{g(\mathbb{N})}(j). \tag{6.12}$$

The following result, from a paper by Bicknell-Johnson (Theorem 2.1 of [1]), shows that Stern's sequence is a subsequence of $(R(n))$.

Lemma 6.2. *For all j, $R(g(j)) = a_{j+1}$.*

Theorem 6.3. *For (R_n), $P(0, m) = 1$ for all m.*

Proof. For each blockhead word $\Omega \in \mathbf{B}$, we define a "block" $\Omega(\mathbf{Z} \cup \mathbf{\Lambda}) := \{\Omega\omega : \omega \in \mathbf{Z} \cup \mathbf{\Lambda}\}$ and note that for every ρ in that block, $[\Omega]$ divides $[\rho]$. Let $|\omega|$ be the length of the word ω. Note that every Zeckendorf word of length at least 3 that does not represent 1 appears in one of the blocks.

In a block $\Omega(\mathbf{Z} \cup \mathbf{\Lambda})$, the number of words ω of length $n + |\Omega|$ is approximately F_n so, asymptotically,

$$P(0, [\Omega]) \geq \delta := \frac{1}{\phi^{|\Omega|}}. \tag{6.13}$$

But this is true of all blocks and, since the entire set of words is a union of blocks, we have that $P(0, [\Omega])$ is at least δ plus δ times what remains and, in general,

$$P(0, [\Omega]) \geq 1 - (1 - \delta)^n \text{ for all } n. \tag{6.14}$$

Hence $P(0, [\Omega]) = 1$ for each Ω and, by Lemma 6.2, since $g(2n) = a_{2n+1}$ takes on all positive integer values, the result follows. ∎

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