Please send all communications concerning ADVANCED PROBLEMS AND SOLUTIONS to FLORIAN LUCA, SCHOOL OF MATHEMATICS, UNIVERSITY OF THE WITWATERSRAND, PRIVATE BAG 3, WITS 2050, JOHANNESBURG, SOUTH AFRICA or by e-mail at the address florian.luca@wits.ac.za as files of the type tex, dvi, ps, doc, html, pdf, etc. This department especially welcomes problems believed to be new or extending old results. Proposers should submit solutions or other information that will assist the editor. To facilitate their consideration, all solutions sent by regular mail should be submitted on separate signed sheets within two months after publication of the problems.

**PROBLEMS PROPOSED IN THIS ISSUE**

**H-854** Proposed by D. M. Bătinețu-Giurgiu, Bucharest, Romania and Neculai Stanciu, Buzău, Romania

Compute
\[
\lim_{n \to \infty} \left( \lim_{x \to \infty} \left( (f(x + 1)) \frac{L_n}{(x+1)F_{n+1}} - (f(x)) \frac{L_n}{x F_{n+1}} \right)^{\frac{L_{n+1}}{L_{n+1}}} \right),
\]
where \( f : \mathbb{R}^* \mapsto \mathbb{R}^* \) is a function that satisfies \( \lim_{x \to \infty} f(x + 1)/(xf(x)) = a \in \mathbb{R}^* \).

**H-855** Proposed by Robert Frontczak, Stuttgart, Germany

Let \((T_n)_{n \geq 0}\) be the sequence of Tribonacci numbers given by
\[
T_0 = 0, \quad T_1 = T_2 = 1, \quad T_n = T_{n-1} + T_{n-2} + T_{n-3} \quad \text{for} \quad n \geq 3.
\]
Define the functions
\[
G_{FT}(z) = \sum_{n=0}^{\infty} F_n T_n z^n \quad \text{and} \quad G_{LT}(z) = \sum_{n=0}^{\infty} L_n T_n z^n.
\]
Show that for \( k \geq 1 \), we have
\[
G_{FT}(2^{-2k}) = \frac{2^{4k}(2^{6k} - 2^{2k} - 1)}{2^{12k} - 2^{10k} - 2^{8k+2} - 2^{6k+2} - 2^{6k} - 2^{4k+1} + 2^{2k} - 1}
\]
and
\[
G_{LT}(2^{-2k}) = \frac{2^{4k}(2^{6k} + 2^{4k+1} + 2^{2k} + 3)}{2^{12k} - 2^{10k} - 2^{8k+2} - 2^{6k+2} - 2^{6k} - 2^{4k+1} + 2^{2k} - 1}.
\]

**H-856** Proposed by Robert Frontczak, Stuttgart, Germany

Let \( T_n \) denote the \( n \)th triangular number; i.e., \( T_n = n(n + 1)/2 \). Show that
\[
\sum_{n=0}^{\infty} T_n \cdot \frac{F_n}{2^{n+2}} = F_7 \quad \text{and} \quad \sum_{n=0}^{\infty} T_n \cdot \frac{L_n}{2^{n+2}} = L_7.
\]
Let $T_n$ be the $n$th Tribonacci number given by $T_0 = T_1 = 0$, $T_2 = 1$, and for $n \geq 3$, $T_n = T_{n-1} + T_{n-2} + T_{n-3}$. For all $n \geq 2$, prove that

$$F_{n-2} = \sum_{i=1}^{n-1} (-1)^{i-1} \sum_{s_1+\cdots+s_i=n} T_{s_1-1}T_{s_2-1}\cdots T_{s_i-1}.$$ 

SOLUTIONS

A formula for $\pi^2$ involving Fibonacci numbers

Proposed by Hideyuki Ohtsuka, Saitama, Japan

(Vol. 56, No. 2, May 2018)

Prove that

$$\frac{\pi^2}{8} = \sum_{n=1}^{\infty} \tan^{-1} \frac{1}{F_n} \tan^{-1} \frac{1}{F_{n+1}}.$$ 

Solution by Jason L. Smith, Richland Community College, Decatur, Ill.

