ADVANCED PROBLEMS AND SOLUTIONS

EDITED BY
ROBERT FRONTCZAK

Please send all communications concerning ADVANCED PROBLEMS AND SOLUTIONS to Robert Frontczak, LBBW, Am Hauptbahnhof 2, 70173 Stuttgart, Germany, or by email at the address robert.frontczak@lbbw.de. This department especially welcomes problems believed to be new or extending old results. Proposers should submit solutions or other information that will assist the editor. To facilitate their consideration, all solutions sent by regular mail should be submitted on separate signed sheets within two months after publication of the problems.

PROBLEMS PROPOSED IN THIS ISSUE

H-936 Proposed by Andrés Ventas, Santiago de Compostela, Spain

Let $\lceil c_0; c_1, c_2, \dots \rceil = c_0 - \frac{1}{c_1} - \frac{1}{c_2} - \frac{1}{c_3} - \cdots$ denote a ceiling continued fraction with numerators of convergents $p_i = c_i p_{n-1} - p_{n-2}$; $p_0 = c_0$; $p_{-1} = 1$. Let $n!! = n(n-2)(n-4)\cdots$ denote the double factorial with $0!! = 1$. Also, we have the Engel expansion of $e^{1/r} = \{1, 1r, 2r, 3r, 4r, \dots\}$. Prove the expression

$$e^{-1/r} = \left\lceil 1; 1r+1, \frac{(i-2)!!(ir+1)}{(i-1)!!r}, \frac{(i-2)!!(ir+1)}{(i-1)!!}, \dots \right\rceil.$$

H-937 Proposed by Toyesh Prakash Sharma, Agra, India

For $n > 0$, show that

$$\sqrt[n]{F_1^{F_1} \cdot F_2^{F_2} \cdots F_n^{F_n}} + \sqrt[n]{L_1^{L_1} \cdot L_2^{L_2} \cdots L_n^{L_n}} \ge 2\Big(\frac{F_{n+3}-2}{n}\Big)^{\frac{F_{n+3}-2}{n}}.$$

H-938 Proposed by Hideyuki Ohtsuka, Saitama, Japan

Prove that

$$\sum_{n=1}^{\infty} \frac{1}{(F_n F_{n+1})^2} = \sum_{n=1}^{\infty} \frac{1}{(F_n F_{n+2})^2} + \frac{13 - 5\sqrt{5}}{2}.$$

H-939 Proposed by Kunle Adegoke, Ile-Ife, Nigeria and the editor

Let $\mathcal{P}$ be the parity function; that is, $\mathcal{P}(s) = s \mod 2$, and m, p, r be integers with $m \ne r$. For $N \ge 1$, show that

$$\sum_{n=1}^{N} \frac{F_{p(2n+m+r)}}{F_{p(n+m)}^2 F_{p(n+r)}^2} = \frac{1}{F_{p(m-r)}} \sum_{n=1}^{m-r} \left(\frac{1}{F_{p(n+r)}^2} - \frac{1}{F_{p(n+N+r)}^2}\right), \quad \text{if } p \text{ is even } \underline{\text{or}}\ \mathcal{P}(m) = \mathcal{P}(r),$$

and

$$\sum_{n=1}^{N} \frac{(-1)^{n-1} F_{p(2n+m+r)}}{F^2_{p(n+m)} F^2_{p(n+r)}}$$

$$= \frac{1}{F_{p(m-r)}} \sum_{n=1}^{m-r} \frac{(-1)^{n-1}}{F^2_{p(n+r)}} + \frac{(-1)^{N-1}}{F_{p(m-r)}} \sum_{n=1}^{m-r} \frac{(-1)^{n-1}}{F^2_{p(n+N+r)}}, \quad \text{if } p \text{ is odd } \underline{\text{and}}\ \mathcal{P}(m) \neq \mathcal{P}(r).$$