Note this inverse tangent identity among Fibonacci numbers [1]:

$$\tan^{-1} \left( \frac{1}{F_{2m}} \right) = \tan^{-1} \left( \frac{1}{F_{2m+1}} \right) + \tan^{-1} \left( \frac{1}{F_{2m+2}} \right).$$

For brevity, denote the sum to be evaluated by $S$ and use $t_n := \tan^{-1}(1/F_n)$, so that $S = \sum_{n \geq 1} t_n t_{n+1}$. Reindex the sum as

$$S = t_1 t_2 + \sum_{m \geq 1} (t_{2m} t_{2m+1} + t_{2m+1} t_{2m+2}) = t_1 t_2 + \sum_{m \geq 1} t_{2m+1} (t_{2m} + t_{2m+2}).$$

Using the arctangent identity above, we can replace the odd-indexed factor inside the summation with $t_{2m+1} = t_{2m} - t_{2m+2}$, so

$$S = t_1 t_2 + \sum_{m \geq 1} (t_{2m} - t_{2m+2}) (t_{2m} + t_{2m+2}) = t_1 t_2 + \sum_{m \geq 1} (t_{2m}^2 - t_{2m+2}^2).$$

The above summation is telescopic in which only the $m = 1$ term survives. Therefore,

$$S = t_1 t_2 + \frac{t_2^2}{8} = \tan^{-1} \left( \frac{1}{F_1} \right) \tan^{-1} \left( \frac{1}{F_2} \right) + \left( \tan^{-1} \left( \frac{1}{F_2} \right) \right)^2 = 2(\tan^{-1}(1))^2 = \frac{\pi^2}{8}.$$ 


Also solved by Brian Bradie, Pridon Davlianidze, Dmitry Fleischman, Raphael Schumacher, and the proposer.
Some inequalities with Fibonacci numbers

H-822 Proposed by D. M. Bătinețu-Giurgiu, Bucharest and Neculai Stanciu, Buzău, Romania (Vol. 56, No. 2, May 2018)

Prove the following inequalities:

(a) \( F_n^2 F_{n+2}^2 + F_{n+1}^2 F_{n+3}^2 + (F_n + F_{n+2})^2 > 2\sqrt{6} F_n F_{n+1} F_{n+2} \);

(b) \( F_{n+2}^2 + (F_n + F_{n+2})^2 + F_{n+3}^2 > 4\sqrt{6} F_n F_{n+1} F_{n+2} \);

(c) \( L_n^2 + (L_n + L_{n+2})^2 + L_{n+3}^2 > 4\sqrt{6} L_n L_{n+1} L_{n+2} \);

(d) \( \sqrt{2}\sqrt{1 + F_n^4} + \sum_{k=1}^{n-1} \sqrt{(F_k^4 + 1)(F_{k+1}^4 + 1)} > 2F_n F_{n+1} \) for \( n > 1 \).

Solution by Wei-Kai Lai, University of South Carolina Salkehatchie, Walterboro, S.C.

(a) Applying \( F_{n+2} = F_n + F_{n+1} \) and \( F_{n+3} = F_n + 2F_{n+1} \), we can rewrite the claimed inequality as:

\[
\frac{F_n(F_n + F_{n+1})^2}{F_n + 2F_{n+1}} + \frac{F_{n+1}(F_n + 2F_{n+1})^2}{2F_n + F_{n+1}} + (2F_n + F_{n+1})^2 > 2\sqrt{6} F_n F_{n+1} (F_n + F_{n+1}).
\]

To make the calculation easier, we let \( a := F_n, b := F_{n+1} \). So, the above inequality becomes

\[
\frac{a(a+b)^2}{a+2b} + \frac{b(a+2b)^2}{2a+b} + (2a+b)^2 > 2\sqrt{6} ab(a+b).
\]

Multiplying by \( (a+2b)(2a+b) \) and expanding all products, we get

\[
5a^4 + 17a^3b + 20a^2b^2 + 13ab^3 + 5b^4 > \sqrt{6} ab(2a^3 + 7a^2b + 7ab^2 + 2b^3).
\]

After squaring both sides, we get

\[
25a^8 + 170a^7b + 489a^6b^2 + 810a^5b^3 + 892a^4b^4 + 690a^3b^5 + 369a^2b^6 + 130ab^7 + 25b^8
\]

\[
> 24a^7b + 168a^6b^2 + 462a^5b^3 + 636a^4b^4 + 462a^3b^5 + 168a^2b^6 + 24ab^7,
\]

which is clearly true.

(b) Let \( a := F_n, b := F_{n+1}, c := F_{n+2}, \) and \( d := F_{n+3} \). We want to prove that

\[
c^2 + (a+c)^2 + d^2 > 4\sqrt{6} abc.
\]

Since \( d = b + c \), we have \( c^2 + (a+c)^2 + d^2 = a^2 + b^2 + 3c^2 + 2ac + 2bc \). Inserting \( c = a + b \) into the products \( ac \) and \( bc \), we have

\[
a^2 + b^2 + 3c^2 + 2ac + 2bc = 3a^2 + 2b^2 + 3c^2 + 4ab.
\]

Applying the AM-GM inequality twice, we get

\[
3a^2 + 3b^2 + 3c^2 + 4ab \geq 3c^2 + 3(36)^{1/3}(ab) \geq 6^{3/3} \sqrt{ab}.
\]

It is easy to check that \( 6^{3/3} > 4\sqrt{6} \). Therefore, we have proved the claimed inequality.

(c) The proof in (b) is still valid if \( a := L_n, b := L_{n+1}, c := L_{n+2}, \) and \( d := L_{n+3} \).

(d) Since \( F_1 = 1 \), we may rewrite the claimed inequality as a cyclic form:

\[
\sqrt{(F_1^4 + 1)(F_2^4 + 1)} + \cdots + \sqrt{(F_{n-1}^4 + 1)(F_n^4 + 1)} + \sqrt{(F_n^4 + 1)(F_1^4 + 1)} > 2F_n F_{n+1}.
\]
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Because the square mean (SM) is greater than or equal to the arithmetic mean (AM), we have

$$\sqrt{(F_1^2 + 1)(F_2^2 + 1)} + \cdots + \sqrt{(F_{n-1}^2 + 1)(F_n^2 + 1)} + \sqrt{(F_n^2 + 1)(F_1^2 + 1)} \geq \frac{(F_1^2 + 1)(F_2^2 + 1)}{2} + \cdots + \frac{(F_{n-1}^2 + 1)(F_n^2 + 1)}{2} + \frac{(F_n^2 + 1)(F_1^2 + 1)}{2}.$$  

We therefore only need to prove that

$$(F_1^2 + 1)(F_2^2 + 1) + \cdots + (F_{n-1}^2 + 1)(F_n^2 + 1) + (F_n^2 + 1)(F_1^2 + 1) > 4F_nF_{n+1}.$$  

Since $F_nF_{n+1} = \sum_{i=1}^{n} F_i^2,$ the above inequality is equivalent to

$$(F_1^2 F_2^2 + 1) + \cdots + (F_{n-1}^2 F_n^2 + 1) + (F_n^2 F_1^2 + 1) > 2 \sum_{i=1}^{n} F_i^2.$$  

However, the above inequality can be transformed into

$$(F_1^2 F_2^2 - F_1^2 - F_2^2 + 1) + \cdots + (F_{n-1}^2 F_n^2 - F_{n-1}^2 - F_n^2 + 1) + (F_n^2 F_1^2 - F_n^2 - F_1^2 + 1)$$  

$$= (F_1^2 - 1)(F_2^2 - 1) + \cdots + (F_{n-1}^2 - 1)(F_n^2 - 1) + (F_n^2 - 1)(F_1^2 - 1) > 0.$$  

The proof is then complete. Note that the equality does not occur in either of the above inequalities or in the SM-AM inequality for $n \geq 3.$

**Also solved by Kenneth B. Davenport, Dmitry Fleichman, and the proposers.**

Some summation formulas with general recurrences

**H-823 Proposed by Hideyuki Ohtsuka, Saitama, Japan**  
(Vol. 56, No. 2, May 2018)

Given an integer $r \geq 2$, define the sequence \( \{G_n\}_{n \geq -r+1} \) by

$$G_n = G_{n-1} + G_{n-2} + \cdots + G_{n-r} \quad \text{for} \quad n \geq 1$$

with arbitrary \( G_0, G_{-1}, G_{-2}, \ldots, G_{-r+1} \). For an integer \( n \geq 1 \), prove that