Show also the Lucas counterparts

$$\sum_{n=1}^{N} \frac{F_{p(2n+m+r)}}{L^2_{p(n+m)} L^2_{p(n+r)}} = \frac{1}{5} \frac{1}{F_{p(m-r)}} \sum_{n=1}^{m-r} \left(\frac{1}{L^2_{p(n+r)}} - \frac{1}{L^2_{p(n+N+r)}} \right), \quad \text{if } p \text{ is even } \underline{\text{or}}\ \mathcal{P}(m) = \mathcal{P}(r),$$

and

$$\sum_{n=1}^{N} \frac{(-1)^{n-1} F_{p(2n+m+r)}}{L^2_{p(n+m)} L^2_{p(n+r)}}$$

$$= \frac{1}{5} \frac{1}{F_{p(m-r)}} \sum_{n=1}^{m-r} \frac{(-1)^{n-1}}{L^2_{p(n+r)}} + \frac{1}{5} \frac{(-1)^{N-1}}{F_{p(m-r)}} \sum_{n=1}^{m-r} \frac{(-1)^{n-1}}{L^2_{p(n+N+r)}}, \quad \text{if } p \text{ is odd } \underline{\text{and}}\ \mathcal{P}(m) \neq \mathcal{P}(r).$$

H-940 Proposed by the editor

Prove that

$$\sum_{n=1}^{\infty} \frac{n+1}{\binom{4n}{2n}} = \frac{1}{2250} \left(390 + 125\sqrt{3}\pi - 99\sqrt{5} \ln \alpha^2 \right),$$

where $\alpha = (1+\sqrt{5})/2$ is the golden section.

SOLUTIONS

H-906 (corrected) Kenny B. Davenport, Dallas, PA

Prove the identity

$$3 \left(\sum_{k=1}^{n} F_{k-3} F_{k-2} F_{k-1} F_{k+1} F_{k+2} F_{k+3} - \sum_{k=1}^{n} F_k^6 \right) = 4(-1)^n (F_n F_{n+1})^2 - 11 F_n F_{n+1} + 12 D_n,$$

where $D_n = (1 - (-1)^n)/2$.

Solution by Hans J. H. Tuenter, Toronto, Canada

Recall Catalan's identity, $F_n^2 - F_{n-d} F_{n+d} = (-1)^{n-d} F_d^2$, where n and d are arbitrary integers. Although discovered F_{12} years ago, see [2, p. 314], the identity is still relevant today. Consider $s_n = F_{n-3} F_{n-2} F_{n-1} F_{n+1} F_{n+2} F_{n+3}$. A triple application of Catalan's identity gives

$$s_n = \prod_{d=1}^{3} \left(F_n^2 - (-1)^{n-d} F_d^2 \right) = F_n^6 + 4(-1)^n F_n^4 - F_n^2 - 4(-1)^n.$$

Proving Davenport's identity thus reduces to determining a closed-form expression for

$$S_n = \sum_{k=0}^{n} \left(s_k - F_k^6 \right) = 4 \sum_{k=0}^{n} (-1)^k F_k^4 - \sum_{k=0}^{n} F_k^2 - 4 \sum_{k=0}^{n} (-1)^k,$$

where we included the zero index, so that S_n is defined for all nonnegative integers n. The sum of squared Fibonacci numbers is an old chestnut that goes back to Édouard Lucas (1877), who gives $\sum_{k=0}^{n} F_k^2 = F_n F_{n+1}$. For the alternating sum of fourth powers of Fibonacci numbers, we use

$$3\sum_{k=0}^{n}(-1)^k F_k^4 = (-1)^n F_{n-2}F_nF_{n+1}F_{n+3} = (-1)^n(F_nF_{n+1})^2 - 2F_nF_{n+1},$$

where the first part is taken from [1, equation (5.6)], and the second part follows from the identity $F_{n-2}F_{n+3} = F_nF_{n+1} - 2(-1)^n$, easily proven by writing it as $F_{n-2}F_{n+2} = F_{n-1}F_{n+1} - 2(-1)^n$ and applying Catalan's identity to the products. Combining the components gives

$$3S_n = 4(-1)^n(F_nF_{n+1})^2 - 11F_nF_{n+1} - 12\delta_{n\,\text{is even}},$$

where δ_c is the Kronecker delta that is one, when the condition c holds, and zero otherwise. To return to Davenport's formulation, where the summation starts at index 1, we need to deduct $3S_0 = -12$ from both sides of our identity for S_n. This completes the proof of Problem H-906.

As a general comment, one might have been tempted to start with the Gelin-Cesàro identity (1880), $F_n^4 - F_{n-2}F_{n-1}F_{n+1}F_{n+2} = 1$. However, after this, one would still have to use Catalan's identity, in some shape or form. Although it is not common knowledge, the Gelin-Cesàro identity is nothing other than a double application of Catalan's identity in disguise. For Problem H-906, a triple application cuts right to the chase.

Editor's Note: The F_{12} in Tuenter's write-up is in relation to the year 2023, when the solution was written and submitted.

References

[1] R. S. Melham, *Alternating sums of fourth powers of Fibonacci and Lucas numbers*, The Fibonacci Quarterly, **38.3** (2000), 254–259.

[2] H. J. H. Tuenter. *Fibonacci summation identities arising from Catalan's identity*, The Fibonacci Quarterly, **60.4** (2022), 312–319.

Also solved by Michel Bataille, Brian Bradie, Charles K. Cook and Michael R. Bacon (jointly), Dmitry Fleischman, Hideyuki Ohtsuka, Raphael Schumacher, Yüksel Soykan, Albert Stadler, David Terr, and the proposer.

H-907 Proposed by Andrés Ventas, Santiago de Compostela, Spain, and Curtis Cooper, Warrensburg, MO

Let $n \geq 0$ be an integer. Prove that

$$F_{3n} = \sum_{i=0}^{\lfloor (n-1)/2 \rfloor} \binom{n-i-1}{i} 2^{2n-1-4i}.$$

First Solution by Hans J. H. Tuenter, Toronto, Canada

Let $F_n(x)$ be the Fibonacci polynomials, defined by the recurrence $F_{n+2}(x) = xF_{n+1}(x) + F_n(x)$, with initial conditions $F_0(x) = 0$ and $F_1(x) = 1$. Recall the closed-form expression for the Fibonacci polynomials

$$F_{n+1}(x) = \sum_{i=0}^{\lfloor n/2 \rfloor} \binom{n-i}{i} x^{n-2i},$$

and we see that Problem H-907 is equivalent to proving that $F_{3n} = 2F_n(4)$. It is well known that the Fibonacci numbers satisfy the lacunary recursion

$$F_{a(n+2)} = L_aF_{a(n+1)} - (-1)^aF_{an},$$

where a is an arbitrary integer; the recursion is easily proven using the Binet formulae for the Fibonacci and Lucas numbers. For fixed, odd integer values of a, consider the sequences F_{an}/F_a and $F_n(L_a)$. They follow the same recursion and have the same initial values, so that the sequences are the same; thus $F_{an}/F_a = F_n(L_a)$. Because $F_3 = 2$ and $L_3 = 4$, this completes the proof of Problem H-907. One also sees that we are dealing with a special case of the more general identity

$$F_{a(n+1)} = F_a \sum_{i=0}^{\lfloor n/2 \rfloor} \binom{n-i}{i} L_a^{n-2i},$$

which is valid for nonnegative integers n and odd integers a. In practice, one may assume a to be positive, by virtue of the reflection formulae $F_{-n} = (-1)^{n+1}F_n$ and $L_{-n} = (-1)^n L_n$. The Lucas numbers follow the same lacunary recurrence as the Fibonacci numbers. For odd integers a, a similar reasoning leads to $L_{an} = L_n(L_a)$, where $L_n(x)$ are the Lucas polynomials, defined by the recurrence $L_{n+2}(x) = xL_{n+1}(x) + L_n(x)$, and initial conditions $L_0(x) = 2$ and $L_1(x) = x$. Using the closed-form expression for the Lucas polynomials, gives the identity