(i) \[ \sum_{k=1}^{n} G_k G_{k+r} = \sum_{k=1}^{r} k(r - k - 1) + r + 1 \sum_{i=1}^{k} (G_{n+i-k} G_{n+i} - G_{i-k} G_i); \]

(ii) \[ \sum_{k=1}^{n} G_k G_{k+r+1} = \sum_{k=1}^{r} k(r - k - 1) + 2r \sum_{i=1}^{k} (G_{n+i-k} G_{n+i} - G_{i-k} G_i). \]

Solution by the proposer

Let $S_m := \sum_{k=1}^{n} G_k G_{k+m}$ and $A_k := \sum_{i=1}^{k} (G_{n+i-k} G_{n+i} - G_{i-k} G_i)$. We use the identity

$$S_0 = \sum_{k=1}^{r} k(r - k - 1) + 2 \sum_{i=1}^{r} A_k \quad \text{(see [1])}.$$  

For $m \geq 0$, since

$$2G_m = G_{m-1} + \cdots + G_{m-r+1} + G_{m-r} = G_{m+1} + G_{m-r},$$

we have

$$2S_m = S_{m+1} + S_{m-r}. \quad \text{(1)}$$  

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For $1 \leq k \leq r$, we have
\[
S_k - S_{-k} = \sum_{i=1}^{n} (G_i G_{i+k} - G_{i-k} G_i) = \sum_{i=1}^{k} (G_{n+i-k} G_{n+i} - G_{i-k} G_i) = A_k. \tag{2}
\]

(i) We have
\[
S_r = S_0 + \sum_{k=1}^{r} (S_k - S_{k-1}) = S_0 + \frac{1}{2} \sum_{k=1}^{r} (S_k - S_{k-1-r}) \quad \text{(by (1))}
\]
\[
= S_0 + \frac{1}{2} \sum_{k=1}^{r} (S_k - S_{-k}) = \sum_{k=1}^{r} \frac{k(r - k - 1) + 2}{2(r - 1)} A_k + \frac{1}{2} \sum_{k=1}^{r} A_k \quad \text{(by (2))}
\]
\[
= \sum_{k=1}^{r} \frac{k(r - k - 1) + r + 1}{2(r - 1)} A_k.
\]

(ii) We have
\[
S_{r+1} = 2S_r - S_0 \quad \text{(by (1))}
\]
\[
= 2 \sum_{k=1}^{r} \frac{k(r - k - 1) + r + 1}{2(r - 1)} A_k - \sum_{k=1}^{r} \frac{k(r - k - 1) + 2}{2(r - 1)} A_k
\]
\[
= \sum_{k=1}^{r} \frac{k(r - k - 1) + 2r}{2(r - 1)} A_k.
\]


Also solved by Dmitry Fleischman.

**Identities with generalized Fibonomial coefficients**

**H-824 Proposed by Hideyuki Ohtsuka, Saitama, Japan**
(Vol. 56, No. 2, May 2018)

Define the generalized Fibonomial coefficient \(^{(n)}_k F_r\) by
\[
\binom{n}{k}_{F_r} = \frac{F_{rn} F_{r(n-1)} F_{r(n-2)} \cdots F_{r(n-k+1)}}{F_{rk} F_{r(k-1)} F_{r(k-2)} \cdots F_r}
\]
for $0 < k \leq n$,

with \(^{(n)}_0 F_r = 1\) and \(^{(n)}_k F_r = 0\) (otherwise). For positive integers $n$, $r$, and $s$, find closed form expressions for the sums

(i) \[ \sum_{i+j=2s-1} (-1)^{(r+1)i} \binom{n-1}{i}_{F,r} \binom{n+1}{j}_{F,r}; \]

(ii) \[ \sum_{i+j=2s} (-1)^i \binom{n-1}{i}_{F,r} \binom{n+1}{j}_{F,r}. \]
Solution by the proposer

Let $m$, $n$, $r$, and $s$ be positive integers. We use the identity

$$
\sum_{k=0}^{n} (-1)^{\frac{rk(k+1)}{2} + mk} \binom{n}{k} z^k = \prod_{k=1}^{n} \left( 1 + (-1)^m \alpha^{r(n-k+1)} \beta^{rk} z \right)
$$

(see [1]).