$$L_{an} = \sum_{i=0}^{\lfloor n/2 \rfloor} \frac{n}{n-i} \binom{n-i}{i} L_a^{n-2i},$$

which is valid for positive integers n and odd integers a. For even integers a, one can show that $F_{an}/F_a = (-\mathbf{i})^{n-1}F_n(\mathbf{i}L_a)$ and $L_{an} = (-\mathbf{i})^n L_n(\mathbf{i}L_a)$, where $\mathbf{i}$ is the imaginary unit defined by $\mathbf{i}^2 = -1$. This gives the identities

$$F_{a(n+1)} = F_a \sum_{i=0}^{\lfloor n/2 \rfloor} \binom{n-i}{i} (-1)^i L_a^{n-2i} \quad \text{and} \quad L_{an} = \sum_{i=0}^{\lfloor n/2 \rfloor} \frac{n}{n-i} \binom{n-i}{i} (-1)^i L_a^{n-2i},$$

which are valid for nonnegative and positive integers n, respectively, and even integers a. Observe that, when we replace $(-1)^i$ by $(-1)^{(a+1)i}$ in these last two identities, one effectively combines the formulae for the even and odd cases of a.

Second Solution by Yüksel Soykan, Zonguldak, Turkey

We use the following generating function of Fibonacci numbers (k fixed):

$$\sum_{n=0}^{\infty} F_{nk} y^n = \frac{yF_k}{1 - (F_{k+1} + F_{k-1})y + (-1)^k y^2} = \frac{yF_k}{1 - L_k y + (-1)^k y^2}.$$

Then, we get

$$\begin{aligned}
\sum_{n=0}^{\infty} F_{nk} y^n &= yF_k \sum_{i=0}^{\infty} \sum_{n=0}^{i} \binom{i}{n} L_k^{i-n} (-(-1)^k)^n y^{i+n} \\
&= F_k \sum_{n=0}^{\infty} \sum_{i=0}^{\lfloor (n-1)/2 \rfloor} \binom{n-1-i}{i} L_k^{n-1-2i} (-(-1)^k)^i y^n \\
&= \sum_{n=0}^{\infty} F_k \sum_{i=0}^{\lfloor (n-1)/2 \rfloor} \binom{n-1-i}{i} L_k^{n-1-2i} (-(-1)^k)^i y^n.
\end{aligned}$$

Now, we read off the coefficient of y^n :

$$F_{nk} = F_k \sum_{i=0}^{\lfloor (n-1)/2 \rfloor} \binom{n-1-i}{i} L_k^{n-1-2i} (-(-1)^k)^i.$$

By setting $k = 3$ in the above formula, we get the required result.

Also solved by Michel Bataille, Won Kyun Jeong, Hideyuki Ohtsuka, Ángel Plaza, Albert Stadler, Seán M. Stewart, and the proposers.

H-908 Proposed by D. M. Bătineţu Giurgiu, Bucharest, Romania, and Neculai Stanciu, Buzău, Romania

Prove that in every triangle ABC with area F and altitudes $h_a,\ h_b,\ h_c$ perpendicular to the sides $a,\ b,\ c$, respectively, the following inequalities hold:

(i) $\dfrac{a^{F_n}b^{F_{n+2}}}{h_a^{F_{n+1}}} + \dfrac{b^{F_n}c^{F_{n+2}}}{h_b^{F_{n+1}}} + \dfrac{c^{F_n}a^{F_{n+2}}}{h_c^{F_{n+1}}} \geq 2^{F_n+F_{n+2}}\sqrt{3}^{2-F_{n+2}}F^{F_n}$;

(ii) $\dfrac{a^{F_n^2}b^{F_{2n+1}}}{h_a^{F_{n+1}^2}} + \dfrac{b^{F_n^2}c^{F_{2n+1}}}{h_b^{F_{n+1}^2}} + \dfrac{c^{F_n^2}a^{F_{2n+1}}}{h_c^{F_{n+1}^2}} \geq 2^{F_{2n+1}+F_n^2}\sqrt{3}^{2-F_{2n+1}}F^{F_n^2}$.

Solution by Wei-Kai Lai, Walterboro, SC

We first establish the inequality $ab + bc + ca \geq 4\sqrt{3}F$. According to exercise 238 in [1, p. 208],

$$a^2 + b^2 + c^2 \geq 4\sqrt{3}F + (a-b)^2 + (b-c)^2 + (c-a)^2.$$

Since

$$\begin{aligned} a^2 + b^2 + c^2 - (a-b)^2 - (b-c)^2 - (c-a)^2 &= 2ab + 2bc + 2ca - a^2 - b^2 - c^2 \\ &= ab + bc + ca - 1/2[(a-b)^2 + (b-c)^2 + (c-a)^2] \leq ab + bc + ca, \end{aligned}$$

the inequality $ab + bc + ca \geq 4\sqrt{3}F$ is then proved. A proof of inequality (i) follows.

It is known that $h_a = 2F/a$, $h_b = 2F/b$, $h_c = 2F/c$, so the claimed inequality (i) can be written as

$$\frac{a^{F_n+F_{n+1}}b^{F_{n+2}}}{(2F)^{F_{n+1}}} + \frac{b^{F_n+F_{n+1}}c^{F_{n+2}}}{(2F)^{F_{n+1}}} + \frac{c^{F_n+F_{n+1}}a^{F_{n+2}}}{(2F)^{F_{n+1}}} \geq 3\frac{2^{F_n+F_{n+2}}F^{F_n}}{\sqrt{3}^{F_{n+2}}},$$

or

$$\frac{(ab)^{F_{n+2}} + (bc)^{F_{n+2}} + (ca)^{F_{n+2}}}{3} \geq \Big(\frac{4F}{\sqrt{3}}\Big)^{F_{n+2}}.$$

According to Jensen's inequality,

$$\frac{(ab)^{F_{n+2}} + (bc)^{F_{n+2}} + (ca)^{F_{n+2}}}{3} \geq \Big(\frac{ab+bc+ca}{3}\Big)^{F_{n+2}},$$

so we only need to prove that $(ab + bc + ca)/3 \geq 4F/\sqrt{3}$, which is apparently true according to the inequality we proved earlier. Similarly, inequality (ii) can be written as

$$\frac{a^{F_n^2+F_{n+1}^2}b^{F_{2n+1}}}{(2F)^{F_{n+1}^2}} + \frac{b^{F_n^2+F_{n+1}^2}c^{F_{2n+1}}}{(2F)^{F_{n+1}^2}} + \frac{c^{F_n^2+F_{n+1}^2}a^{F_{2n+1}}}{(2F)^{F_{n+1}^2}} \geq 3\frac{2^{F_n^2+F_{2n+1}}F^{F_n^2}}{\sqrt{3}^{F_{2n+1}}}.$$

Since $F_n^2 + F_{n+1}^2 = F_{2n+1}$ [2, Corollary 5.4, p.79], the above inequality can be simplified to

$$\frac{(ab)^{F_{2n+1}} + (bc)^{F_{2n+1}} + (ca)^{F_{2n+1}}}{3} \geq \Big(\frac{4F}{\sqrt{3}}\Big)^{F_{2n+1}}.$$

Applying Jensen's inequality again we have

$$\frac{(ab)^{F_{2n+1}} + (bc)^{F_{2n+1}} + (ca)^{F_{2n+1}}}{3} \geq \Big(\frac{ab+bc+ca}{3}\Big)^{F_{2n+1}}.$$

Since $(ab + bc + ca)/3 \geq 4F/\sqrt{3}$, inequality (ii) is then proved.