We have

$$
\begin{align*}
\sum_{l=0}^{2n} \left( \sum_{i+j=l} (-1)^{\frac{r(i+1)}{2}} \binom{n+1}{i} \binom{n-1}{j} z^i \right) z^j &= \prod_{k=1}^{n+1} \left( 1 - \alpha^{r(n-k+2)} \beta^{rk} z \right) \prod_{k=1}^{n-1} \left( 1 + (-1)^r \alpha^{r(n-k)} \beta^{rk} z \right) \\
&= \prod_{k=0}^{n} \left( 1 - \alpha^{r(n-k+1)} \beta^{rk} z \right) \prod_{k=1}^{n-1} \left( 1 + (-1)^r \alpha^{r(n-k)} \beta^{rk} z \right) \\
&= \prod_{k=0}^{n} \left( 1 - (-1)^r \alpha^{rn} z \right) \prod_{k=1}^{n-1} \left( 1 - \alpha^{2r(n-k)} \beta^{2rk} z^2 \right) \\
&= \left( 1 - (-1)^r \alpha^{rn} z + (-1)^{rn} z^2 \right) \sum_{k=0}^{n-1} \binom{n-1}{k} z^{2k} \quad (3)
\end{align*}
$$

(by $\alpha \beta = -1$ and $L_{rn} = \alpha^{rn} + \beta^{rn}$).

(i) In (3), by comparing the coefficient of $z^{2s-1}$, we have

$$
\sum_{i+j=2s-1} (-1)^{\frac{i^2+i^2+j^2+3j+3i+2}{2}} \binom{n+1}{i} \binom{n-1}{j} = (-1)^{r+s} L_{rn} \binom{n-1}{s-1} F_{2r}.
$$

By $j = 2s - 1 - i$, we have

$$
(-1)^{\frac{i^2+i^2+j^2+3j+3i+2}{2}} = (-1)^{r^2+i+rs+s-2rsi+2rs^2} = (-1)^{(r+1)i+(r+1)s}.
$$

Therefore, we obtain

$$
\sum_{i+j=2s-1} (-1)^{(r+1)i} \binom{n+1}{i} \binom{n-1}{j} = (-1)^{(r+1)s} L_{rn} \binom{n-1}{s-1} F_{2r}.
$$

(ii) In (3), by comparing the coefficient of $z^{2s}$, we have

$$
\sum_{i+j=2s} (-1)^{\frac{i^2+i^2+j^2+3j+3i+2}{2}} \binom{n+1}{i} \binom{n-1}{j} = (-1)^s \binom{n-1}{s} F_{2r} - (-1)^{rn+s} \binom{n-1}{s-1} F_{2r}.
$$
By $j = 2s - i$, we have

$$(-1)^{\frac{1}{2}(i^2+i+j^2+j)}i+rs = (-1)^{i+ri^2-ri+3rs-2rsi+2rs^2} = (-1)^{i+rs}.$$ 

Therefore, we obtain

$$\sum_{i+j=2s} (-1)^i \binom{n-1}{i}_{F;r} \binom{n+1}{j}_{F;r} = (-1)^{(r+1)s} \left( \binom{n-1}{s}_{F;2r} - (-1)^{rn} \binom{n-1}{s-1}_{F;2r} \right).$$


Also solved by Dmitry Fleischman.

Errata: There are some typos in the Advanced Problem Section of Volume 58 Number 1, February 2020, Pages 89–95 as follows:

(i) Page 89, Line -1: The exponent “$1-(n-k)-1$” in “$2^{1-(n-k)-1}$” should be “$1-(n-k)$”.

(ii) Page 90, Lines 4-5: In the left sides of (i) and (ii), the denominators should be under a square root “$\sqrt{...}$”.

(iii) Page 92, Line 10: The numerator “$F_{n+4} - F_{n+1}$” should be “$F_{n+1} - F_n$”.

(iv) Page 94, Line 7: The expression “$F - n^b$” should be “$F^n$”.

The Editor apologizes for these typos.