References

[1] Z. Cvetkovski, *Inequalities: Theorems, Techniques and Selected Problems*, Springer, New York, 2012.
[2] T. Koshy, *Fibonacci and Lucas Numbers with Applications*, Wiley, New York, 2001.

Also solved by Michel Bataille, Brian Bradie, Dmitry Fleischman, Albert Stadler, Nicuşor Zlota, and the proposers.

H-909 Proposed by Michel Bataille, Rouen, France

Let n be a positive integer. For each integer k in $[0, n/2]$, let

$$U_{n,k} = \sum_{j=0}^{k} \binom{k}{j} \frac{4^j}{n-j} \qquad \text{and} \qquad V_{n,k} = \sum_{j=0}^{k} \binom{n}{j} (-5)^j.$$

Prove that

$$n \sum_{k=0}^{\lfloor n/2 \rfloor} \binom{n+1}{2k+1} U_{n,k} = 2^n L_n = \sum_{k=0}^{\lfloor n/2 \rfloor} \binom{n+1}{2k+1} \frac{(-1)^k V_{n,k}}{\binom{n-1}{k}}.$$

Solution by Albert Stadler, Herrliberg, Switzerland

We note that

$$U_{n,k} = \sum_{j=0}^{k} \binom{k}{j} \frac{4^j}{n-j} = \sum_{j=0}^{k} \binom{k}{j} 4^j \int_0^1 t^{n-1-j} dt = \int_0^1 t^{n-1} \left(1 + \frac{4}{t}\right)^k dt.$$

Hence,

$$\begin{aligned}
&\sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n+1}{2k+1} U_{n,k} = \sum_{k=0}^{n+1} \binom{n+1}{k} \left(\frac{1-(-1)^k}{2}\right) U_{n,\frac{k-1}{2}} \\
= \quad & \sum_{k=0}^{n+1} \binom{n+1}{k} \left(\frac{1-(-1)^k}{2}\right) \int_0^1 t^{n-1} \left(\sqrt{1+\frac{4}{t}}\right)^{k-1} dt \\
= \quad & \frac{1}{2} \int_0^1 \frac{t^{n-1}}{\sqrt{1+\frac{4}{t}}} \left(\left(1 + \sqrt{1+\frac{4}{t}}\right)^{n+1} - \left(1 - \sqrt{1+\frac{4}{t}}\right)^{n+1} \right) dt \\
\underbrace{=}^{t=u^2} \quad & \int_0^1 \frac{u^{2n-1}}{\sqrt{1+\frac{4}{u^2}}} \left(\left(1 + \sqrt{1+\frac{4}{u^2}}\right)^{n+1} - \left(1 - \sqrt{1+\frac{4}{u^2}}\right)^{n+1} \right) du \\
= \quad & \int_0^1 \frac{u^{n-1}}{\sqrt{u^2+4}} \left(\left(u + \sqrt{u^2+4}\right)^{n+1} - \left(u - \sqrt{u^2+4}\right)^{n+1} \right) du \\
\underbrace{=}^{u=x-\frac{1}{x}} \quad & \int_1^{\alpha} \frac{\left(x - \frac{1}{x}\right)^{n-1}}{x + \frac{1}{x}} \left((2x)^{n+1} - \left(-\frac{2}{x}\right)^{n+1} \right) \left(1 + \frac{1}{x^2}\right) dx \\
= \quad & 2^{n+1} \int_1^{\alpha} \left(x - \frac{1}{x}\right)^{n-1} \left(x^n + (-1)^n x^{-n-2}\right) dx
\end{aligned}$$

$$
\begin{aligned}
&= 2^{n+1}\int_1^{\alpha} x\left(x^2-1\right)^{n-1}dx + 2^{n+1}(-1)^n\int_1^{\alpha}\frac{1}{x^3}\left(1-\frac{1}{x^2}\right)^{n-1}dx\\
&= 2^{n+1}\int_1^{\alpha} x\left(x^2-1\right)^{n-1}dx - 2^{n+1}\int_{\frac{1}{\alpha}}^{1} x\left(x^2-1\right)^{n-1}dx\\
&\overbrace{=}^{x=\sqrt{y}} 2^n\int_1^{\alpha^2}(y-1)^{n-1}dy - 2^n\int_{\frac{1}{\alpha^2}}^{1}(y-1)^{n-1}dy\\
&= \frac{2^n}{n}\left(\alpha^2-1\right)^n + \frac{2^n}{n}\left(\frac{1}{\alpha^2}-1\right)^n = \frac{2^n}{n}\alpha^n + \frac{2^n}{n}\beta^n = \frac{2^n}{n}L_n.
\end{aligned}
$$

For the second part, it is sufficient to prove that

$$(-1)^k n\binom{n-1}{k}U_{n,k} = V_{n,k},\ 0\le k\le n-1,\ n\ge 1.$$

For that purpose, it is sufficient to prove that $(-1)^0 n\binom{n-1}{0}U_{n,0} = V_{n,0}$ (which is clearly true) and that $(-1)^k n\binom{n-1}{k}U_{n,k} - (-1)^{k-1}n\binom{n-1}{k-1}U_{n,k-1} = V_{n,k} - V_{n,k-1}$, which is equivalent to each of the following lines:

$$(-1)^k n\binom{n-1}{k}\sum_{j=0}^{k}\binom{k}{j}\frac{4^j}{n-j} - (-1)^{k-1}n\binom{n-1}{k-1}\sum_{j=0}^{k-1}\binom{k-1}{j}\frac{4^j}{n-j} = \binom{n}{k}(-5)^k$$

$$(n-k)\sum_{j=0}^{k}\binom{k}{j}\frac{4^j}{n-j} + k\sum_{j=0}^{k-1}\binom{k-1}{j}\frac{4^j}{n-j} = 5^k$$

$$(n-k)\sum_{j=0}^{k}\binom{k}{j}4^j\int_0^1 t^{n-1-j}dt + k\sum_{j=0}^{k-1}\binom{k-1}{j}4^j\int_0^1 t^{n-1-j}dt = 5^k$$

$$(n-k)\int_0^1 t^{n-1}\left(1+\frac{4}{t}\right)^k dt + k\int_0^1 t^{n-1}\left(1+\frac{4}{t}\right)^{k-1}dt = 5^k$$

$$n\int_0^1 t^{n-1}\left(1+\frac{4}{t}\right)^k dt - 4k\int_0^1 t^{n-2}\left(1+\frac{4}{t}\right)^{k-1}dt = 5^k$$

$$\int_0^1 \frac{d}{dt}\left(t^n\left(1+\frac{4}{t}\right)^k\right)dt = 5^k,$$

which is clearly true.

Also solved by Dmitry Fleischman, and the proposer.

H-910 Proposed by Robert Frontczak, Stuttgart, Germany

Show the following identity involving Lucas numbers

$$\sum_{k=1}^{\infty}\frac{L_{k+1}}{k(k+1)^2 2^{k+1}} = 1 - \frac{\pi^2}{12} + \frac{\ln(2)}{2}(2\ln(2)-3) - \ln(\alpha)(2\ln(\alpha)-\sqrt{5}).$$

Solution by Won Kyun Jeong, Daegu, Korea

Since $\frac{1}{k(k+1)^2} = \frac{1}{k(k+1)} - \frac{1}{(k+1)^2}$, we have

$$\sum_{k=1}^{\infty}\frac{L_{k+1}}{k(k+1)^2 2^{k+1}} = \sum_{k=1}^{\infty}\frac{L_{k+1}/2^{k+1}}{k(k+1)} - \sum_{k=1}^{\infty}\frac{L_{k+1}/2^{k+1}}{(k+1)^2}.$$

Note that
$$\sum_{k=1}^{\infty}\frac{x^{k+1}}{k(k+1)}=\sum_{n=1}^{\infty}\frac{1}{k}\int_0^x t^k dt = -\int_0^x \ln(1-t)dt = -(x-1)\ln(1-x)+x.$$
By Binet's formula, we have
$$\begin{aligned}\sum_{k=1}^{\infty}\frac{L_{k+1}/2^{k+1}}{k(k+1)} &= \sum_{k=1}^{\infty}\frac{(\frac{\alpha}{2})^{k+1}}{k(k+1)}+\sum_{k=1}^{\infty}\frac{(\frac{\beta}{2})^{k+1}}{k(k+1)}\\ &= \frac{\alpha}{2}-\left(1-\frac{\alpha}{2}\right)\ln\left(1-\frac{\alpha}{2}\right)+\frac{\beta}{2}-\left(1-\frac{\beta}{2}\right)\ln\left(1-\frac{\beta}{2}\right)\\ &= \frac{1}{2}+\sqrt{5}\ln\alpha-\frac{3}{2}\ln 2.\end{aligned}$$
Next, we can calculate
$$\sum_{k=1}^{\infty}\frac{L_{k+1}/2^{k+1}}{(k+1)^2}=\sum_{k=1}^{\infty}\frac{(\frac{\alpha}{2})^{k+1}}{(k+1)^2}+\sum_{k=1}^{\infty}\frac{(\frac{\beta}{2})^{k+1}}{(k+1)^2}=\mathrm{Li}_2\left(\frac{\alpha}{2}\right)+\mathrm{Li}_2\left(\frac{\beta}{2}\right)-\frac{1}{2},$$
where
$$\mathrm{Li}_2(x)=\sum_{n=1}^{\infty}\frac{x^n}{n^2},\qquad (-1\le x\le 1)$$
is the dilogarithm function. For x, y, $x+y<1$, the identity holds (Abel, see [1]):
$$\mathrm{Li}_2\left(\frac{x}{1-x}\cdot\frac{y}{1-y}\right)=\mathrm{Li}_2\left(\frac{x}{1-y}\right)+\mathrm{Li}_2\left(\frac{y}{1-x}\right)-\mathrm{Li}_2(x)-\mathrm{Li}_2(y)-\ln(1-x)\ln(1-y).$$
If $x=\frac{\alpha}{2}, y=\frac{\beta}{2}$, then
$$\mathrm{Li}_2(-1)=\mathrm{Li}_2(-\beta)+\mathrm{Li}_2(-\alpha)-\mathrm{Li}_2\left(\frac{\alpha}{2}\right)-\mathrm{Li}_2\left(\frac{\beta}{2}\right)+(\ln 2+2\ln\alpha)(2\ln\alpha-\ln 2).$$
It follows that
$$\mathrm{Li}_2\left(\frac{\alpha}{2}\right)+\mathrm{Li}_2\left(\frac{\beta}{2}\right)=\frac{\pi^2}{12}+2\ln^2\alpha-\ln^2 2,$$
since $\mathrm{Li}_2(-1)=-\frac{\pi^2}{12}, \mathrm{Li}_2(-\alpha)=-\frac{\pi^2}{10}-\ln^2\alpha$, and $\mathrm{Li}_2(-\beta)=\frac{\pi^2}{10}-\ln^2\alpha$. Finally, we obtain
$$\sum_{k=1}^{\infty}\frac{L_{k+1}/2^{k+1}}{(k+1)^2}=\frac{\pi^2}{12}+2\ln^2\alpha-\ln^2 2-\frac{1}{2},$$
and the required result follows.

References

[1] F. M. S. Lima, *On the accessibility of Khoi's dilogarithm identity involving the golden ratio*, Vietnam J. Math., **45** (2017), 619–623.

Also solved by Brian Bradie, Dmitry Fleischman, Albert Stadler, Seán M. Stewart, and the proposer.